2. Basics

2.1 Reminder: QED and gauge invariance

- gauge symmetry is a fundamental principle that determines the form of the Lagrangian
- consider $\psi(x)$ (complex-valued Dirac-field)

we now demand the theory to be invariant under local phase transformations: $\boxed{\psi(x) \to e^{i\alpha(x)}\psi(x)}$

Q: which Lagrangian terms can we construct that are invariant?

A1: terms that are also invariant under global trafos

e.g. $\underline{\bar\psi(x)\psi(x)}$ ((recall Dirac-adjoint $\bar\psi \equiv \psi^\dagger\gamma^0$))

A2: for terms with derivatives, we need some preparation:

((recall: derivative in eg. n^μ-direction def'd as differential quotient))

$$\frac{\psi(x+\varepsilon n)-\psi(x)}{\varepsilon} \xrightarrow{\varepsilon\to 0} n^\mu\partial_\mu\psi(x)$$

↑ ——— ↑ feel completely different phase transformation ! ↯

($U(x,x)=1$; $\psi(y)$, $U(y,x)\psi(x)$ transform the same)

⇒ for meaningful comparison, introduce a compensating (scalar) phase factor, transforming as $U(y,x) \to e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)}$

⇒ def. covariant derivative $\frac{\psi(x+\varepsilon n)-U(x+\varepsilon n,x)\psi(x)}{\varepsilon} \xrightarrow{\varepsilon\to 0} n^\mu D_\mu\psi(x)$

for infinitesimal separation of y, x, expand:

$$U(x+\varepsilon n,x) \approx 1 \underbrace{-ie}_{\text{definition}} \varepsilon n^\mu A_\mu(x) + \mathcal{O}(\varepsilon^2)$$

↑ new vector field ! "connection"

⇒ cov. deriv.: $D_\mu\psi(x) = \partial_\mu\psi(x) + ie A_\mu(x)\psi(x)$

where $A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x)$ (consistent w/ U-transformation)

now: $\boxed{D_\mu\psi(x) \to \overset{\text{(check!)}}{\ldots} = e^{i\alpha(x)}D_\mu\psi(x)}$

transforms the same way as the field $\psi(x)$ ✓✓

⇒ $\underline{\bar\psi(x)D_\mu\psi(x)}$ also invariant.

- summary 1: local phase rotation symmetry
  $\Rightarrow$ def. of covariant derivative
  and existence of vector field A_μ (connection)
  and transformation properties of A_μ

$\rightarrow$ all terms that are globally ($\psi \to e^{i\alpha}\psi$, α const) invariant are also locally invariant if we replace all $\partial_\mu \to D_\mu$.

- how about (locally invariant) kinetic terms for A_μ?

(a) construction using $U(y,x)$

($u(x,x)=0$ from $U(x,x)=1$)

U is pure phase: $U(y,x) = e^{iu(y,x)}$, $u(y,x) \in \mathbb{R}$

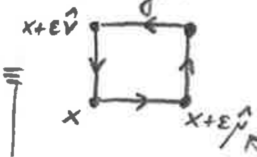
assume $U(y,x) = [U(x,y)]^+$ $\left(\!\left(\Rightarrow u(y,x) = -u(x,y) \right)\!\right)$

$\Rightarrow u(y,x) = \sum_{n=0}^{\infty} (y-x)^{2n+1} f_n(y+x)$ is odd under $y \leftrightarrow x$

$\rightsquigarrow$ can write $U(x+\varepsilon n, x) = e^{-ie\varepsilon n^\mu A_\mu(x+\frac{\varepsilon n}{2})} + O(\varepsilon^3)$

now, use this for comparing phase products around a small square

$U(x) \equiv$



(unit vector in μ-direction, e.g. $\hat{1}$)

$$= U(x, x+\varepsilon\hat{\nu})\, U(x+\varepsilon\hat{\nu}, x+\varepsilon\hat{\mu}+\varepsilon\hat{\nu})\, U(x+\varepsilon\hat{\mu}+\varepsilon\hat{\nu}, x+\varepsilon\hat{\mu})\, U(x+\varepsilon\hat{\mu}, x)$$

$$= e^{-ie\varepsilon\left\{-A_\nu(x+\frac{\varepsilon\hat{\nu}}{2}) - A_\mu(x+\frac{\varepsilon\hat{\mu}}{2}+\varepsilon\hat{\nu}) + A_\nu(x+\varepsilon\hat{\mu}+\frac{\varepsilon\hat{\nu}}{2}) + A_\mu(x+\frac{\varepsilon\hat{\mu}}{2})\right\} + O(\varepsilon^3)}$$

$$\approx 1 - ie\varepsilon\,\frac{\varepsilon}{2}\left\{\cancel{-\partial_\nu A_\nu(x)} \cancel{- \partial_\mu A_\mu(x)} - 2\partial_\nu A_\mu + 2\partial_\mu A_\nu(x) \cancel{+ \partial_\nu A_\nu(x)} \cancel{+ \partial_\mu A_\mu(x)}\right\} + O(\varepsilon^3)$$

$$= 1 - ie\varepsilon^2 \underbrace{\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right)}_{\equiv F_{\mu\nu}(x)} + O(\varepsilon^3)$$

(area of square)

$\equiv F_{\mu\nu}(x)$ $\hat{=}$ electromagnetic field strength tensor

but $U(x)$ is locally invariant by construction!

$\Rightarrow F_{\mu\nu}(x)$ is a locally invariant function of A_μ!

(6) construction using D_μ

Since (see above) $\psi \to e^{i\alpha(x)}\psi$, $D_\mu\psi \to e^{i\alpha(x)} D_\mu\psi$

it also follows that $D_\mu D_\nu \psi \to e^{i\alpha(x)} D_\mu D_\nu \psi$

or $[D_\mu, D_\nu]\psi \to e^{i\alpha(x)} [D_\mu, D_\nu]\psi$ (*)

now, note that

$$[D_\mu,D_\nu]\psi = \overset{0}{\cancel{[\partial_\mu,\partial_\nu]}}\psi + ie\left([\partial_\mu,A_\nu]+[A_\mu,\partial_\nu]\right)\psi - e^2\overset{0}{\cancel{[A_\mu,A_\nu]}}\psi$$

$$= ie\left(\partial_\mu A_\nu\psi + \cancel{\partial_\mu A_\nu\psi} - \cancel{A_\nu\partial_\mu\psi} + \cancel{A_\mu\partial_\nu\psi} - \partial_\nu A_\mu\psi - \cancel{\partial_\nu A_\mu\psi}\right)$$

$$= ie\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\cdot\psi$$ has no derivative acting outside ()!

$\Rightarrow$ $[D_\mu, D_\nu] = ie F_{\mu\nu}$

$\Rightarrow$ in (*), $F_{\mu\nu}$ is just a multiplicative factor, must be invariant.

- can now write the most general locally invariant Lagrangian (for the electron field ψ and its associated vector field A_μ)

$$\mathcal{L}_{QED} = \bar\psi\left(i\gamma^\mu D_\mu - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - c\,\varepsilon^{\mu\nu\sigma\rho}F_{\mu\nu}F_{\sigma\rho}$$

remarks: ▶ used operators of dimension ≤ 4 here

in general, there are many additional gauge-invariant op's, e.g.:

$\mathcal{L}_5 \sim \bar\psi[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\psi$

$\mathcal{L}_6 \sim (\bar\psi\psi)^2$, $(\bar\psi\gamma^5\psi)^2$, ...

→ all these are non-renormalizable (see later) interactions

(see e.g. § 12.1 in Peskin/Schroeder) → irrelevant for physics, in Wilsonian sense

▶ the coefficient $c \equiv 0$ if we postulate invariance under (discrete) P, T symmetries

→ then only 2 free parameters in $\mathcal{L}$: m, e (hidden in D_μ)

- summary 2: local phase rotation symmetry of electron field ψ

$\Rightarrow$ existence + transformation properties of em. vector potential A_μ

$\Rightarrow$ most general (4d, renormalizable, T or P invariant) Lagrangian is unique : Maxwell-Dirac-Lagrangian !