6

Fy par, $\frac{1}{()()} = \int_0^1 dx \frac{1}{[(1-x)k^2 + x(k^2+2kq+q^2-m^2)]^2} = \int_0^1 dx \frac{1}{[(k+xq)^2 + \underbrace{x(1-x)q^2 - xm^2}_{\equiv -\Delta}]^2}$

shift $k \to k - xq$

$$= -g^2 \frac{N^2-1}{2N}\int \frac{d^dk}{(2\pi)^d}\int_0^1 dx \frac{(2-d)(\overset{\to 0\ (\text{odd})}{\not k} + (1-x)\not q) + md}{(k^2-\Delta)^2}$$

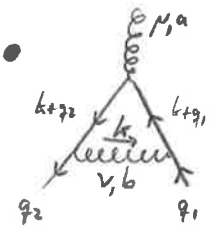
Wick rotate, use basic tadpole from pg 31

$$= -ig^2 \frac{N^2-1}{2N}\int_0^1 dx\, \underbrace{I_2^0(xm^2 - x(1-x)q^2)}_{\approx \frac{1}{\varepsilon}\frac{1}{(4\pi)^2}}\{(2-d)(1-x)\not q + md\}$$

, see pg. 34

$$\approx i\frac{g^2}{16\pi^2}\frac{N_c^2-1}{2N_c}\frac{1}{\varepsilon}\{\underbrace{\boxed{1}}_{\to\xi}\not q - \underbrace{\boxed{4}}_{\to(3+\xi)}m + O(\varepsilon)\}$$

for general value of ξ

- 

$$= \int \frac{d^dk}{(2\pi)^d}(ig)^3 T^bT^aT^b \frac{\gamma^\nu i(\not k+\not q_2+m)\gamma^\mu i(\not k + \not q_1 + m)\gamma_\nu}{((k+q_2)^2-m^2)\,((k+q_1)^2-m^2)}\frac{(-i)}{k^2}$$

$$\gamma^{\mu\nu\rho}{}_\mu = (2g^{\mu\nu} - \gamma^{\nu\mu})\gamma^{\rho}{}_\mu = 2\underbrace{\gamma^{\rho\nu}}_{=2g^{\rho\nu}-\gamma^{\nu\rho}} - \gamma^\nu\underbrace{\gamma^{\mu\rho}{}_\mu}_{=(2-d)\gamma^\rho} = 4g^{\nu\rho} - (4-d)\gamma^{\nu\rho}$$

$$\gamma^{\mu\nu\rho\sigma}{}_\mu = (2g^{\mu\nu}-\gamma^{\nu\mu})\gamma^{\rho\sigma}{}_\mu = 2\underbrace{\gamma^{\rho\sigma}}_{\cancel{=2g^{\rho\sigma}-\gamma^{\sigma\rho}}}\gamma^\nu - \gamma^\nu(\cancel{4g^{\rho\sigma}} - (4-d)\gamma^{\rho\sigma}) = -2\gamma^{\sigma\rho\nu} + (4-d)\gamma^{\nu\rho\sigma}$$

$$T^b_{ij}T^a_{jk}T^b_{kl} = T^a_{jk}\,\tfrac12\left(\delta_{il}\delta_{jk} - \tfrac1N\delta_{ij}\delta_{kl}\right) = \tfrac12\delta_{il}\overset{0}{\cancel{T^a_{jj}}} - \tfrac{1}{2N}T^a_{il} = -\tfrac{1}{2N}T^a$$

$$= g^3\left(-\tfrac{1}{2N}T^a\right)\int\frac{d^dk}{(2\pi)^d}\frac{N^\mu}{((k+q_2)^2-m^2)((k+q_1)^2-m^2)k^2}$$

$$N^\mu = m^2(2-d)\gamma^\mu + m(k+q_2)_\nu(4g^{\nu\mu}-(4-d)\gamma^{\nu\mu}) + m(k+q_1)_\nu(4g^{\mu\nu}-(4-d)\gamma^{\mu\nu})$$
$$+ (k+q_2)_\nu(k+q_1)_\sigma[-2\gamma^{\sigma\mu\nu} + (4-d)\gamma^{\nu\mu\sigma}]$$

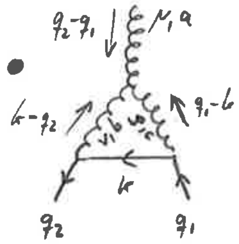
Superficial degree of divergence of this integral: $\frac{k^6}{k^6} \to \log$!

$\Rightarrow$ look at $k \gg \{q_1, q_2, m\}$ only, to extract this leading log div

$$\sim g^3\left(-\tfrac{1}{2N}T^a\right)\int\frac{d^dk}{(2\pi)^d}\frac{[-2\gamma^{\sigma\mu\nu}+(4-d)\gamma^{\nu\mu\sigma}]k_\nu k_\sigma}{(k^2)^3} + \text{const.}$$

numerator $\to [\dots]\frac{g_{\nu\sigma}k^2}{d} = \frac{k^2}{d}[-2\gamma^{\sigma\mu}{}_\sigma + (4-d)\gamma^{\nu\mu}{}_\nu] \overset{(\text{pg. }34)}{=} \frac{k^2}{d}(2-d)^2\gamma^\mu$

$$= g^3\left(-\tfrac{1}{2N}T^a\right)\frac{(2-d)^2}{d}\gamma^\mu\underbrace{\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^2}}_{= i I_2^0(0)\ \text{(after Wick rot.)}}$$

- 

$$= \int \frac{d^dk}{(2\pi)^d} (ig)^2 T^b T^c \frac{\gamma_\nu\, i(\not k+m)\,\gamma_\rho \quad (-i) \quad (-i)}{(k^2-m^2) \quad (k-q_2)^2 \quad (k-q_1)^2}\, g f^{abc} \ast$$

$$\ast \left[(2q_2-q_1-k)^\rho g^{\mu\nu} + (2k-q_1-q_2)^\mu g^{\nu\rho} + (2q_1-q_2-k)^\nu g^{\rho\mu}\right]$$

$$T^bT^c f^{abc} = \frac{1}{2}[T^b,T^c]f^{abc} = \frac{i}{2}\underbrace{f^{bcd}\,T^d f^{abc}}_{= N_c\delta^{ad}\ \text{(see pg. 33)}} = \frac{i}{2}N_c T^a$$

again, extract only leading log div, $k \gg \{q_1,q_2,m\}$

$$\sim -g^3 \frac{N_c}{2} T^a \int \frac{d^dk}{(2\pi)^d}\frac{\gamma_\nu \not k\gamma_\rho\left[-k^\rho g^{\mu\nu}+2k^\mu g^{\nu\rho}-k^\nu g^{\rho\mu}\right]}{(k^2)^3} + \text{const.}$$

replace $k^\mu k^\nu \to \frac{g^{\mu\nu}k^2}{d}$

numerator $\sim \gamma_\nu\gamma_\sigma\gamma_\rho\left[-g^{\sigma\rho}g^{\mu\nu}+2g^{\sigma\mu}g^{\nu\rho}-g^{\sigma\nu}g^{\rho\mu}\right]$

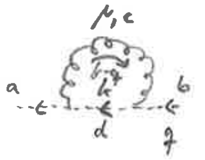
$$= -\gamma^{\mu\rho}{}_\rho + 2\gamma^{\rho\mu}{}_\rho - \gamma_\nu{}^{\nu\mu} \overset{(\text{pg. }34)}{=} -d\gamma^\mu + 2(2-d)\gamma^\mu - d\gamma^\mu = 4(1-d)\gamma^\mu$$

$$= -g^3\, 2N_c T^a \frac{1-d}{d}\gamma^\mu \underbrace{\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^2}}_{= i I_2^0(0)\ \text{(after Wick rot.)}}$$

- sum of last two diagrams

 $+$  $= -ig^3 T^a\gamma^\mu I_2^0(0)\left\{\frac{1}{2N}\frac{(2-d)^2}{d} + 2N\frac{1-d}{d}\right\} + \text{const.}$

$d=4-2\varepsilon$: $\approx -igT^a\gamma^\mu \frac{g^2}{16\pi^2}\frac{1}{\varepsilon}\left\{\underbrace{\boxed{1}}_{\xi}\frac{1}{2N} - \frac{3}{4}\underbrace{\boxed{2}}_{(\xi+1)}N + O(\varepsilon)\right\}$ for general value of ξ

- 

$$= \int\frac{d^dk}{(2\pi)^d}\left(-gf^{acd}q^\mu\right)\left(-gf^{dcb}k_\mu\right)\frac{i}{k^2}\frac{-i}{(k-q)^2}$$

$$= -g^2 \underbrace{f^{acd}f^{cdb}}_{=N_c\delta^{ab}\ (\text{pg. }33)} \int \frac{d^dk}{(2\pi)^d}\frac{q\cdot k}{k^2(k-q)^2}$$

denominator invariant under $k\to q-k$

write numerator $k = \frac{1}{2}k+\frac{1}{2}k \to \frac{1}{2}k + \frac{1}{2}(q-k) = \frac{1}{2}q$

$$= -g^2N_c\delta^{ab}\frac{q^2}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2(k-q)^2}$$

extract leading log div, $k\gg q$; Wick rot ; use tadpole

$$\sim -ig^2N_c\delta^{ab}\frac{q^2}{2}I_2^0(0) + \text{const.}$$

$d=4-2\varepsilon$: $\approx -i\frac{g^2}{16\pi^2}\delta^{ab}q^2\frac{1}{\varepsilon}\left\{\underbrace{\boxed{2}}_{(3-\xi)}\frac{N_c}{4}+O(\varepsilon)\right\}$ for general value of ξ

3.3 one-loop counterterms in QCD

$\rightsquigarrow$ now, renormalize the theory.

use freedom of redefining fields, parameters/couplings

schematically: $\phi_B = \sqrt{Z_\phi}\,\phi_R \quad , \quad \phi \in \{\psi, A, c\}$ $\qquad B \equiv$ "bare"

$\lambda_B = Z_\lambda \lambda_R \quad , \quad \lambda \in \{m, g, \xi\}$ $\qquad R \equiv$ "renormalized"

where the multiplicative renormalization factors Z_i depend on the renormalized parameters (and the dimension d), and are taken to be dimension<u>less</u>, $Z_i = 1 + \delta Z_i \quad , \quad \delta Z_i \sim g^2$ (see below)

- recall (pg. 28)

$$\mathcal{L}_B = \bar\psi_B (i\gamma^\mu \underbrace{D_\mu}_{\partial_\mu - i g_B A_\mu^a T^a} - m_B)\psi_B - \frac{1}{4}\underbrace{F^a_{\mu\nu}}_{\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c} F^{a\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 + \bar c^a(-\partial^\mu \underbrace{D_\mu^{ac}}_{\partial_\mu \delta^{ac} + g f^{abc} A_\mu^b}) c^c$$

(in this line, all ψ, A, c, m, g, ξ should have an index $\underline{B}$)

$$= \overset{Z_\psi}{\bar\psi i \not\partial \psi} - \overset{Z_m Z_\psi}{m\bar\psi\psi} + \overset{Z_g Z_\psi Z_A^{1/2}}{g\bar\psi\gamma^\mu A_\mu^a T^a \psi}$$

$$- \overset{Z_A}{\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2} - \overset{Z_A Z_\xi^{-1}}{\frac{1}{2\xi}(\partial^\mu A_\mu^a)^2} - \overset{Z_g Z_A^{3/2}}{\frac{1}{2} g f^{abc} A_\mu^b A_\nu^c (\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu})}$$

$$- \overset{Z_g^2 Z_A^2}{\frac{1}{4} g^2 f^{abc} A_\mu^b A_\nu^c f^{ade} A^{d\mu} A^{e\nu}} - \overset{Z_c}{\bar c^a \partial^\mu \partial_\mu c^a} - \overset{Z_g Z_c Z_A^{1/2}}{g f^{abc} \bar c^a \partial^\mu A_\mu^b c^c}$$

(here, without Z's : index B ; <u>with</u> Z's : index R for all ψ, A, c, m, g, ξ)

$$= \mathcal{L}_R + \underbrace{\mathcal{L}^{c.t.}}$$ $\longleftarrow$ counterterms

$$= (Z_\psi - 1)\bar\psi\Big(i\not\partial - \frac{Z_m Z_\psi - 1}{Z_\psi - 1} m\Big)\psi + (Z_g Z_\psi Z_A^{1/2} - 1)$$

$$- (Z_A - 1)\Big[\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + \frac{Z_A Z_\xi^{-1} - 1}{Z_A - 1}\,\frac{1}{2\xi}(\partial^\mu A_\mu^a)^2\Big] + (Z_g Z_A^{3/2} - 1)$$

$$+ (Z_g^2 Z_A^2 - 1) \quad - (Z_c - 1)\,\bar c\,\partial^\mu\partial_\mu c \quad + (Z_g Z_c Z_A^{1/2} - 1)$$

(now, all indices are $\underline{R}$, and omitted)