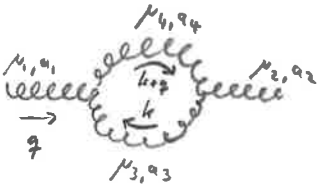
• 2^{nd} diagram : calculation parallels the one above !



symmetry factor   Feynman gauge   color indices a_i

$$= \frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\left(\frac{-i}{k^2}\right)\left(\frac{-i}{(k+q)^2}\right) g^2 f^{134} f^{243} \times$$

Lorentz indices μ_i

$$\times\left[(q-k)^4 g^{13} + (2k+q)^1 g^{34} + (-k-2q)^3 g^{41}\right]\times$$

$$\times\left[(-k-2q)_3\, g^2{}_4 + (2k+q)^2 g_{34} + (-k+q)_4\, g_3{}^2\right]$$

Feynman parameters ; shift $k\to k-xq$; $\Delta = -x(1-x)q^2$;
in numerator, linear $k\to 0$, $k^\mu k^\nu \to \frac{g^{\mu\nu}k^2}{d}$

$$= \frac12 g^2 f^{134} f^{234}\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-\Delta)^2}\times$$

$$\times\Big\{ g^{12}\left[k\cdot k\; 6\,\frac{d-1}{d} + q\cdot q\left((2-x)^2+(1+x)^2\right)\right]$$

$$- q^1q^2\left[(2-d)(1-2x)^2 + 2(1+x)(2-x)\right]\Big\}$$

Wick rotate, use basic 1-loop tadpole integrals, $I_2^1 = I_2^0\,\frac{d}{d-2}\,\Delta$

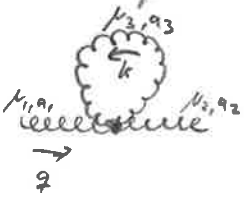
$$= i g^2 f^{a_1a_3a_4} f^{a_2a_3a_4}\int_0^1 dx\, I_2^0(-x(1-x)q^2)\;\frac12\Big\{g^{\mu_1\mu_2}q^2\left[6\,\frac{d-1}{2-d}\,x(1-x)+5-2x+2x^2\right]$$

$$- q^{\mu_1}q^{\mu_2}\left[(2-d)(1-2x)^2+2(1+x)(2-x)\right]\Big\}$$

• 3^{rd} diagram: template for

: $f^{12\circ}f^{\circ 34}(g_{13}g_{24}-g_{14}g_{23}) + 1324 + 1423$

12   33   1323   1323 ← here



$$= \frac12\int\frac{d^dk}{(2\pi)^d}\left(\frac{-i}{k^2}\right)(-ig^2)\left\{f^{12\circ}\cancel{f^{\circ33}}^{\,0}(..) + 2f^{13\circ}f^{\circ23}\underbrace{(g^{12}g^3{}_3 - g^{13}g_3{}^2)}_{=\,g^{12}(d-1)}\right\}$$

symm. factor

$$= -g^2 f^{134}f^{234}g^{12}(d-1)\underbrace{\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}}_{\text{would be zero}} \times \underbrace{\frac{k^2+2kq+q^2}{(k+q)^2}}_{\text{try to make it look like 2nd diag}}$$

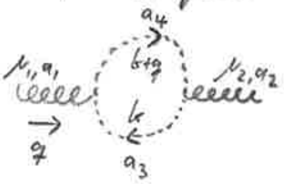
Fy par, shift $k\to k-xq$, $\Delta = -x(1-x)q^2$, numerator $k=0$

$$= -g^2 f^{134}f^{234}g^{12}(d-1)\int_0^1dx\int\frac{d^dk}{(2\pi)^d}\frac{k^2+(1-x)^2q^2}{(k^2-\Delta)^2}$$

Wick rotate , use basic 1-loop tadpole ints

$$= ig^2 f^{a_1a_3a_4}f^{a_2a_3a_4}\int_0^1dx\; I_2^0(-x(1-x)q^2)\; g^{\mu_1\mu_2}q^2(1-d)\left[\frac{d}{2-d}\,x(1-x)+(1-x)^2\right]$$

• 4^{th} diagram



closed loop of anticomm. fields — color — Lorentz

$$= -\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2}\frac{i}{(k+q)^2}(-g)^2 f^{413}(k+q)^1 f^{324}k^2$$

$$= -g^2 f^{134}f^{234}\int\frac{d^dk}{(2\pi)^d}\frac{(k+q)^1k^2}{k^2(k+q)^2}$$

Fy par, shift $k\to k-xq$, $\Delta\equiv -x(1-x)q^2$, numerator $k=0$, $k^\mu k^\nu\to \frac{g^{\mu\nu}k^2}{d}$

$$= -g^2 f^{134}f^{234}\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-\Delta)^2}\left\{\frac{g^{12}k^2}{d}-x(1-x)q^1q^2\right\}$$

Wick rotate, use 1-loop tadpole integrals

$$= ig^2 f^{a_1a_3a_4}f^{a_2a_3a_4}\int_0^1 dx\, I_2^0(-x(1-x)q^2)\; x(1-x)\left\{-g^{\mu_1\mu_2}q^2\frac{1}{2-d}+q^{\mu_1}q^{\mu_2}\right\}$$

• Sum diagrams 2+3+4

$$2+3+4 = ig^2 f^{134}f^{234}\int_0^1 dx\, I_2^0(-x(1-x)q^2)\;*$$

$$*\Big\{g^{\mu_1\mu_2}q^2\Big[3\frac{d-1}{2-d}x(1-x)+\tfrac{1}{2}(5-2x+2x^2)+(1-d)\frac{d}{2-d}x(1-x)+(1-d)(1-x)^2-\frac{1}{2-d}x(1-x)\Big]$$

$$-q^{\mu_1}q^{\mu_2}\Big[\frac{2-d}{2}(1-2x)^2+(1+x)(2-x) \qquad\qquad -x(1-x)\Big]\Big\}$$

from 

note that first line is invariant under $x\to 1-x$

$\{\dots\}$ is polynomial $\sim 1+x+x^2$

re-express this in terms of $a\equiv 1-2x$ which is odd under $x\to 1-x$

((i.e. $x^2=\frac{1}{4}(a^2-2a+1)$, $x=\frac{1}{2}(1-a)$))

$$= ig^2 f^{134}f^{234}\int_0^1 dx\, I_2^0(-x(1-x)q^2)\left\{g^{12}q^2\Big[(1-\tfrac{d}{2})a^2+\cancel{\tfrac{1-d}{2}a}+2\Big]-q^1q^2\Big[(1-\tfrac{d}{2})a^2+2\Big]\right\}$$

integrates to 0 (odd under $x\to 1-x$)

$$= ig^2\underbrace{f^{a_1a_3a_4}f^{a_2a_3a_4}}_{=N_c\delta^{a_1a_2}\ \text{for } SU(N_c)}\left(g^{\mu_1\mu_2}q^2-q^{\mu_1}q^{\mu_2}\right)\int_0^1 dx\, I_2^0(-x(1-x)q^2)\Big[(1-\tfrac{d}{2})(1-2x)^2+2\Big]$$

((wow: same Lorentz-structure as  !))

derivation: from Lie algebra (pg. 9) $[T^a,T^b]=if^{abc}T^c$

$$\Rightarrow f^{abc} = -2i\,\mathrm{tr}([T^a,T^b]T^c)$$
$$= -2i\left\{(T^a)_{ij}(T^b)_{jk}(T^c)_{ki}-(T^b)_{ij}(T^a)_{jk}(T^c)_{ki}\right\}$$

then use Fierz identity (pg. 9) $(T^a)_{ij}(T^a)_{kl}=\frac{1}{2}\left(\delta_{il}\delta_{kj}-\frac{1}{N_c}\delta_{ij}\delta_{kl}\right)$,

normalization (pg. 9) $\mathrm{tr}(T^aT^b)=\frac{1}{2}\delta^{ab}$, and $\delta_{ii}=N_c$

• summing up all four diagrams (see pg. 31, 33)

$$= i g^2 \delta^{a_1 a_2}\left(g^{\mu_1\mu_2} q^2 - q^{\mu_1} q^{\mu_2}\right) \int_0^1 dx \, *$$

$$* \left\{ -4x(1-x) \sum_f I_2^0\left(m_f^2 - x(1-x) q^2\right) + N_c \left[\left(1-\tfrac{d}{2}\right)(1-2x)^2 + 2\right] I_2^0\left(-x(1-x) q^2\right) \right\}$$

now, use (pg. 31) $I_2^0(\Delta) = \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-d/2}} = \frac{1}{2-\frac{d}{2}} \frac{\Gamma(3-\frac{d}{2})}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-d/2}}$

$$\overset{d=4-2\varepsilon}{\approx} \frac{1}{\varepsilon} \frac{1}{(4\pi)^2} + O(\varepsilon^0)$$

$$\Rightarrow \{ \dots \} \approx \frac{1}{\varepsilon} \frac{1}{(4\pi)^2} \left\{ -4x(1-x) N_f + N_c \left[-(1-2x)^2 + 2\right] \right\} + O(\varepsilon^0)$$

$$\Rightarrow \int_0^1 dx \{ \dots \} \approx \frac{1}{\varepsilon} \frac{1}{(4\pi)^2} \left\{ -\frac{2}{3} N_f + \frac{5}{3} N_c \right\} + O(\varepsilon^0)$$

$$\approx i \frac{g^2}{16\pi^2} \delta^{a_1 a_2} \left(g^{\mu_1\mu_2} q^2 - q^{\mu_1} q^{\mu_2}\right) \frac{1}{\varepsilon} \left\{ \boxed{\tfrac{5}{3}} N_c - \frac{2}{3} N_f + O(\varepsilon) \right\}$$

$\hookrightarrow \left(\frac{13}{6} - \frac{\xi}{2}\right)$ for general value of gauge parameter ξ

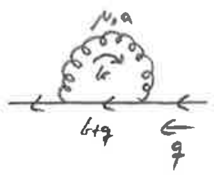
$\rightsquigarrow$ cannot (yet) take limit $\varepsilon \to 0$

need to remove the $\frac{1}{\varepsilon}$ divergence by a counterterm (see pg 29; § 3.3 below)

$\rightsquigarrow$ want to first compute other 1-loop divergences

3.2 more 1-loop divergences in QCD

$\rightarrow$ goal: evaluate 1-loop diagrams (again, in dim. reg. $d = 4 - 2\varepsilon$ and Feynman gauge $\xi = 1$)

• 

$$= \int \frac{d^d k}{(2\pi)^d} \left(i g \gamma^\mu T^a\right) \frac{i(\not k + \not q + m)}{(k+q)^2 - m^2 + i\varepsilon} \left(i g \gamma_\mu T^a\right) \left(\frac{-i}{k^2}\right)$$

$$\gamma^\mu \gamma_\mu = \gamma^\mu \gamma^\nu g_{\mu\nu} = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} g_{\mu\nu} = \frac{1}{2} 2 g^{\mu\nu} \mathbb{1} g_{\mu\nu} = d \mathbb{1}$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = \{\gamma^\mu, \gamma^\nu\}\gamma_\mu - \gamma^\nu \gamma^\mu \gamma_\mu = 2\gamma^\nu - \gamma^\nu d = (2-d)\gamma^\nu$$

$$T^a_{ij} T^a_{jk} = \frac{1}{2}\left(\delta_{ik}\delta_{jj} - \frac{1}{N_c}\delta_{ij}\delta_{jk}\right) = \frac{N_c^2 - 1}{2 N_c} \delta_{ik}$$

$$= -g^2 \frac{N_c^2 - 1}{2 N_c} \mathbb{1}_{N \times N} \int \frac{d^d k}{(2\pi)^d} \frac{(2-d)(\not k + \not q) + m d \, \mathbb{1}_{Dirac}}{(k^2)\left((k+q)^2 - m^2\right)}$$