2.5 QCD Feynman rules

→ recall from above remarks that for a perturbative treatment, we need to read off propagators (cf pg 21) and vertices (cf pg 23) from the Lagrangian $\mathcal{L}$.

→ $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$

$\mathcal{L}_I$: interactions ⇒ vertices

$\mathcal{L}_0$: bilinear in fields ⇒ propagators

→ recall: $\mathcal{L}_{QCD} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar\psi(i\not{D} - m)\psi$ (pg 18)

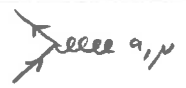
$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$ (pg 17)

$D_\mu = \partial_\mu - ig A^a_\mu T^a$

$= \mathcal{L}_0 + \bar\psi A^a_\mu g\gamma^\mu T^a \psi - g f^{abc} g^{\nu\rho}(\partial^\mu A^a_\nu) A^b_\mu A^c_\rho - \frac{1}{4} g^2 f^{eab} f^{ecd} g^{\mu\rho} g^{\nu\sigma} A^a_\mu A^b_\nu A^c_\rho A^d_\sigma$

- quark-gluon vertex: from $e^{i\int \mathcal{L}_I}$

$$\boxed{i \cdot g\gamma^\mu T^a \;\hat{=}}$$

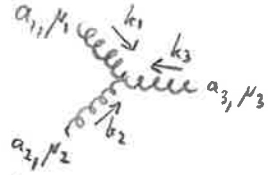


- 3-gluon vertex: need to fix conventions

in Fourier space, $\partial_\mu \to -ik_\mu$ ⇒ $i \cdot (-g f^{abc} g^{\nu\rho})(-ik^\mu)$

symmetrize this wrt A: 3! possible permutations

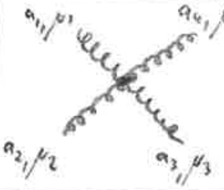
$\left(A(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{A}(k) \right)$



$$\hat{=} \; g f^{123} \{ (k_1 - k_2)_3 \, g_{12} + (k_2 - k_3)_1 \, g_{23} + (k_3 - k_1)_2 \, g_{31} \}$$

color indices $a_1, \dots$; Lorentz indices $\mu_1, \dots$

- 4-gluon vertex: $i \cdot \left(-\frac{1}{4} g^2 f^{e12} f^{e34} g^{13} g^{24}\right)$, 4! possible permutations (sets of 4 are equal)



$$\hat{=} -ig^2 \{ f^{12e} f^{e34} (g_{13} g_{24} - g_{14} g_{23}) + (1324) + (1423) \}$$

→ for the propagators, need to look at

$$S_0 = \int d^4x\, \mathcal{L}_0 = \int d^4x \left\{ \tfrac{1}{2} A_\mu^a \delta^{ab} (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu^b + \sum_{\text{flavor}} \bar\psi_f (i\not\partial - m_f) \psi_f \right\}$$

- mini-review: anticommuting (Grassmann) numbers

$\{\theta, \eta\} = 0 \Rightarrow \theta^2 = 0$, Taylor $f(\theta) = a + b\theta$ terminates!

integrals: $\int d\theta = 0$, $\int d\theta\, \theta \equiv 1$

complex Grassmann #s: $\theta = \theta_1 + i\theta_2$, $\theta^* = \theta_1 - i\theta_2$, $(\theta\eta)^* = \eta^*\theta^* = -\theta^*\eta^*$

complex Gauss int: $\int d\theta^* d\theta\, e^{-\theta^* b \theta} = \int d\theta^* d\theta\, (1 - \theta^* b \theta) = \int d\theta^* d\theta\, (1 + \theta\theta^* b) = b$

another one: $\int d\theta^* d\theta\, \theta\theta^*\, e^{-\theta^* b\theta} = \int d\theta^* d\theta\, \theta\theta^* = 1$

higher dim Gauss int: $\left(\prod_i \int d\theta_i^* d\theta_i \right) e^{-\theta_i^* B_{ij} \theta_j} = (\quad) e^{-\sum_i \theta_i^* b_i \theta_i} = \prod_i b_i = \det B$

↑ hermitean; diagonalize by unitary trafo.

derivatives: $\partial_\theta \theta = 1$; e.g. $\partial_\theta \eta\theta = -\partial_\theta \theta\eta = -\eta$ etc.

- quark propagator: consider one quark flavor, $\mathcal{L}_0 \ni \bar\psi (i\not\partial - m)\psi$

Grassmann-valued source fields

$$Z^q_{free}[\bar\eta, \eta] = \int \mathcal{D}\bar\psi \mathcal{D}\psi\, e^{i\int d^4x [\bar\psi (i\not\partial - m + i\varepsilon)\psi + \bar\eta\psi + \bar\psi\eta]}$$

shift ψ to complete square (see pg. 22), $\psi \to \psi + S\eta$ symbolically

$$= Z^q_{free}[0,0]\, e^{-\int d^4x \int d^4y\, \bar\eta(x) S_F(x-y) \eta(y)}$$

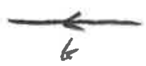
where $S_F(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{i}{\not k - m + i\varepsilon}$

((is again Greens fct: $(i\not\partial - m + i\varepsilon) S_F(x-y) = i\delta^{(4)}(x-y)$))

now e.g.
$$\langle 0 | T \psi(x_1) \bar\psi(x_2) | 0 \rangle = \frac{\int \mathcal{D}\bar\psi \mathcal{D}\psi\, \psi(x_1)\bar\psi(x_2)\, e^{i\int d^4x \mathcal{L}}}{\int \mathcal{D}\bar\psi \mathcal{D}\psi\, e^{i\int d^4x \mathcal{L}}}$$

$$= \frac{1}{Z^q_{free}[0,0]} \left(-i\delta_{\bar\eta(x_1)}\right)\left(+i\delta_{\eta(x_2)}\right) Z^q_{free}[\bar\eta, \eta] \Big|_{\bar\eta = \eta = 0}$$

$$= S_F(x_1 - x_2)$$ Feynman propagator ✓



$$\boxed{\quad \hat{=} \frac{i}{\not k - m + i\varepsilon} = \frac{i(\not k + m)}{k^2 - m^2 + i\varepsilon}}$$

(($(\not k - m)(\not k + m) = \not k \not k - m^2$, $\not k \not k = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} k_\mu k_\nu = k^2$))

→ for given propagator, will have same problem as in QED:
$(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)$ has no inverse (see pg 21)
need Faddeev-Popov gauge fixing

- <u>mini-review</u>: defining the functional integral of a gauge theory

$$Z \equiv \int \mathcal{D}\phi\, G(\phi) \quad ; \; \phi \text{ some gauge-fields } ((A_\mu^a)) \quad \left(\left(G(A) = e^{i\int d^4x \left(-\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu}\right)} \right)\right)$$

gauge invariance: $G(\phi) = G(\phi_\Lambda)$, $\int \mathcal{D}\phi = \int \mathcal{D}\phi_\Lambda$ $\quad \left(\left((A_\mu)_\Lambda = V(A_\mu + \frac{i}{g}\partial_\mu)V^\dagger,\ V = e^{iT\alpha^a} \right)\right)$ see §2.3

$$= \int \mathcal{D}\phi\, G(\phi)\, \underbrace{\Delta(\phi,b) \int \mathcal{D}\Lambda\, \delta(f(\phi_\Lambda) - b)}_{\equiv 1 \text{ (defines } \Delta)}$$

arbitrary field (b); "gauge condition" (f) $\quad ((f^a(A) = \partial^\mu A_\mu^a))$ covariant gauge

note: $\Delta(\phi,b) = \Delta(\phi_\Lambda, b)$ owing to $\int \mathcal{D}\Lambda$ in its definition

$$= \int \mathcal{D}\Lambda \int \mathcal{D}\phi_\Lambda\, G(\phi_\Lambda)\, \Delta(\phi_\Lambda, b)\, \delta(f(\phi_\Lambda) - b)$$
used gauge invariance of $\int \mathcal{D}\phi, G, \Delta$ (at each Λ)

$$= \left(\int \mathcal{D}\Lambda\right) \int \mathcal{D}\phi\, G(\phi)\, \Delta(\phi,b)\, \delta(f(\phi) - b)$$
renamed int variable $\phi_\Lambda \to \phi$

↑ "volume" of gauge orbit factors out; <u>cancels</u> in expectation values!

now average over b, with <u>weight</u> $B(b)$ $\quad \left(\left(B(b) = e^{-\frac{i}{2\xi}\int d^4x\, b^a b^a} \right)\right)$

$$= \frac{\left(\int \mathcal{D}\Lambda\right)}{\left(\int \mathcal{D}b\, B(b)\right)} \int \mathcal{D}\phi\, G(\phi) \int \mathcal{D}b\, B(b)\, \Delta(\phi,b)\, \delta(f(\phi) - b)$$

$$= \frac{\left(\int \mathcal{D}\Lambda\right)}{\left(\int \mathcal{D}b\, B(b)\right)} \int \mathcal{D}\phi\, G(\phi)\, B(f(\phi))\, \Delta(\phi, f(\phi))$$
used δ-fct

now compute the "<u>Faddeev-Popov determinant</u>" Δ from its definition:

$$\Delta(\phi, f(\phi)) = \left[\int \mathcal{D}\Lambda\, \delta(f(\phi_\Lambda) - f(\phi))\right]^{-1}$$
cross δ-peak with infinit. gauge trafo

$$= \left[\int \mathcal{D}\Lambda\, \delta\left(\Lambda \mathcal{F}(\phi) + \mathcal{O}(\Lambda^2)\right)\right]^{-1}$$
$f(\phi_\Lambda) \approx f(\phi) + \Lambda\, \mathcal{F}(\phi) + \mathcal{O}(\Lambda^2)$

$$= \left[\frac{1}{\det \mathcal{F}(\phi)} \int \mathcal{D}\Lambda\, \delta(\Lambda)\right]^{-1} = \det \mathcal{F}(\phi)$$

$$= \frac{\left(\int \mathcal{D}\Lambda\right)}{\left(\int \mathcal{D}b\, B(b)\right)} \int \mathcal{D}\phi\, G(\phi)\, B(f(\phi))\, \det \mathcal{F}(\phi)$$
, where $\mathcal{F}(\phi) = (\partial_\phi f(\phi))(\partial_\Lambda \phi_\Lambda)_{\Lambda=0}$

$\left(\left(\mathcal{F}(A) = \partial^\mu \left(f^{abc} A_\mu^b + \frac{1}{g}\delta^{ac}\partial_\mu\right), \text{ pg } 17 \right)\right)$

note: in QED, $(A_\mu)_\Lambda = A_\mu - \frac{1}{e}\partial_\mu \alpha$ (pg 12),
so $\mathcal{F}$ does <u>not</u> depend on A, hence $\det \mathcal{F}$ cancels in correlators.
⇒ in QCD, $\det \mathcal{F}(A)$ remains <u>inside</u> the functional integral.