4.3.2 virtual corrections: $\sigma_{e^+e^-\to q\bar q}$ at $\mathcal{O}(\alpha_s)$

structure of cross section computation:

$$|\mathcal{M}_0 + \alpha_s \mathcal{M}_1 + \mathcal{O}(\alpha_s^2)|^2 = |\mathcal{M}_0|^2 + \alpha_s(\mathcal{M}_1\mathcal{M}_0^* + \mathcal{M}_0\mathcal{M}_1^*) + \mathcal{O}(\alpha_s^2)$$

$\to$ need to compute interference term $\mathcal{M}_1\mathcal{M}_0$ only.

recall $\mathcal{M}_0 =$ 

$\alpha_s\mathcal{M}_1 =$ 

(a) (b) (c)

- in dimensional regularization,  $\sim \int d^dk \, \frac{1}{k^2}\frac{\not k + \not p}{(k+p)^2} = \not p \, I(p^2)$

with $\dim[I(p^2)] = m^{d-4}$

but $p^2 = 0$ (due to onshell condition), so no scale $\Rightarrow$ zero

$\Rightarrow \mathcal{M}_{1(a)} = 0 = \mathcal{M}_{1(b)}$ in dim. reg.

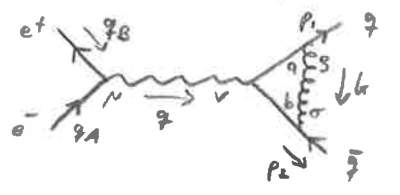
- for diagram (c), need to do some computation

$$\sigma_{e^+e^-\to q\bar q} \overset{\text{pg. 49}}{=} \frac{1}{2s}\left(\prod_{i=1}^{2}\int \frac{d^{d-1}p_i}{(2\pi)^{d-1}2E_{p_i}}\right)(2\pi)^d\,\delta^{(d)}(p_1+p_2-q_A-q_B)\,\langle|\mathcal{M}|^2\rangle$$

structure of $\alpha\mathcal{M}_{1(c)}$ is similar to $\mathcal{M}_0$!

recall (pg 44) $\mathcal{M}_0 =$  $= \frac{e^2 Q_q}{q^2}\,\bar v_B \gamma^\mu u_A \, \bar u_1 \gamma_\mu v_2$

$$\langle|\mathcal{M}_0|^2\rangle \overset{\text{do spin sums}}{=} \frac{e^4 Q_q^2}{4q^4}\,\mathrm{tr}(\not q_B \gamma^\mu \not q_A \gamma^{\nu(\dagger)})\,\mathrm{tr}(\not p_1 \gamma_\mu \not p_2 \gamma_\nu^{(\dagger)})$$

now, $\alpha_s\mathcal{M}_1 =$ 

use Feynman gauge $\xi = 1$, massless quarks $m_q = 0$

$$= \bar v_B(-ie\gamma^\mu)u_A\left(-\frac{ig_{\mu\nu}}{q^2}\right)\bar u_1(ig_s\gamma^g T^a)\int\frac{d^dk}{(2\pi)^d}\,\frac{i(\not k + \not p_1)}{(k+p_1)^2}(-ieQ_q\gamma^\nu)\frac{i(\not k - \not p_2)}{(k-p_2)^2} \,*$$

$$*\,(ig_s\gamma^\sigma T^b)\,v_2\left(\frac{-i\delta^{ab}g_{g\sigma}}{k^2}\right)$$

Fierz, pg. 9

use $T^aT^a = T^a_{ij}T^a_{jk} \overset{\downarrow}{=} \frac{1}{2}\left(\delta_{ik}\delta_{jj} - \frac{1}{N}\delta_{ij}\delta_{jk}\right) = \frac{\delta_{ik}}{2}\left(N - \frac{1}{N}\right) = \mathbb{1}\, C_F$

$$= \frac{e^2 Q_q}{q^2}\, \bar{v}_B \gamma^\mu u_A\, \bar{u}_1 \underbrace{g_s^2 C_F \mathbb{1} \int \frac{d^dk}{(2\pi)^d} \gamma^\rho \frac{\not k + \not p_1}{(k+p_1)^2} \gamma_\mu \frac{\not k - \not p_2}{(k-p_2)^2} \gamma_\rho \frac{1}{k^2}}_{\equiv \Lambda_\mu} v_2$$

$$\Rightarrow \langle \alpha_s (M_1 M_0^\dagger + M_0 M_1^\dagger) \rangle = \frac{e^4 Q_q^2}{4 q^4}\, tr(\not p_B \gamma^\mu \not p_A \gamma^\nu)\, tr(\not p_1 \Lambda_\mu \not p_2 \gamma_\nu) + c.c.$$

• note that IF the contribution of Λ_μ inside the 2nd trace can be reduced to $\Lambda_\mu = \gamma_\mu \cdot \alpha_s C_F \frac{\Lambda(q^2)}{2}$

THEN $\langle \alpha_s (M_1 M_0^\dagger + M_0 M_1^\dagger)\rangle = \langle |M_0|^2 \rangle \cdot \alpha_s C_F\, \mathrm{Re}[\Lambda(q^2)]$

and σ follows without additional work!

→ the idea is to profit from the onshell conditions, such that if there is a term, say, $\not p_1 \gamma_\mu \not p_1$ in Λ_μ it gets killed by $\not p_1 \Lambda_\mu = \not p_1 \not p_1 \gamma_\mu \not p_1 = \frac{1}{2}\{\gamma_\rho, \gamma_\nu\} p_1^\rho p_1^\nu \gamma_\mu \not p_1 = p_1^2 \gamma_\mu \not p_1 = 0$

• $$\Lambda_\mu = 4\pi \mu^{2\varepsilon} \alpha_s C_F \underbrace{\gamma^\rho{}_{\sigma\mu\nu\rho}} \underbrace{\int \frac{d^dk}{(2\pi)^d} \frac{(k+p_1)^\sigma (k-p_2)^\nu}{(k+p_1)^2 (k-p_2)^2 k^2}}_{\equiv I^{\sigma\nu}}$$

pg. 35

$= -2\gamma_{\nu\mu\sigma} + (4-d)\gamma_{\sigma\mu\nu}$ ①

or $= -4(\gamma_\nu g_{\mu\sigma} - \gamma_\mu g_{\nu\sigma} + \gamma_\sigma g_{\nu\mu}) + (6-d)\gamma_{\sigma\mu\nu}$ ②

or $= 2(4-d)(\gamma_\sigma g_{\mu\nu} - \gamma_\mu g_{\nu\sigma} + \gamma_\nu g_{\sigma\mu}) - (6-d)\gamma_{\nu\mu\sigma}$ ③

• solve the loop integral $I^{\sigma\nu}$ via Feynman parameters (see pg. 30)

$(q^2 = (p_1+p_2)^2 = 2p_1p_2)$

$$I^{\sigma\nu} = \Gamma(3) \int_0^1 dx_1 \ldots dx_3\, \delta(1-x_1-x_2-x_3) \int \frac{d^dk}{(2\pi)^d} \frac{(k+p_1)^\sigma (k-p_2)^\nu}{[x_1(k^2+2kp_1+\overset{0}{p_1^2}) + x_2(k^2-2kp_2+\overset{0}{p_2^2}) + x_3 k^2]^3}$$

denominator $[\ldots] = [(k + x_1p_1 - x_2p_2)^2 + x_1x_2 q^2]$ (used δ fct for k^2-prefactor)

shift $k \to k - x_1p_1 + x_2p_2$, linear terms in k integrate to zero

$$= \Gamma(3) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^dk}{(2\pi)^d} \frac{k^\sigma k^\nu - [(1-x_1)p_1 + x_2p_2]^\sigma [x_1p_1 + (1-x_2)p_2]^\nu}{(k^2 + x_1x_2q^2)^3}$$

momentum integrals are standard (see pg 31),
after $k^\sigma k^\nu \to \frac{g^{\sigma\nu} k^2}{d}$ and Wick rotation : $I_3'(-x_1 x_2 q^2)$, $I_3^0(-x_1 x_2 q^2)$

$$= \frac{1}{(4\pi)^{d/2}} \frac{1}{(-q^2)^{2-d/2}} \left\{ \frac{g^{\sigma\nu}}{d} \frac{d}{2} \Gamma(2-\tfrac{d}{2}) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{(x_1 x_2)^{2-d/2}} + \Gamma(3-\tfrac{d}{2}) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{[\;]^\sigma[\;]^\nu}{(x_1 x_2)^{3-d/2}} \frac{1}{(-q^2)} \right\}$$

perform the integrals over Fy parameters, e.g. ~~via~~ Mathematica , for Re(d)>4

$$= \frac{1}{(4\pi)^{d/2}} \frac{1}{(-q^2)^{2-d/2}} \left\{ g^{\sigma\nu} \frac{1}{2} \Gamma(2-\tfrac{d}{2}) \frac{\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2})}{(\frac{d}{2}-1)\,\Gamma(d-1)} + \Gamma(3-\tfrac{d}{2}) \frac{\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2})}{(\frac{d}{2}-2)\,\Gamma(d-1)} \left[p_1^\sigma p_1^\nu + p_2^\sigma p_2^\nu + \frac{8-4d+d^2}{(d-4)(d-2)} p_1^\sigma p_2^\nu + \frac{d-4}{d-2} p_2^\sigma p_1^\nu \right] \frac{1}{(-q^2)} \right\}$$

- $$\Lambda_\mu = 4\pi\mu^{2\varepsilon} \alpha_s C_F \, \underline{\gamma^\rho \sigma_{\mu\nu\rho}} \Big\{ \underbrace{I_1 g^{\sigma\nu}}_{\textcircled{1}} + \underbrace{I_2 p_1^\sigma p_1^\nu}_{\textcircled{1}\,\to 0} + \underbrace{I_3 p_2^\sigma p_2^\nu}_{\textcircled{1}\,\to 0} + \underbrace{I_4 p_1^\sigma p_2^\nu}_{\textcircled{2}\,\to 4\gamma_\mu p_1 p_2} + \underbrace{I_5 p_2^\sigma p_1^\nu}_{\textcircled{3}\,\to -2(4-d)\gamma_\mu p_2\cdot p_1} \Big\}$$

$I^{\sigma\nu}$; coeffs I_i from above

$\textcircled{1} \to (2-d)\gamma^\nu{}_{\mu\nu} \overset{\text{pg } 34}{=} (2-d)^2 \gamma_\mu$

have now specified terms that survive in product $\not p_1 \Lambda_\mu \not p_2$
(in particular, $0 = \{ \not p_1 \gamma_\mu \not p_1 , \not p_2 \gamma_\mu \not p_2 , \not p_1 \gamma_\mu \not p_2 , \not p_2 p_{1\mu} , \not p_1 p_{2\mu} \}$)

$$= 4\pi\mu^{2\varepsilon} \alpha_s C_F \frac{1}{(4\pi)^{d/2}} \frac{1}{(-q^2)^{2-d/2}} \left\{ \frac{1}{2}\Gamma(2-\tfrac{d}{2}) \frac{\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2})}{(\frac{d}{2}-1)\Gamma(d-1)} (2-d)^2 \gamma_\mu + \Gamma(3-\tfrac{d}{2}) \frac{\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2})}{(\frac{d}{2}-2)\Gamma(d-1)} \left[0 + 0 + \frac{8-4d+d^2}{(d-4)(d-2)} 2\gamma_\mu \boxed{\frac{2p_1\cdot p_2}{-q^2}}^{-1} + \frac{d-4}{d-2} (d-4) \gamma_\mu \boxed{\frac{2p_1\cdot p_2}{-q^2}}^{-1} \right] \right\}$$

$$= \frac{1}{2}\gamma_\mu \frac{\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{-q^2} \right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \left\{ \left(\frac{1}{\varepsilon} + 1 + O(\varepsilon) \right) + 0 + 0 + \left(-\frac{2}{\varepsilon^2} - \frac{4}{\varepsilon} - 9 + O(\varepsilon) \right) + O(\varepsilon) \right\}$$

$d = 4-2\varepsilon$

$$= \frac{1}{2}\gamma_\mu \frac{\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{s} \right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \pi^2 + O(\varepsilon) + i\pi\left(-\frac{2}{\varepsilon} - 3 + O(\varepsilon) \right) \right)$$

from $(-1)^\varepsilon$; but irrelevant

$\left(\begin{array}{l} (-1)^\varepsilon = e^{i\pi\varepsilon} \\ \mathrm{Re}[(-1)^\varepsilon] = \cos(\pi\varepsilon) \end{array} \right)$

- collecting , we finally have

$$\sigma_{e^+e^-\to q\bar q} = \sigma^{(\text{tree})}_{e^+e^-\to q\bar q} \cdot \left\{ 1 + \frac{\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{s} \right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} \left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \pi^2 + O(\varepsilon) \right) + O(\alpha_s^2) \right\}$$