→ using $\bar{v}_B \equiv \bar{v}(q_B, s_B)$ etc for brevity, this reads

$$\mathcal{M} = \frac{-ie^2 Q_q g_s}{q^2} \varepsilon_\lambda^{*a} T^a \; \bar{v}_B \gamma_\mu u_A \bar{u}_1 \underbrace{\left\{ \gamma^\lambda \frac{\not p_1 + \not p_3}{(p_1+p_3)^2} \gamma^\mu + \gamma^\mu \frac{-\not p_2 - \not p_3}{(p_2+p_3)^2} \gamma^\lambda \right\}}_{\equiv S^{\lambda\mu}} v_2$$

→ for cross section, need $|\mathcal{M}|^2$

as in §4.1, unpolarized e^+, e^- beams ⇒ average over incoming spins

spin-blind detector ⇒ sum over outgoing spins

$$\langle |\mathcal{M}|^2 \rangle = \sum_{pol} \frac{1}{4} \sum_s \mathcal{M}\mathcal{M}^+$$

$$= \left(\frac{e^2 Q_q g_s}{q^2}\right)^2 \underbrace{\sum_{pol} \varepsilon_\lambda^{*a} \varepsilon_\sigma^b}_{= -g_{\lambda\sigma}\delta^{ab}} \underbrace{(T^a_{ij})(T^b_{ij})^+}_{= tr(T^a T^b) = \frac{1}{2}\delta^{ab}} \frac{1}{4} \sum_{s_{AB12}} \bar{v}_B \gamma_\mu \underbrace{u_A \bar{u}_A}_{\sum_{s_A} = \not q_A} \gamma_\nu^+ v_B \; \bar{u}_1 S^{\lambda\mu} \underbrace{v_2 \bar{v}_2}_{\sum_{s_2} = \not p_2} (S^{\sigma\nu})^+ u_1 \quad \text{etc}$$

(completeness rel., Fey gauge) (normalization)

perform spin-sums via completeness rels, see pg. 44

(T hermitean, γ hermitean)

$$= \frac{e^4 Q_q^2 g_s^2}{4 q^4} \frac{N_c^2 - 1}{2} \underbrace{tr(\not q_B \gamma_\mu \not q_A \gamma_\nu)}_{\equiv L_{\mu\nu}} \underbrace{tr(\not p_1 S^{\lambda\mu} \not p_2 S^\nu{}_\lambda)}_{\equiv G^{\mu\nu}}$$

$$L_{\mu\nu} = q_B^\sigma q_A^\beta \, 4 (g_{\sigma\mu} g_{\beta\nu} - g_{\sigma\beta} g_{\mu\nu} + g_{\sigma\nu} g_{\mu\beta}) \quad \text{(see pg. 30)}$$

$$= 4 (q_{B\mu} q_{A\nu} + q_{A\mu} q_{B\nu} - q_A \cdot q_B \, g_{\mu\nu})$$

→ now, total cross section is (pg 44, $m_i = 0$), using ~~dimensional regularization~~,

$$\sigma_{e^+e^- \to q\bar{q}g} = \frac{1}{2s} \left(\prod_{i=1}^{3} \int \frac{d^{d-1} p_i}{(2\pi)^{d-1} 2 E_{p_i}} \right) (2\pi)^d \delta^{(d)}(p_1 + p_2 + p_3 - q_A - q_B) \langle |\mathcal{M}|^2 \rangle$$

$$= -\frac{e^4 g_s^2 \left(\sum_q Q_q^2\right) C_F N_c}{8s (q^2)^2 (2\pi)^{2d-3}} L_{\mu\nu} \underbrace{\left(\prod_{i=1}^{3} \int \frac{d^{d-1} p_i}{2|\vec{p}_i|} \right) \delta^{(d)}(p_1 + p_2 + p_3 - q) G^{\mu\nu}}_{\equiv I^{\mu\nu}(q)}$$

((used $C_F \equiv \frac{N_c^2 - 1}{2N_c}$))

$\rightarrow$ note that (check ?!) $\quad q_\mu I^{\mu\nu}(q) = 0$

$$\Rightarrow I^{\mu\nu}(q) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) I(q^2) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) \frac{g_{\sigma\rho} I^{\sigma\rho}(q)}{d-1}$$

follows from contracting the eq. with $g_{\mu\nu}$

$$\Rightarrow L_{\mu\nu} I^{\mu\nu}(q) = 4\left[q_A\cdot q_B(1-d+1) - \frac{1}{q^2}\left(2\, q\cdot q_B\, q\cdot q_A - q_A\cdot q_B\, q^2\right)\right] \frac{g_{\sigma\rho} I^{\sigma\rho}(q)}{d-1}$$

use $\quad q_A\cdot q_B = \frac{1}{2}(\underbrace{q_A^2}_{=0}+\underbrace{q_B^2}_{=0}) + q_A\cdot q_B = \frac{1}{2}(q_A+q_B)^2 = \frac{q^2}{2} = \frac{s}{2}$

(onshell)

$$= 2q^2(2-d)\frac{g_{\mu\nu} I^{\mu\nu}(q)}{d-1}$$

$\rightarrow$ in $\sigma_{ee\to q\bar q g}$, need now

$$g_{\mu\nu} G^{\mu\nu} = \mathrm{tr}\left(\not p_1 S^{\lambda\nu} \not p_2 S_{\nu\lambda}\right) = \underset{\cdots\ \cdots}{(\text{check ?!})} = 4(d-2)\frac{x_1^2+x_2^2+\frac{d-4}{2}x_3^2}{(1-x_1)(1-x_2)}$$

where $\quad x_i \equiv \dfrac{2p_i\cdot q}{q^2}$

$\rightarrow$ plug these elements into cross section:

$$\sigma_{e^+e^-\to q\bar q g} = + \frac{e^4 g_s^2\left(\sum_q Q_q^2\right) C_F N_c}{s^2\,(2\pi)^{2d-3}}\,\frac{(d-2)^2}{d-1}\,\underbrace{\left(\prod_{i=1}^{3}\int\frac{d^{d-1}p_i}{2|\vec p_i|}\right)\delta^{(d)}(p_{123}-q)}\,\frac{x_1^2+x_2^2+\frac{d-4}{2}x_3^2}{(1-x_1)(1-x_2)}$$

rewrite phase space integral in terms of x_i [see eg. Muta §5.1]

$$= \frac{\pi(\pi s)^{d-3}}{4\Gamma(d-2)}\left(\prod_{i=1}^{3}\int_0^1\frac{dx_i}{(1-x_i)^{\frac{4-d}{2}}}\right)\delta(2-x_1-x_2-x_3)$$

$\left(g_s^2 = 4\pi\mu^{2\varepsilon}\alpha_s\right)$

$d=4-2\varepsilon$

$$= \alpha^2\alpha_s\left(\sum_q Q_q^2\right) C_F N_c\,\frac{2}{s}\left(\frac{4\pi\mu}{s}\right)^{2\varepsilon}\frac{(1-\varepsilon)^2}{(3-2\varepsilon)\Gamma(2-2\varepsilon)}\cdot I$$

where
$$I \equiv \left(\prod_{i=1}^{3}\int_0^1\frac{dx_i}{(1-x_i)^{\varepsilon}}\right)\delta(2-x_1-x_2-x_3)\,\frac{x_1^2+x_2^2-\varepsilon x_3^2}{(1-x_1)(1-x_2)}$$

$$= \int_0^1 dx_1\int_{1-x_1}^1 dx_2\,\frac{x_1^2+x_2^2-\varepsilon(2-x_1-x_2)^2}{(1-x_1)^{1+\varepsilon}(1-x_2)^{1+\varepsilon}(x_1+x_2-1)^{\varepsilon}}$$

eg Mathematica

$$= \frac{2(2-6\varepsilon+5\varepsilon^2-2\varepsilon^3)\,\Gamma^3(1-\varepsilon)}{\varepsilon^2\,\Gamma(3-3\varepsilon)} \approx \frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+\left(\frac{19}{2}-\pi^2\right)+\mathcal{O}(\varepsilon)$$

→ compare result with $\sigma_{e^+e^- \to q\bar{q}}$ in <u>same</u> regularization:

repeat calculation of $\left| \text{(diagram } e^+e^- \to q\bar{q}) \right|^2$ (see § 4.1) in d-dimensions,

$$\sigma_{e^+e^- \to q\bar{q}} \overset{d=4-2\varepsilon}{=} \frac{4\pi\alpha^2}{3s} \left(\sum_q Q_q^2 \right) N_c \left(\frac{4\pi}{s} \right)^\varepsilon \frac{3(1-\varepsilon)\Gamma(2-\varepsilon)}{(3-2\varepsilon)\Gamma(2-2\varepsilon)}$$

$$\Rightarrow \sigma_{e^+e^- \to q\bar{q}g} = \sigma_{e^+e^- \to q\bar{q}} \cdot \frac{\alpha_s C_F}{2\pi} \left(\frac{4\pi\mu^2}{s} \right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} I$$

- note that I is <u>divergent</u> for $\varepsilon \to 0$.

  not a desaster (see pg 48): $\lim_{\varepsilon \to 0}$ (real + virtual corr.) should exist.
  ↳ compute next.

- physical <u>origin</u> of divergences:

  in a 'naive' calculation $(d=4)$, $I = \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$

  ⤳ divergences come from $x_1 = 1$ and $x_2 = 1$

but $1 - x_1 = 1 - \frac{2p_1 \cdot q}{q^2} = \frac{(q-p_1)^2 - p_1^2}{q^2} = \frac{(p_2+p_3)^2 - p_1^2}{q^2} = \frac{2p_2 \cdot p_3 + p_2^2 + p_3^2 - p_1^2}{q^2}$

onshell: $p_i^2 = 0 \Rightarrow p_i = E_i(1, \vec{e}_i)$

$$= \frac{2E_2E_3(1 - \vec{e}_2\vec{e}_3)}{q^2}$$

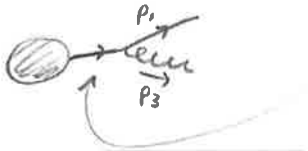
⤳ divergences originate from $\begin{cases} \vec{e}_1 \cdot \vec{e}_3 \to 1 & : q, g \\ \vec{e}_2 \cdot \vec{e}_3 \to 1 & : \bar{q}, g \end{cases}$ <u>collinear</u> ($\theta_{13} \to 0$)

and from $E_3 \to 0$: g <u>soft</u>

both are referred to as <u>infrared</u> singularities.

another view of the same fact:

divergences originate from diverging propagators



$$\sim \frac{1}{(p_1+p_3)^2} = \frac{1}{2p_1 \cdot p_3} = \frac{1}{2E_1E_3(1 - \vec{e}_1\vec{e}_3)} = \frac{1}{2E_1E_3(1 - \cos\theta_{13})}$$

collinear limit, $\theta \to 0$: $\approx \frac{1}{2E_1E_3\theta^2} \Rightarrow |\mathcal{M}|^2 \sim \frac{\theta^2}{\theta^4}$ ← from numerators

soft limit, $E_3 \to 0$: interference term in $\left| \text{(diagram)} + \text{(diagram)} \right|^2$

gives $|\mathcal{M}|^2 \sim \frac{p_1 \cdot p_2}{p_1 \cdot p_3 \; p_2 \cdot p_3} \approx \frac{1}{E_3^2}$

in phase space integral, $\frac{d^3p_3}{2E_3} = \frac{1}{2E_3} E_3^2 dE_3 \sin\theta \, d\theta \, d\varphi \sim E_3 \, dE_3 \; \theta d\theta$

$\Rightarrow$ logarithmic singularities in both limits.