4.3.3 Result

let $\sigma_0 \equiv \sigma_{e^+e^- \to q\bar{q}}$ (tree level) denote the leading order (LO) result
(see pg. 45; d-dim result on pg 51)

then, in dimensional regularization, real and virtual QCD-corrections are

LO (pg. 51) $$\sigma_{e^+e^- \to q\bar{q}g} = \sigma_0 \frac{\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{s}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \left(\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{19}{2} - \pi^2 + \mathcal{O}(\varepsilon)\right) + \mathcal{O}(\alpha_s^2)$$

NLO (pg. 54) $$\sigma_{e^+e^- \to q\bar{q}} = \sigma_0 \left\{ 1 + \frac{\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{s}\right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \pi^2 + \mathcal{O}(\varepsilon)\right) + \mathcal{O}(\alpha_s^2) \right\}$$

the sum, needed for the hadronic cross section, is finite;
can take $\varepsilon \to 0$

NLO $$\Rightarrow \sigma_{e^+e^- \to \text{hadrons}} = \sigma_0 \left\{ 1 + \frac{\alpha_s C_F}{2\pi} \frac{3}{2} + \mathcal{O}(\alpha_s^2) \right\} \overset{N_c=3}{=} \sigma_0 \left\{ 1 + \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2) \right\}$$

in QCD, $N_c = 3$, $C_F = \frac{N_c^2-1}{2N} = \frac{4}{3}$

Before using this results, a couple of remarks:

- cancellation of soft and collinear divergences between the real and virtual gluon diagrams is not accidental.
  They are in fact guaranteed by theorems (Bloch/Nordsieck, Kinoshita/Lee/Nauenberg (KLN)): suitably defined inclusive quantities will be IR safe in the massless limit.
  (($\sigma_{e^+e^- \to \text{hadrons}}$ is such a quantity; $\sigma_{e^+e^- \to q\bar{q}}$ is not))
  → proof in QCD: see e.g. [Collins/Soper, Ann Rev Nucl Sci 37 (1987) 383]
- our result would be worthless if it depended on our choice of regularization procedure, dim. reg.
  Proof of independence is beyond this lecture; but demonstrate it by comparing with gluon mass regularization scheme (m_g = gluon mass)

$$\sigma_{e^+e^- \to q\bar{q}g} = \sigma_0 \frac{\alpha_s C_F}{2\pi} \left(\ln^2 \frac{s}{m_g^2} - 3 \ln \frac{s}{m_g^2} + 7 - \frac{\pi^2}{3} + \mathcal{O}(\varepsilon)\right)$$

$$\sigma_{e^+e^- \to q\bar{q}} = \sigma_0 \left\{ 1 + \frac{\alpha_s C_F}{2\pi} \left(-\ln^2 \frac{s}{m_g^2} + 3 \ln \frac{s}{m_g^2} - \frac{11}{2} + \frac{\pi^2}{3} + \mathcal{O}(\varepsilon)\right)\right.$$

$$\Rightarrow \sigma_{e^+e^- \to \text{hadrons}} = \sigma_0 \left\{ 1 + \frac{\alpha_s C_F}{2\pi} \frac{3}{2} + \mathcal{O}(\alpha_s^2) \right\}$$

→ individual cross sections completely different; sum scheme independent!

- at this order (NLO), and for $m_q = 0$, our computation of the QCD correction is independent of the nature of the exchanged weak boson (we took the photon only, cf pg. 48).

→ generalization: the $(1 + \frac{\alpha_s}{\pi})$ result is valid also for $R_{Z\,peak}$, see pg. 47

ratios: $$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \underbrace{N_c\Big(\sum_f Q_f^2\Big)}_{s \ll m_Z^2\text{ ; otherwise see § 4.2}}\Big(1 + \frac{\alpha_s C_F}{2\pi}\frac{3}{2} + O(\alpha_s^2)\Big)$$

$$R_{Z\,peak} = \frac{\sigma(e^+e^- \to Z \to \text{hadrons})}{\sigma(e^+e^- \to Z \to \mu^+\mu^-)}\Bigg|_{s = m_Z^2} = 19.984\Big(1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2)\Big)$$

note that NLO correction is positive.

α_s :
- comparing with the (correctly scaled) experimental result

$$R_{Z\,peak}(\text{LEP}) = 20.767 \pm 0.025 \qquad (\text{see pg. } 47)$$

⇒ our first measurement of α_s : $\alpha_s(m_Z) = 0.123 \pm 0.004$

((compare with "world average" from PDG (2012): 0.1184 ± 0.0007))

- as another determination of α_s, let us compare to data taken by PETRA (DESY), at $\sqrt{s} \approx 34$ GeV

$$R(s \approx (34\,\text{GeV})^2, \text{PETRA}) = 3.88 \pm 0.03$$

we would predict ($udsc\underset{4.5\,\text{GeV}}{b}$ − $\underset{172\,\text{GeV}}{t}$ too heavy)

$$R((34\,\text{GeV})^2) = \underbrace{3\Big(2\big(\tfrac{2}{3}\big)^2 + 3\big(-\tfrac{1}{3}\big)^2\Big)}_{= \frac{11}{3} \approx 3.667}\Big(1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2)\Big)$$

or, including Z, $R((34\,\text{GeV})^2) = 3.69\Big(1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2)\Big)$

⇒ our second measurement of α_s : $\alpha_s(34\,\text{GeV}) = 0.162 \pm 0.026$

- recall (§ 3.4, pg. 42) that α_s is running!

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln(\mu^2/\mu_0^2)}, \quad \text{with } \beta_0 = \frac{11}{3}N_c - \frac{2}{3}N_f \overset{N_c = 3}{=} 11 - \frac{2}{3}N_f$$

$$\Leftrightarrow \frac{1}{\alpha_s(\mu_1)} - \frac{1}{\alpha_s(\mu_2)} = \frac{\beta_0}{4\pi}\ln\Big(\frac{\mu_1^2}{\mu_2^2}\Big)$$

⇒ 2^{nd} measurement translates into $\alpha_s(m_Z = 91.2\,\text{GeV}) = 0.135 \pm 0.018$

→ so far, our NLO correction to R shows correct qualitative features.

→ but what about higher orders?

4.3.4 Higher-order QCD corrections to R(s)

write $R(s) = 3\left(\sum_f Q_f^2\right) \cdot K_{QCD}$

where $K_{QCD} = 1 + 1 \cdot \frac{\alpha_s(\mu)}{\pi} + \sum_{n\geq 2} \underbrace{C_n\left(\frac{s}{\mu^2}\right)} \cdot \left(\frac{\alpha_s(\mu)}{\pi}\right)^n$

↑ result of our computation; from $\left\{\begin{array}{ll}\text{tree-level} & q\bar{q}g \\ \text{one-loop} & q\bar{q}\end{array}\right\}$ final state

the functions $\underbrace{C_n\left(\frac{s}{\mu^2}\right)}$ follow from higher-order computations:

e.g. C_2 from $\left\{\begin{array}{ll}\text{tree-level} & q\bar{q}gg,\ q\bar{q}q\bar{q} \\ \text{one-loop} & q\bar{q}g \\ \text{two-loop} & q\bar{q}\end{array}\right\}$ final states

etc...

→ note that in our computation, there were no UV divergences ((in fact, those in  +  cancel exactly)), so we did not need to renormalize, hence our coefficient did not depend on the renormalization scale μ: $C_1\left(\frac{s}{\mu^2}\right) = 1$.

→ in higher orders, we will encounter UV divs, hence $C_{n\geq 2}$ are renormalization scheme dependent.
If we could sum the whole series, it would be μ-indep.
In a truncated series, μ-dependence is of higher order.

⇒ μ-dependence of $C_n\left(\frac{s}{\mu^2}\right)$ is fixed by knowing μ-dependence of $\alpha_s(\mu)$ (pg 38: $\mu^2 \partial_{\mu^2} \alpha_s = -\frac{\beta_0}{4\pi}\alpha_s^2 - \frac{\beta_1}{(4\pi)^2}\alpha_s^3 - \ldots$)

$$\Rightarrow C_2\left(\frac{s}{\mu^2}\right) = C_2(1) + C_1(1) \boxed{\frac{\beta_0}{4} \ln\left(\frac{\mu^2}{s}\right)} \equiv L$$

$$C_3\left(\frac{s}{\mu^2}\right) = C_3(1) + C_1(1) L^2 + \left[C_1(1)\frac{\beta_1}{4\beta_0} + 2C_2(1)\right] L$$

etc. (check ?!)

$$\left(\left(\beta_1 = \frac{2}{3}\left(17N_c^2 - 5N_c N_f - 3C_F N_f\right) \overset{N_c=3}{=} 102 - \frac{38}{3}N_f\right)\right)$$

- C_2 and C_3 have been computed [Samuel/Surguladze, PRL 66 (1991) 560] [Gorishny/Kataev/Larin, PL B 259 (1991) 144]

(here $N_c = 3$, ren. scale set to $\mu = \sqrt{s}$, $\overline{MS}$ scheme)

$$C_2(1) = \left(\frac{365}{24} - 11\zeta(3)\right) + \left(-\frac{11}{12} + \frac{2}{3}\zeta(3)\right)N_f$$
$$\approx 1.986 - 0.115\, N_f$$

$$C_3(1) = \left(\frac{87029}{288} - \frac{1103}{4}\zeta(3) + \frac{275}{6}\zeta(5)\right) + \left(-\frac{7847}{216} + \frac{262}{9}\zeta(3) - \frac{25}{9}\zeta(5)\right)N_f$$
$$+ \left(\frac{151}{162} - \frac{19}{27}\zeta(3)\right)N_f^2 - \frac{\pi^2}{432}(33-2N_f)^2 + \eta\left(\frac{55}{72} - \frac{5}{3}\zeta(3)\right)$$
$$\approx -6.637 - 1.200\, N_f - 0.005\, N_f^2 - 1.240\, \eta$$

where $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ is the Riemann Zeta function $\quad \zeta(3) \approx 1.202057$, $\zeta(5) \approx 1.036928$

and $\eta = \dfrac{\left(\sum_q Q_q\right)^2}{3\left(\sum_q Q_q^2\right)}$, $\sum_q$ over all quarks with $m_q \ll \sqrt{s}$ (effectively massless)

((for $R_{Z\text{peak}}$, QCD corrections are again the same, except that $\eta \to \dfrac{\left(\sum_q V_q\right)^2}{3\sum_q (V_q^2 + A_q^2)}$))

- having a few orders of the perturbative series, can now discuss convergence

→ coefficients are scheme-dependent, so can try to find an "optimal" scheme (from the point of view of convergence), examples are:

FAC (fastest apparent convergence)
choose scale $\mu = \mu_{FAC}$ such that $R^{(1)}(\mu_{FAC}) = R^{(n)}(\mu_{FAC})$

PMS (principle of minimal sensitivity)
choose $\mu = \mu_{PMS}$ such that $\mu\partial_\mu R^{(n)}(\mu)\big|_{\mu_{PMS}} = 0$
[Stevenson, PLB 100 (1981) 61]

BLM ([Brodsky/Lepage/Mackenzie, PR D28 (1983) 228])
absorb all N_f-terms into α_s via β fct

etc ...

for $R^{(3)}(s)$, these are $\{\mu_{FAC}, \mu_{PMS}, \mu_{BLM}\} \overset{(N_f=5 \text{ massless flavors})}{\approx} \{0.692, 0.587, 0.708\}\sqrt{s}$

→ μ-variation does get weaker, comparing $R^{(1)}(s)$, $R^{(2)}(s)$, $R^{(3)}(s)$