

1.4 Elements of group theory

- the color charge introduced above can be treated much more rigorously.
 - symmetry at work.
 - before (re-) learning the connection symmetry \leftrightarrow charge from QED (cf. § 2.1), let us review some basic facts about the theory of continuous symmetry groups
 - (much) more detail e.g. in [H. Georgi: Lie Algebras in Particle Physics] or at <http://www.physik.uni-bielefeld.de/~v.laine/symmetrien/cover.html>
- our provisional color assignment to gluons (cf. § 1.3) $g_1 \dots g_8$ can be rewritten in a different basis (just different linear combinations) of 3×3 -matrices (label e.g. rows by colors, columns by anticolors $\begin{pmatrix} r & g & b \\ g & \dots & \dots \\ b & \dots & \dots \end{pmatrix}$)

$$T^1 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad T^2 = \frac{1}{2} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad T^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad T^4 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$T^5 = \frac{1}{2} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad T^6 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad T^7 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad T^8 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

→ actually, $T^a = \frac{1}{2} \lambda^a$, where λ^a are "Gell-Mann matrices", $a = \begin{matrix} 1 \dots 3 \\ \dots \\ 1 \dots 8 \end{matrix}$

→ they form a possible representation of the infinitesimal generators of the "special unitary group" $SU(3)$, the fundamental representation

→ some important properties (check?!):

$$[T^a, T^b] = i f^{abc} T^c$$

Lie algebra

↑ antisymmetric structure constants

$$\{T^a, T^b\} = \frac{1}{3} \delta^{ab} \mathbb{1}_{3 \times 3} + d^{abc} T^c$$

↑ symmetric structure constants

$$\Rightarrow T^a T^b = \frac{1}{2} \left(\frac{1}{3} \delta^{ab} \mathbb{1}_{3 \times 3} + (d^{abc} + i f^{abc}) T^c \right)$$

$$T_{ij}^a T_{kl}^a = \frac{1}{2} \left(\delta_{ik} \delta_{jl} - \frac{1}{3} \delta_{ij} \delta_{kl} \right)$$

Fierz identity

→ often we will need traces

$$\text{Tr}(T^a) = 0$$

$$\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$$

normalization

etc.

("generalization of Pauli matrices")

→ could calculate f^{abc} by multiplying Lie algebra with T^d , then taking trace:

$$f^{abc} = \frac{2}{i} (\text{Tr}(T^a T^b T^c) - \text{Tr}(T^b T^a T^c))$$

result (check?!): $f^{123} = 1$, $f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}$,
 $f^{458} = f^{678} = \frac{\sqrt{3}}{2}$; rest by antisymmetry

• from a more general viewpoint, we have just seen one example of a broader mathematical concept: representations of Lie Groups

math: group contains abstract entities that obey certain algebraic rules

QM: interested in groups of unitary operators acting on vector space of states

here: interested in continuously generated groups

contain elements arbitrarily close to identity.

can reach general group element by repeated action of infinitesimal ones

$$g(\alpha) = 1 + i\alpha^a T^a + \mathcal{O}(\alpha^2)$$

↑ ↑ Hermitian op's; "generators" of symm. group
group parameters

a group with this structure is called a "Lie group"

• the set T^a spans space of infinitesimal group transformations

⇒ commutator is a linear combination of generators

$$[T^a, T^b] = i f^{abc} T^c$$

vector space spanned by generators + commutator ≡ Lie algebra

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0 \quad \text{Jacobi identity}$$

$$\Rightarrow f^{abc} f^{ced} + f^{ace} f^{edb} + f^{ade} f^{ebc} = 0$$

• for us, symmetry ≡ unitary transformation of a set of fields

→ interested in Lie groups with finite # of generators: "compact"

• classification of Lie Algebras (group of phase rotations)

→ if one T^a commutes with all others: Abelian subgroup, $\psi \rightarrow e^{i\alpha} \psi$, $U(1)$

→ if set of T^a 's cannot be divided into two mutually commuting sets: "simple"

→ general Lie algebra ≡ direct sum of non-Abelian simple components + additional Abelian generators

→ $SU(N)$ ($UU^\dagger=1, \det U=1$), $SO(N)$ ($RR^T=1, \det R=1$), $Sp(N)$; $G_2, F_4, E_{6,7,8}$
is complete set of compact simple Lie groups!

(Killing; Cartan)
ca. 1894

1.5 Notation and conventions

• natural units $\hbar = c = k_B = 1$

$$\Rightarrow [\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1} = \text{GeV}^{-1}$$

• vectors + tensors

indices $\mu = 0, 1, 2, 3$ or t, x, y, z

metric tensor $g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$

four-vectors $x^\mu = (x^0, \vec{x})$; $\partial_\mu = \partial_{x^\mu} = (\partial_0, \vec{\nabla})$

totally antisymmetric tensor $\epsilon^{0123} \equiv 1$ ($\Rightarrow \epsilon_{0123} = -1, \epsilon^{1230} = -1$ etc)

• matrices

Pauli $\sigma^i \sigma^j = \delta^{ij} \mathbb{1}_{2 \times 2} + i \epsilon^{ijk} \sigma^k$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

$$\text{standard basis: } \gamma^0 = \begin{pmatrix} \mathbb{1}_{2 \times 2} & 0 \\ 0 & -\mathbb{1}_{2 \times 2} \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$$

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

• Einstein summation convention

$$\text{e.g. } p_\mu x^\mu \equiv \sum_{\mu=0}^3 p_\mu x^\mu = p_0 x^0 + (-\vec{p}) \cdot \vec{x} = p^0 x^0 - \vec{p} \cdot \vec{x}$$

$$\text{e.g. } A^\alpha T_\alpha \equiv \sum_\alpha A^\alpha T_\alpha$$