# 4. QCD in $e^+e^-$ - annihilation

→ want to compare basic properties of (perturbative) QCD with experiment.

→ consider $e^+e^- \to$ hadrons

- total cross section $\sigma \sim \frac{\alpha_{em}^2}{s}$

  calculate $\alpha_s$ corrections, $\sigma \to \sigma \cdot (1 + \alpha_s)$

  renormalization scheme dependence enters at $\alpha_s^2$

  inclusive cross section: $(1 + \alpha_s + \alpha_s^2 + \alpha_s^3)$ known,

  high precision QCD result!

  non-perturbative corrections expected to be small

  ⇒ used as one of the most precise measurements of $\alpha_s$.

- QCD predicts "jet" structure for final-state hadrons

  define jet cross sections

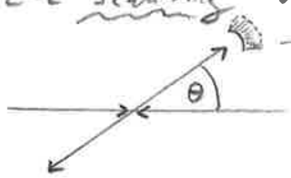
  calculate them, compare with experiment

  ⇒ can also be used to measure $\alpha_s$,

  and to test/"see" triple-gluon-vertex.

## 4.1 $e^+e^- \to$ hadrons at leading order

reminder: 2→2 scattering in CMS (center of mass system)



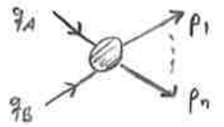

detector measures in

$d\Omega = \sin\theta \, d\theta \, d\varphi$

(see also PDG Booklet §43: kinematics)

spherical coords.: $\int d\Omega = \int_0^\pi d\theta \, \sin\theta \int_0^{2\pi} d\varphi = \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\varphi = 4\pi$

total cross section $\sigma = \int d\Omega \left(\frac{d\sigma}{d\Omega}\right) \Leftarrow$ differential cross section remember: Fermi's golden rule



$$\sigma_{2\to n} = \frac{1}{F}\int d\Phi_n \, |\mathcal{M}|^2$$

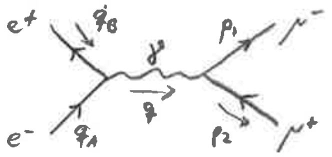
amplitude, e.g. from Feynman diags

phase space integral

$$\left(\prod_{i=1}^{n}\int \frac{d^3p_i}{(2\pi)^3\, 2E_{\vec p_i}}\right)(2\pi)^4\,\delta^{(4)}\Big(\sum_{i=1}^{n} p_i - q_A - q_B\Big)$$

where
$$F = 4\sqrt{(q_A\cdot q_B)^2 - m_A^2 m_B^2}$$
$$= 2\sqrt{(s-m_A^2-m_B^2)^2 - 4m_A^2 m_B^2}\,, \quad s = (q_A+q_B)^2 \quad \big(= (E_A+E_B)^2 \text{ in CMS}\big)$$
$$= 4(E_A+E_B)\,|\vec q_A| \quad \text{in CMS}, \quad q = (E, \vec q)$$

remember: amplitude $e^+e^- \to \mu^+\mu^-$ [see, e.g., Peskin/Schroeder §5.1]

$\xrightarrow{t}$ 

$$= \bar v(q_B,s_B)(-ie\gamma^\mu)\,u(q_A,s_A)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar u(p_1,s_1)(-ie\gamma^\nu)\,v(p_2,s_2)$$

spin: +/-

Dirac spinor for incoming particle

start with unpolarized beam of $e^+, e^-$

⇝ average over spin states $s_A, s_B$: $\frac{1}{4}\sum_{s_A=+,-}\sum_{s_B=+,-}$

detector does not measure spin of final state

⇝ sum over spins $s_1, s_2$

$$\frac{1}{4}\sum_{s_i=+/-}|\mathcal{M}|^2 = \frac{1}{4}\sum_s \left| \quad \right|^2 = \frac{1}{4}\sum_s\left|\frac{e^2}{q^2}\bar v_B\gamma u_A\, \bar u_1\gamma v_2\right|^2$$

$$= \frac{e^4}{4q^4}\sum_s \bar v_B\gamma u_A\bar u_A\gamma v_B\;\bar u_1\gamma v_2\bar v_2\gamma u_1$$

do spin sums via completeness rels

$$\sum_s u_A\bar u_A = \not q_A + m_e\,, \quad \sum_s v_2\bar v_2 = \not p_2 - m_\mu$$

$$= \frac{e^4}{4q^4}\,\mathrm{tr}\Big((\not q_B - m_e)\gamma^\mu(\not q_A + m_e)\gamma^\nu\Big)\,\mathrm{tr}\Big((\not p_1 + m_\mu)\gamma_\mu(\not p_2 - m_\mu)\gamma_\nu\Big)$$

$$= \frac{e^4}{4q^4}\,4\Big[q_B^\mu q_A^\nu + q_A^\mu q_B^\nu - g^{\mu\nu}(q_A\cdot q_B + m_e^2)\Big]\,4\Big[p_{1\mu}p_{2\nu} + p_{2\mu}p_{1\nu} - g_{\mu\nu}(p_1\cdot p_2 + m_\mu^2)\Big]$$

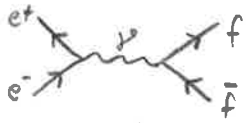
$$= \frac{8e^4}{q^4}\Big(q_A\cdot p_1\; q_B\cdot p_2 + q_A\cdot p_2\; q_B\cdot p_1 + m_e^2\, p_1\cdot p_2 + m_\mu^2\, q_A\cdot q_B + 2m_e^2 m_\mu^2\Big)$$

CMS: $\vec q_B = -\vec q_A$, $\vec p_2 = -\vec p_1$ ; E-cons.: $2E = E_1 + E_2 = 2E_1$

$\Rightarrow E_A = E_B = E_1 = E_2 = E$ , $\vec q_A^{\,2} + m_e^2 = \vec p_1^{\,2} + m_\mu^2$

$$= \frac{e^4}{E^4}\Big(E^4 + m_e^2E^2 + m_\mu^2E^2 + (E^2 - m_e^2)(E^2 - m_\mu^2)\cos^2\theta\Big) \qquad \measuredangle(\vec q_A, \vec p_1)$$

generalization: amplitude $e^+e^- \to f\bar{f}$



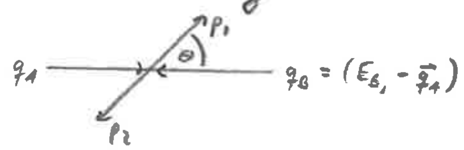

where $f \neq e$, $f \in \left\{ \begin{matrix} \nu_e, \nu_\mu, \nu_\tau, u, c, t \\ \mu, \tau, d, s, b \end{matrix} \right\}$

use charge $Q_f = \left\{ \begin{matrix} 0, 0, 0, +\frac{2}{3}, +\frac{2}{3}, +\frac{2}{3} \\ -1, -1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3} \end{matrix} \right\}$

$$\Rightarrow \langle |\mathcal{M}|^2 \rangle \equiv \sum_{\text{colors}} \sum_f \frac{1}{4} \sum_s |\mathcal{M}|^2 = \sum_{\substack{\text{colors} \\ f}} \frac{Q_f^2 e^4}{E^4} \Big( E^4 + m_e^2 E^2 + m_f^2 E^2 + (E^2 - m_e^2)(E^2 - m_f^2)\cos^2\theta \Big)$$

reminder: phase space integration for $2 \to 2$ scattering

$A + B \to 1 + 2$ in CMS:




$$d\sigma_{2\to 2} = \frac{1}{F} d\Phi_2 |\mathcal{M}|^2 \quad , \quad \text{use } \delta^{(3)} \text{ for } \vec{p}_2\text{-integral}$$

$$= \frac{1}{(8\pi)^2} \frac{|\mathcal{M}|^2}{|\vec{q}_A|(E_A + E_B)} d^3\vec{p}_1 \frac{\delta\left(\sqrt{m_1^2 + \vec{p}_1^{\,2}} + \sqrt{m_2^2 + \vec{p}_1^{\,2}} - E_A - E_B\right)}{\sqrt{m_1^2 + \vec{p}_1^{\,2}}\sqrt{m_2^2 + \vec{p}_1^{\,2}}}$$

use that $|\mathcal{M}|^2 = |\mathcal{M}|^2(\vec{q}_A, \vec{q}_B, \vec{p}_1, \vec{p}_2) = |\mathcal{M}|^2(|\vec{q}_A|, |\vec{p}_1|, \cos\theta)$

spherical coords, $\varrho \equiv |\vec{p}_1|$, $d^3\vec{p}_1 = \varrho^2 d\varrho \underbrace{\sin\theta \, d\theta \, d\varphi}_{= d\Omega}$

do $\int d\varrho$ using $\delta$ fct

$$\frac{d\sigma_{2\to 2}}{d\Omega} = \frac{1}{(8\pi)^2} \frac{|\vec{p}_1|}{|\vec{q}_A|} \frac{|\mathcal{M}|^2(|\vec{q}_A|, |\vec{p}_1|, \cos\theta)}{(E_A + E_B)^2} \cdot \Theta(\sqrt{s} - m_1 - m_2)$$

total cross section

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \int_{-1}^{1} d(\cos\theta) \frac{2\pi}{(8\pi)^2} \frac{\sqrt{E^2 - m_f^2}}{\sqrt{E^2 - m_e^2}} \frac{\langle |\mathcal{M}|^2 \rangle}{(2E)^2} \Theta(E - m_f)$$

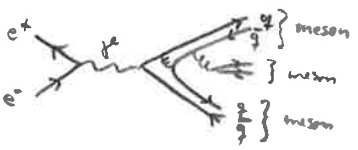
$$= \sum_{\substack{\text{colors} \\ f}} \frac{\pi}{3} \frac{Q_f^2 \alpha_{em}^2}{E^2} \frac{\sqrt{1 - m_f^2/E^2}}{\sqrt{1 - m_e^2/E^2}} \left(1 + \frac{m_e^2}{2E^2}\right)\left(1 + \frac{m_f^2}{2E^2}\right) \Theta(E - m_f)$$

$$\approx \frac{\pi}{3} \frac{\alpha_{em}^2}{E^2} \cdot \sum_{\substack{\text{colors} \\ f}} Q_f^2 \, \Theta(E - m_f), \text{ for } E \gg m_f \gg m_e$$

$$\Rightarrow R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \frac{\sum_q \sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} \approx N_c \sum_q Q_q^2 \, \Theta(E - m_q)$$

$\sum_{\text{colors}}$

$$\approx N_c \left\{ \left(\tfrac{2}{3}\right)_u^2 + \left(-\tfrac{1}{3}\right)_d^2 + \left(-\tfrac{1}{3}\right)_s^2 + \left(\tfrac{2}{3}\right)_c^2 + \left(-\tfrac{1}{3}\right)_b^2 + \left(\tfrac{2}{3}\right)_t^2 \right\} = N_c \left\{ \tfrac{4}{9}, \tfrac{5}{9}, \underset{uds}{\tfrac{2}{3}}, \underset{+c}{\tfrac{10}{9}}, \underset{+b}{\tfrac{11}{9}}, \underset{+t}{\tfrac{5}{3}} \right\}$$

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$\Rightarrow N_c \stackrel{!}{=} 3$