2.2 Generalization : Yang-Mills Lagrangian

geometric construction can be generalized :

invariance under local phase rotations

$\rightarrow$ invariance under any (continuous) symmetry group

here, use 3d rotation group (O(3) or SU(2)) for brevity

$\rightarrow$ in the end, simple generalization to arbitrary local symmetry.

- consider $\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$ (doublet of Dirac fields)

demand invariance under local 3d rotations : $\boxed{\psi(x) \to \underbrace{e^{i\frac{\sigma^i}{2}\alpha^i(x)}}_{\equiv V(x)}\psi(x)}$

(matrix = ? $e^{\text{matrix}} = \sum_{n=0}^{\infty} \frac{(\text{Matrix})^n}{n!}$)

((where σ^i = Pauli matrices $= \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$; $\sum_{i=1}^{3}$ suppressed))

Q: construct invariant Lagrangian ?

$\rightarrow$ need again a covariant derivative !

$\rightarrow$ now, compensating phase factor has to be a matrix, with transformation $U(y,x) \to V(y)\,U(y,x)\,V^+(x)$

$\rightarrow$ again, $U(x,x)=1$ and $U^+U = UU^+ = \mathbb{1}$ unitary

$\Rightarrow$ can expand in terms of (hermitean: $\sigma^+ = \sigma$) SU(2)-generators:

$$U(x+\varepsilon n, x) \approx \mathbb{1} + \underbrace{ig}_{\text{convention}}\, \varepsilon n^\mu \underbrace{A_\mu^i \frac{\sigma^i}{2}}_{\text{new (matrix-valued) vector field}} + O(\varepsilon^2)$$

('$-$' different from QED, since we take $e<0$, $g>0$)

$\Rightarrow$ covariant derivative : $\left(\left(n^\mu D_\mu \psi = \lim_{\varepsilon\to 0} \frac{\psi(x+\varepsilon n) - U(x+\varepsilon n, x)\psi(x)}{\varepsilon} \right)\right)$

$$D_\mu = \partial_\mu - ig A_\mu^i \frac{\sigma^i}{2}$$

where $A_\mu^i(x)\frac{\sigma^i}{2} \longrightarrow V(x)\left(A_\mu^i(x)\frac{\sigma^i}{2} + \frac{i}{g}\partial_\mu\right)V^+(x)$ (consistency w/ U-trafo)

now, infinitesimally, $\psi \to \left(1 + i\alpha^i \frac{\sigma^i}{2} + \dots\right)\psi$

$$D_\mu\psi \to \underset{\text{- - - -}}{(\text{check!})} = \left(1 + i\alpha^i\frac{\sigma^i}{2}\right) D_\mu \psi$$

again, transforms the same way as field $\psi(x)$ ✓

((also valid for finite transformations (check!)))

$\Rightarrow$ again, $\bar\psi(x)\left(i\gamma^\mu D_\mu - m\right)\psi(x)$ is locally invariant.

- gauge-invariant terms containing A_μ^i only?

  → here, use construction via D_μ:

  from above, we have $[D_\mu, D_\nu]\psi(x) \to V(x)\,[D_\mu, D_\nu]\psi(x)$ (**)

now, note that

$$[D_\mu, D_\nu]\psi = \overset{\circ}{[\partial_\mu, \partial_\nu]\psi} - ig\left([\partial_\mu, A_\nu^i \tfrac{\sigma^i}{2}] + [A_\mu^i \tfrac{\sigma^i}{2}, \partial_\nu]\right)\psi - g^2 \underbrace{[A_\mu^i \tfrac{\sigma^i}{2}, A_\nu^j \tfrac{\sigma^j}{2}]}\psi$$

does not vanish, as in QED → $= A_\mu^i A_\nu^j [\frac{\sigma^i}{2}, \frac{\sigma^j}{2}] = A_\mu^i A_\nu^j\, i\varepsilon^{ijk}\frac{\sigma^k}{2}$

$(\sigma^i\sigma^j = \delta^{ij}\mathbb{1} + i\varepsilon^{ijk}\sigma^k)$

$$= -ig\,\underbrace{\left(\partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g\,\varepsilon^{ijk} A_\mu^j A_\nu^k\right)}_{\equiv F_{\mu\nu}^i(x)} \frac{\sigma^i}{2}\psi$$

$\equiv F_{\mu\nu}^i(x)$ = (non-Abelian) field strength tensor

→ as before, $[D_\mu, D_\nu]$ is not a derivative, but a constant (matrix)!

⇒ from (**), the field strength is <u>not invariant</u> now,

but transforms as $F_{\mu\nu}^i \frac{\sigma^i}{2} \to V(x)\, F_{\mu\nu}^i \frac{\sigma^i}{2} V^\dagger(x)$

⇒ can construct the locally invariant terms from <u>traces</u> (using cyclicity and $V^\dagger V = \mathbb{1}$)

e.g. $\mathrm{Tr}\left(F_{\mu\nu}^i \frac{\sigma^i}{2} F^{\mu\nu j}\frac{\sigma^j}{2}\right) = \frac{1}{2} F_{\mu\nu}^i F^{\mu\nu i} \equiv \frac{1}{2}(F_{\mu\nu}^i)^2$

- adding up: $\boxed{\mathcal{L} = \bar\psi(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}(F_{\mu\nu}^i)^2}$ Yang-Mills Lagrangian

  - two parameters: m, g
  - variations → equations of motion: Dirac eqn + eqn for vector field

- generalize to other continuous symmetry groups:

  V → $n\times n$ unitary matrices ; $\psi(x)$ is n-plet ; $\psi(x) \to V(x)\psi(x)$

  expand $V(x) \approx \mathbb{1} + i T^a \alpha^a(x) + \mathcal{O}(\alpha^2)$

  ↑ $(T^a)^\dagger = T^a$ set of generators of symmetry group

  all as above, with $\frac{\sigma^i}{2} \to T^a$

  for def. of $F_{\mu\nu}^a$ use $[T^a, T^b] = i f^{abc} T^c$

  ↑ completely antisy. structure const.

- summary:

  invariance of n-plet ψ under local "gauge" transformations $\psi(x) \to V(x)\psi(x)$

  with $V(x) \equiv n\times n$ unitary matrices $= e^{iT^a\alpha^a(x)}$

  where $(T^a)^\dagger = T^a$ are hermitean generators

  with structure constants f^{abc} given by $[T^a, T^b] = if^{abc}T^c$

  $\Rightarrow$ covariant derivative $D_\mu = \partial_\mu - igA_\mu^a T^a$

  contains one vector field for each independent generator of local symmetry

  $A_\mu^a T^a \to V(x)\left(A_\mu^a T^a + \frac{i}{g}\partial_\mu\right)V^\dagger(x)$ guarantees $D_\mu\psi(x) \to V(x)D_\mu\psi(x)$

  $\Rightarrow$ field strength tensor $[D_\mu, D_\nu] = -igF_{\mu\nu}^a T^a$

  ((or $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$))

  transforms as $F_{\mu\nu}^a T^a \to V(x)F_{\mu\nu}^a T^a V^\dagger(x)$

  $\rightsquigarrow$ for later reference: <u>infinitesimal transformations</u>

  $$\psi \to \psi + iT^a\alpha^a(x)\psi + O(\alpha^2)$$

  $$A_\mu^a \to A_\mu^a + \left(f^{abc}A_\mu^b + \frac{1}{g}\delta^{ac}\partial_\mu\right)\alpha^c(x) + O(\alpha^2)$$

  $$F_{\mu\nu}^a \to F_{\mu\nu}^a + f^{abc}F_{\mu\nu}^b\alpha^c(x) + O(\alpha^2)$$

  $\Rightarrow$ most general gauge-invariant renormalizable Lagrangian

  (conserving P, T): $\mathcal{L} = \bar\psi(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}(F_{\mu\nu}^a)^2$

- jargon: <u>Abelian</u> symmetry group of QED

  vs <u>non-Abelian</u> symmetry group of the more general theories above.

  $\to$ <u>non-Abelian gauge theory</u> $\equiv$ QFT associated with a non-commuting local symmetry