2.4 Quantization, path integral (remarks only)

- so far, have seen non-Abelian gauge symmetry at work $\Rightarrow \mathcal{L}_{QCD}$.

  now, work out consequences for particle physics interactions

  $\rightarrow$ need rules for computing Feynman diagrams

  $\rightarrow$ apply rules to compute amplitudes, cross sections

- local gauge symmetry $\Rightarrow$ some Lagrangian dof's are unphysical

  ($\hat{=}$ can be adjusted arbitrarily by gauge trafos)

  (see QFT lecture)

  cf. QED: in functional integral $\int \mathcal{D}A\, e^{iS[A]}$ the photon part

  has $S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right)$

$$= \frac{1}{2} \int d^4x\, A_\mu(x) \left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu\right) A_\nu(x)$$

$$= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(k) \left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right) \tilde{A}_\nu(-k)$$

(FT $A(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{A}(k)$, $\int d^4x\, e^{-ikx} = (2\pi)^4 \delta^{(4)}(k)$)

$\Rightarrow$ for $\tilde{A}_\mu(k) = k_\mu a(k)$, $S = 0 \Rightarrow \int \mathcal{D}A\, e^{i \cdot 0}$ diverges ! ↯

(arbitrary scalar fct)

(($\Leftrightarrow$ ( ) has no inverse : cannot solve $\left(-k^2 g_{\mu\nu} + k_\mu k_\nu\right) \tilde{D}_F^{\nu\rho}(k) = i g_\mu{}^\rho$

for Feynman propag $\tilde{D}_F$))

recall Abelian gauge invariance $A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x)$

$\Rightarrow$ field configurations that are gauge-equivalent to $A_\mu(x) = 0$ did ↯

$\rightarrow$ the way out was Faddeev-Popov gauge fixing

[Phys. Lett. 25B (1967) 29]

result: $S \rightarrow S + \int d^4x \left(-\frac{1}{2\xi} (\partial_\mu A^\mu)^2\right)$

$\Leftrightarrow$ can solve $\left(-k^2 g_{\mu\nu} + (1 - \frac{1}{\xi}) k_\mu k_\nu\right) \tilde{D}_F^{\nu\rho}(k) = i g_\mu{}^\rho$:

$$\tilde{D}_F^{\mu\nu}(k) = \frac{-i}{k^2 + i\varepsilon} \left(g^{\mu\nu} - (1-\xi) \frac{k^\mu k^\nu}{k^2}\right)$$ photon propagator

$\rightarrow$ propagator depends on arbitrary parameter ξ ? ↯

physics does not : QED vertex is such

that ξ drops out of S-matrix elements

(due to the Ward-Takahashi identities)

$\rightsquigarrow$ similar structure in QCD; ξ-cancellations more complicated.

- we will make use of <u>functional methods</u>

  → most useful for interacting QFT's :
  <u>path integral</u> method, relying on functional integration

  → for (many) more details : [QFT lecture]
  [Peskin/Schröder, §9]

- reminder of a <u>functional derivative</u> :

  def. $\delta_{J(x)} J(y) = \delta^{(4)}(x-y)$ or $\delta_{J(x)} \int d^4y \, J(y) \phi(y) = \phi(x)$

  ⇒ can take functional derivatives as usual,

  e.g. $\delta_{J(x)} e^{i\int d^4y \, J(y)\phi(y)} = i\phi(x) \, e^{i\int d^4y \, J(y)\phi(y)}$

  e.g. $\delta_{J(x)} \int d^4y \, (\partial_\mu J(y)) A^\mu(y) = -\partial_\mu A^\mu(x)$ ((after partial integration))

- reminder of the <u>generating functional</u> of correlation functions

  $$Z[J] = \int \mathcal{D}\phi \, e^{i\int d^4x \, [\mathcal{L} + \underbrace{J(x)\phi(x)}_{\text{source term}}]}$$

  such that $\langle 0 | \overset{\text{time ordering}}{T} \phi(x_1)\phi(x_2) | 0 \rangle = \dfrac{\int \mathcal{D}\phi \, \phi(x_1)\phi(x_2) \, e^{i\int d^4x \, \mathcal{L}}}{\int \mathcal{D}\phi \, e^{i\int d^4x \, \mathcal{L}}}$

  $= \dfrac{1}{Z[0]} \left(-i\delta_{J(x_1)}\right)\left(-i\delta_{J(x_2)}\right) Z[J] \Big|_{J=0}$ very elegant!

- to see the elegance of the $Z[J]$ formulation,
  consider a free scalar theory, $\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$

  ⇒ $\int d^4x \, [\mathcal{L}_0 + J\phi] \overset{PI}{=} \int d^4x \, [\frac{1}{2}\phi(-\partial^2 - m^2 + i\varepsilon)\phi + J\phi]$

  ↳ convergence factor for functional integral, $\varepsilon > 0$

  complete the square : $\phi \to \phi + i\int d^4y \, D_F(x-y) J(y)$

  where $(-\partial^2 - m^2 + i\varepsilon) D_F(x-y) = -i\delta^{(4)}(x-y)$, D_F is Greens fct

  $= \int d^4x \, \left[\mathcal{L}_0 + \frac{i}{2} J(x) \int d^4y \, D_F(x-y) J(y) \right]$

  ⇒ $Z_{\text{free}}[J] = Z_{\text{free}}[0] \, e^{-\frac{1}{2}\int d^4x \, d^4y \, J(x) D_F(x-y) J(y)}$

  ⇒ two-point function $\langle 0 | T\phi(x_1)\phi(x_2) | 0 \rangle_{\text{free}} \overset{\text{(check ?!)}}{=} \ldots = D_F(x_1 - x_2)$

  ⇒ four-point function $\langle 0 | T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) | 0 \rangle_{\text{free}} = D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}$

  ((where $D_{ij} \equiv D_F(x_i - x_j)$; ))

  ⇒ etc

- now, consider an interacting theory, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ (e.g. $-\frac{\lambda}{4!}\phi^4$)

look at generating functional again

$$Z[J] = \int \mathcal{D}\phi \, e^{i\int d^4x [\mathcal{L}_0 + \mathcal{L}_I + J\phi]}$$

$$= \int \mathcal{D}\phi \, \underbrace{e^{i\int d^4x \mathcal{L}_I(\phi \to -i\delta_J)}}_{\phi\text{-independent!}} \, \underbrace{e^{i\int d^4x [\mathcal{L}_0 + J\phi]}}_{\text{as in free theory} \Rightarrow \text{shift as above}}$$

$$= e^{i\int d^4x \mathcal{L}_I(\phi \to -i\delta_J)} \cdot e^{-\frac{1}{2}\int d^4x \int d^4y \, J(x) D_F(x-y) J(y)} \cdot \underbrace{\int \mathcal{D}\phi \, e^{i\int d^4x \mathcal{L}_0}}_{= Z_{free}[0]}$$

such that the correlation functions follow from

$$\langle 0 | T \, \mathcal{O}(\phi) | 0 \rangle = \frac{1}{Z[0]} \mathcal{O}(\phi \to -i\delta_J) Z[J] \Big|_{J=0}$$

$$= \frac{\mathcal{O}(\phi \to -i\delta_J) \, e^{i\int d^4x \mathcal{L}_I(\phi \to -i\delta_J)} \, e^{-\frac{1}{2}\int d^4x \int d^4y \, J(x) D_F(x-y) J(y)} \Big|_{J=0} \cdot \cancel{Z_{free}[0]}}{e^{i\int d^4x \mathcal{L}_I(\phi \to -i\delta_J)} \, e^{-\frac{1}{2}\int d^4x \int d^4y \, J(x) D_F(x-y) J(y)} \Big|_{J=0} \cdot \cancel{Z_{free}[0]}}$$

((note: in denominator, sum of "vacuum diagrams"))

→ perturbative expansion (Feynman diagrams) follow from expanding $e^{i\int \mathcal{L}_I}$ in terms of (small) coupling constants (here: λ)

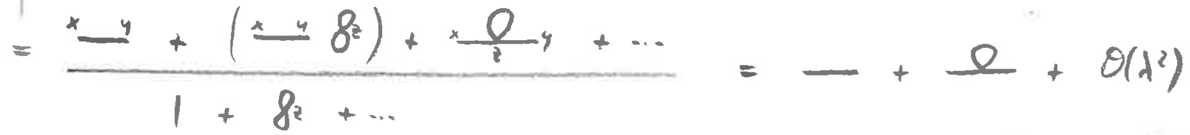
→ all combinatorics for evaluating correlation fcts is just in exponentials!

⇒ two-point function $\langle 0 | T \, \phi(x_1) \phi(x_2) | 0 \rangle =$

$$= \frac{\delta_{iJ(x_1)} \delta_{iJ(x_2)} \left\{ 1 + \int d^4z \left(-\frac{i\lambda}{4!}\right) \delta_{iJ(z)}^4 + \mathcal{O}(\lambda^2) \right\} e^{-\frac{1}{2}\int d^4x \int d^4y \, J(x) D_F(x-y) J(y)} \Big|_{J=0}}{\left\{ \quad \dots \quad \right\} e^{-\frac{1}{2}\,\dots} \Big|_{J=0}}$$

((check!)) $\left(D_{iz} = D_F(x_i - z)\right)$

$$= \frac{D_{12} + \left(-\frac{i\lambda}{4!}\right) \int d^4z \left(3 D_{12} D_{zz} D_{zz} + 12 \, D_{1z} D_{2z} D_{zz} \right) + \mathcal{O}(\lambda^2)}{1 + \left(-\frac{i\lambda}{4!}\right) \int d^4z \; 3 D_{zz} D_{zz} + \mathcal{O}(\lambda^2)}$$



→ this cancellation is actually generic: $\dfrac{(\text{connected pieces}) \cdot e^{(\text{disconnected pieces})}}{e^{(\text{disconnected pieces})}}$

works for all higher correlation fcts as well.