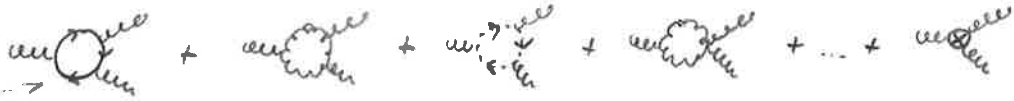
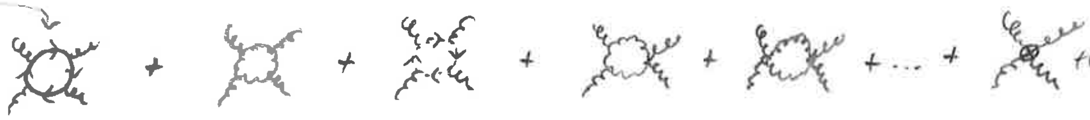
→ could have computed Z_g also from

finite $\overset{?!}{=}$  $+ \, \mathcal{O}(g^5)$

(check ?!)

finite $\overset{?!}{=}$  $+ \, \mathcal{O}(g^5)$

finite $\overset{?!}{=}$  $+ \, \mathcal{O}(g^6)$

- current status (as of May 2013) of Z's:

$$Z \sim 1 + \underbrace{g^2}_{\substack{\text{1-loop} \\ \text{(see above)}}} + g^4 + g^6 + \underbrace{g^8}_{\text{4-loop} \checkmark\checkmark} + \underbrace{\mathcal{O}(g^{10})}_{\text{5-loop ??}}$$

⇒ Nobel 2004 [Gross/Wilczek, Phys. Rev. Letters 30 (1973) 1343
Politzer, Phys. Rev. Letters 30 (1973) 1346]

4-loop: [
Ritbergen/Vermaseren/Larin, Phys. Lett. B 400 (1997) 379 : Z_g
Chetyrkin, Phys. Lett. B 404 (1997) 161 } : Z_m
Vermaseren/Larin/Ritbergen, Phys. Lett. B 405 (1997) 327 }
Chetyrkin/Retey, Nucl. Phys. B 583 (2000) 3 : Z_ψ
Chetyrkin, Nucl. Phys. B 710 (2005) 499 } : Z_c, Z_A
Czakon, Nucl. Phys. B 710 (2005) 485 }
]

3.4 QCD Beta-function, running coupling

⤳ recall that we had regularized QCD dimensionally : $d = 4 - 2\varepsilon$.

dimensional analysis: $e^{i\int d^dx \, \mathcal{L}}$ ⇒ $[\mathcal{L}] \overset{!}{=} d$ (mass-dim: $[m] \equiv 1$)

$$\mathcal{L} \ni m\bar\psi\psi \Rightarrow [\psi] = \tfrac{d-1}{2}$$
$$\mathcal{L} \ni (\partial^\mu A_\mu)^2 \Rightarrow [A_\mu] = \tfrac{d-2}{2}$$
$$\mathcal{L} \ni \bar c \, \partial^2 c \Rightarrow [c] = \tfrac{d-2}{2}$$
$$D_\mu \sim \partial_\mu + g A_\mu \Rightarrow [g] = \tfrac{4-d}{2}$$

$[\partial_\mu] = 1$
$[x] = -1$

$\rightsquigarrow$ we had defined $g_B^2 = Z_g^2 g_R^2$, $Z_g = 1 + \dots$: $[Z_g] = 0$

dim: $(4-d) = (0) + (4-d)$

it is convenient to use integer-dimensional renormalized couplings already in d dimensions

arbitrary mass-scale, $[\mu] = 1$

$$\underline{g_B^2 = Z_g^2 \hat{g}_R^2 \mu^{4-d}} \quad , \quad [\hat{g}_R^2] = 0$$

$\Rightarrow$ in fact, in all our expressions above, $g^2 \to \hat{g}_R^2$ was understood

$$\left(\!\left(Z \sim 1 + \underbrace{g_B^2 \int \frac{d^dk}{k^4}}_{\text{dim-less}} = 1 + \underbrace{Z_g^2}_{\checkmark} \underbrace{\hat{g}_R^2}_{\checkmark} \underbrace{\mu^{4-d} \int \frac{d^dk}{k^4}}_{\text{dim-less}} \right.\right.$$, Z is fct of dim-less parameters! $\Big)\!\Big)$

- QCD Beta-function

immediate consequence : $\hat{g}_R^2$ is a function of μ

$$\mu^2 \partial_{\mu^2} \cdot \Big| \quad g_B^2 = Z_g^2 \hat{g}_R^2 \mu^{4-d} \qquad\qquad \hat{g}_R^2 \equiv g^2 \text{ from here on}$$

$$\Rightarrow \quad 0 = \underbrace{(\mu^2 \partial_{\mu^2} Z_g^2)}_{= (\mu^2\partial_{\mu^2} g^2)(\partial_{g^2} Z_g^2) + (\mu^2 \partial_{\mu^2} \xi)\cancel{(\partial_\xi Z_g^2)}^{\,0}} g^2 \mu^{4-d} + Z_g^2 (\mu^2 \partial_{\mu^2} g^2) \mu^{4-d} + Z_g^2 g^2 \frac{4-d}{2} \mu^{4-d}$$

$$\Leftrightarrow \boxed{\beta(g^2) \equiv \mu^2 \partial_{\mu^2} g^2 = \frac{\frac{d-4}{2} g^2}{1 + g^2 (\partial_{g^2} Z_g^2) Z_g^{-2}}}$$

in $d = 4 - 2\varepsilon$ dimensions, (pg. 39),

$Z_g = 1 - \frac{g^2}{\varepsilon} \frac{a}{2} + O(g^4)$

where $a \equiv \frac{1}{16\pi^2} \left(\frac{11}{3} N_c - \frac{2}{3} N_f \right) \equiv \frac{\beta_0}{16\pi^2}$

$$= \frac{-\varepsilon g^2}{1 - \frac{g^2}{\varepsilon} a + O(g^4)} = -\varepsilon g^2 \left(1 + \frac{g^2}{\varepsilon} a + O(g^4)\right)$$

$$\longrightarrow -a g^4 + O(g^6) \qquad \text{for } \varepsilon \to 0$$

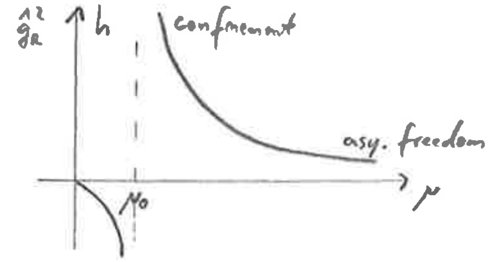
$\rightarrow$ more convenient to use $h \equiv \frac{\hat{g}_R^2}{16\pi^2}$:

$$\beta(h) = \mu^2 \partial_{\mu^2} h = \frac{\frac{d-4}{2} h}{1 + h (\partial_h Z_g^2) Z_g^{-2}} \approx \underbrace{-\beta_0 h^2}_{\text{1-loop}} - \beta_1 h^3 - \beta_2 h^4 \underbrace{- \beta_3 h^5}_{\text{4-loop}} + \underbrace{O(h^6)}_{\text{unknown}}$$

- *running coupling*

solve the differential equation $\mu^2 \partial_{\mu^2} h = -\beta_0 h^2$

soln: $h(\mu) = \dfrac{1}{\text{const.} + \beta_0 \ln(\mu^2)} = \dfrac{1}{\beta_0 \ln(\mu^2/\mu_0^2)}$



$\beta_0 = \frac{11}{3} N_c - \frac{2}{3} N_f > 0$ for $N_f < \frac{11}{2} N_c$

(QCD: $6 < \frac{33}{2}$ ✓)

- *higher-order systematics* of renormalization factors:

in general (schematically), $Z_i \sim 1 + h\frac{1}{\varepsilon} + h^2\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\right) + h^3\left(\frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\right) + \dots$

(in literature, often find $\gamma(h) = -\gamma_0 h - \gamma_1 h^2 - \gamma_2 h^3 - \dots$)

furthermore, existence of the limit $\varepsilon \to 0$ in β-fct (or, analogously, in *"anomalous dimensions"* $\gamma_i \equiv -\mu^2 \partial_{\mu^2} \ln Z_i$) fixes the coefficients of poles $\frac{1}{\varepsilon^{n>1}}$ in terms of those of $\frac{1}{\varepsilon}$.

e.g. $Z_g = 1 + h\frac{z_{11}}{\varepsilon} + h^2\left(\frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon}\right) + h^3\left(\frac{z_{33}}{\varepsilon^3} + \frac{z_{32}}{\varepsilon^2} + \frac{z_{31}}{\varepsilon}\right) + O(h^4)$

$$\Rightarrow \beta(h) \overset{!}{=} h(-\varepsilon) + h^2(2z_{11}) + h^3\left(\frac{4z_{22} - 6z_{11}^2}{\varepsilon} + 4z_{21}\right) + h^4\left(\frac{2(3z_{33} - 11 z_{22} z_{11} + 9 z_{11}^3)}{\varepsilon^2} + \frac{2(3z_{32} - 11 z_{21} z_{11})}{\varepsilon} + 6 z_{31}\right) + O(h^5)$$
$$\equiv -\beta_0 h^2 - \beta_1 h^3 - \beta_2 h^4 + \dots$$

$$\Rightarrow z_{11} = -\frac{\beta_0}{2}, \quad z_{21} = -\frac{\beta_1}{4}, \quad z_{31} = -\frac{\beta_2}{6}, \dots$$

and $z_{22} \overset{!}{=} \frac{3}{2} z_{11}^2 = \frac{3}{8}\beta_0^2$; $z_{32} \overset{!}{=} \frac{11}{3} z_{21} z_{11} = \frac{11}{24}\beta_1\beta_0$; $z_{33} \overset{!}{=} -\frac{5}{16}\beta_0^3$; ...

such that $Z_g = 1 + h\left(-\frac{\beta_0}{2\varepsilon}\right) + h^2\left(\frac{3\beta_0^2}{8\varepsilon^2} - \frac{\beta_1}{4\varepsilon}\right) + h^3\left(-\frac{5\beta_0^3}{16\varepsilon^3} + \frac{11\beta_1\beta_0}{24\varepsilon^2} - \frac{\beta_2}{6\varepsilon}\right) + \dots$

i.e., all information is already encoded in the $\frac{1}{\varepsilon}$ poles ✓

- one often needs *ε-expansions* of Gamma-functions.

(recall pg 31 : $I_n^a(\Delta) \sim \dfrac{\Gamma(a + d/2)\Gamma(n-a-d/2)}{\Gamma(d/2)}$)

Mathematica : $\Gamma(1-\varepsilon) \sim \sum \varepsilon^{n+1}$ PolyGamma$[n,1]$ $= \partial_n^2 \ln(\Gamma(n))$

more useful : $\Gamma(1-\varepsilon) = e^{\gamma_E \varepsilon} \, e^{\sum_{n=2}^{\infty} \frac{\varepsilon^n}{n}\zeta(n)}$

Riemann Zeta $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$ $(n>1)$

Euler-Mascheroni $\gamma_E = -\Gamma'(1) = -\int_0^\infty dx\, e^{-x} \ln(x) = \lim_{N\to\infty}\left(\sum_{k=1}^{N} \frac{1}{k} - \ln(N)\right) \approx 0.5772\dots$