1. Introduction

Nature is extremely strange — but also very beautiful.
We have built a system of understanding (bar):

- QM + Special Rel ⇒ QFT
- objects: space-filling fields;
  excitations: particles
- "Standard Model" = 3 basic conceptual structures
  - gauge system: \( SU(3) \times SU(2) \times U(1) \)
    ~ 3 parameters \( g_i \)
  - gravity system: Einstein–Hilbert action
    + minimal matter coupling
    ~ 2 parameters \( G, \lambda \)
  - Higgs system: no deep principle
    ~ many parameters
  - provision concept?

- (extremely) accurately tested/confirmed
  by many experiments

1.1. QCD Hypothesis

"Zoom" into part of gauge system: QCD

Theory of strong interactions

\& models you know from particle physics (quarks, color, pions)
+ mathematical structure you know from QFT
  (non-Abelian gauge theory, [Yang, Mills 1954])
Some highlights:

- Major feature: asymptotic freedom
QCD shows different forms at long and short distances

Long distance
- Low energy
- Conformal to asymptotic freedom
- Non-perturbative hadronic structure
  - e.g., lattice QCD

Short distance
- High energy
- Interplay to asymptotic freedom
- Non-perturbative
- Lattice QCD
- Perturbative methods

- Rough qualitative picture of asymptotic freedom:
  - Value of \( \alpha_s \) depends on distance (i.e., energy)

\[ \alpha_s = \frac{\alpha_s}{\bar{\alpha}_s} \]

\[ \alpha_s(r) \propto \frac{1}{r} \]

who cares? \[ \alpha_s(Q^2) \approx \left( \frac{-\alpha_s(\mu)}{Q^2} + \frac{1}{\alpha_s(\mu)} \right) \]

Nobel 2004:
- Gross, Politzer, Wilczek

\[ e^\frac{\alpha_s}{\pi} \text{ for } Q \text{ high} \]

\[ e^\frac{\alpha_s}{\pi} \text{ for } Q \text{ low} \]

- Lattice QCD
QCD \( \alpha_s(\mu) = 0.1184 \pm 0.0007 \)

Heavy Quarkonia

Deep Inelastic Scattering

July 2009

\[ Q \gev \]

\begin{align*}
\text{mass [GeV]} & \\
\text{vector meson octet baryon} & \\
\text{decuplet baryon} & \\
\text{octet baryon} & \\
\end{align*}
1.2 reality checks

- **Hadamron spectrum**

  1. (Bound states of quarks; e.g., K, Σ, Π, ρ, ω, η, A, ud, ...)
  2. "sea" world, at long distances, observe not quarks/ghos, but hadrons (mesons 77+9, baryons 799)
  3. problem: only %0.007

  \[ \text{"solve" QCD eyes by computer: lattice QCD} \]

  \[ \text{What one gets are just the observed particle masses} \]

  \[ \text{(no gluons; no fractional changes)} \]

- **upshot:** QCD predicts the heavy lying hadron masses

- **Experimental checks of QCD**

  - **Collider physics**

    - **CEP** (CERN, 1989-2000): e⁺e⁻ → X

      \[ \text{Asymptotic freedom enables use to compute} \]

      \[ \text{interactions of quarks/ghos at short distances;} \]

      \[ \text{detectors are far distance away, see hadrons (not far particles)} \]

      \[ \text{for comparison of theory vs experiment, need also} \]

      \[ \text{infrared safety: check of quantities which are} \]

      \[ \text{independent of long-distance physics, hence QCD-calculable} \]

      \[ \text{factorization: even weak class of processes, can} \]

      \[ \text{be factored into universal long-distance piece} \]

      \[ \text{and process-dependent (lit QCD-calculable) short-distance pc.} \]

    \[ \text{get (QCD!)} \]

    1. \( \text{X} = \mu^{-} - \mu^{-} \) or \( \tau^{-} \) or ...

    2. \( X > 10 \) particles: π, S, P, F, ...

        \[ \text{QCD "jets"} \]

    \[ \text{case (1): no color charge \rightarrow nearly QCD interactions} \]

      \[ \text{simple final state coupling} \quad \text{g} \rightarrow \text{small} \]

      \[ \rightarrow \text{most of the time (99%) nothing happens} \]

      \[ \rightarrow e^{-}\gamma \sim 1\% \quad \text{to check QCD details} \]

      \[ \rightarrow e^{-}\gamma \sim 0.01\% \rightarrow \text{...} \]

    \[ \text{case (2): } X \in \{ \text{muon or beta ray} \} \text{ contrived from quantum field} \]

        \[ \text{observed pattern} : \]

        \[ \text{[Diagram showing pattern]} \]

        \[ \text{flux of energy + momentum \rightarrow "jets"} \]

        \[ \begin{array}{c}
          \text{X} \\
          \rightarrow 2 \text{jets} \quad 90\%
        \end{array} \]

        \[ \begin{array}{c}
          \rightarrow 3 \text{jets} \quad 7\%
        \end{array} \]

        \[ \begin{array}{c}
          \rightarrow 4 \text{ jets} \quad 3\%
        \end{array} \]

- **Perturbative QCD and hard physics**

  \[ \text{we have seen (two) \( \text{(and will calculate later)} \) that} \]

  \[ x_0 (Q) = \]

  \[ \text{for large enough 4-moment squared QCD coupling should be small enough for perturbation theory to converge} \]

  \[ \text{(more precisely: one gets asymptotic expansions which converge only when "highertz" and quark non-perturbative contributions such as "instantons" are also accounted for)} \]

  \[ \text{perturbative QCD is the basis for interpreting most experiments.} \]

  \[ \text{so QCD is the most important hope to learn from.} \]

  \[ \text{e.g., deep inelastic scattering} \]

  \[ \text{[Diagram of deep inelastic scattering]} \]

- **High order**

  \[ \text{[Diagram of high order]} \]

  \[ \text{[Diagram of high order]} \]
QED and search for "New Physics"

Specific example: anomalous magnetic moment of muon $\mu$

- determined experimentally and theoretically (within the SM) with such high precision that it became a very sensitive test for many ideas for "physics beyond the SM"

- $\alpha_y$ (exp) = 116.60208 (26) x 10^{-10}
- $\alpha_y$ (theory) = 116.59186 (48) x 10^{-10}

$\delta$ deviates at 2.3 $\sigma$, not significant yet

- dominated by uncertainty of QED contributions

- strategy for "New Physics" search:

$$\alpha_y \sim \frac{4\pi}{\text{new physics}} + \text{other terms}$$

- typically set rather stringent limits on e.g. the minimal allowed mass of hypothetical new particles;
- obviously, any deviation between $\alpha_y$ (exp) and $\alpha_y$ (theory) could be a signal for new physics.

- does the lack of precision in our QED calculations keep us from clearly "seeing" signals of exciting new physics?

1.3. Color change in QCD

- in addition to its electric charge ($u, c, t \sim \frac{2}{3} + \frac{1}{3}$, $s, d, b \sim \frac{1}{3}$, ...$\mu$, ...)
- each quark carries a color charge.

- 3 possible values, e.g. red, green, blue.

- (experimentally determined, more later; often in quarks 3x3)

- if a quark emits a gluon, its color may or may not change

$$\frac{\text{color}}{\text{ gluon}} \rightarrow \text{9 ways of coupling a gluon between initial and final quark}$$

- $u \rightarrow u^+$, $d \rightarrow d^+$, $d \rightarrow s^-$, $u \rightarrow d^-$, $d \rightarrow u^+$

- $G_G = 0.8$, $G_S = 0.12$ (pert. calc.), $G_G = 1.8$ (non-pert. calc.)

- experimental evidence: from scattering effects in beam that matters (muons, baryons, protons) is composed of quarks, yet those hadrons must be neutral to the strong force.

- stable particles (hadrons) are "colorless";
- more precisely: they are a "color singlet state".

- color singlet quark state go is not needed/observed.

- strength of coupling between 2 quarks as color factors:

QED: $\frac{1}{\epsilon} \sim e, \epsilon_\text{em} = \text{very small}$

QCD: $\frac{1}{\alpha_s} \sim \frac{0.8}{0.3} \sim \frac{1}{3}$

- where $\frac{1}{\alpha_s}$ are normalized to $\frac{1}{3}$ from quark exchange:

$$6 \rightarrow \text{blue}$$

$$6 \rightarrow \bar{\text{blue}}$$

$$\alpha_s \sim \frac{1}{3} (\frac{2}{3}) \rightarrow \frac{1}{3} (\frac{1}{3}) \times \frac{3}{2}$$

- same result due to color symmetry

$$\text{vs. } \frac{1}{6} \text{ red} + \frac{1}{6} \text{ red} \sim \frac{3}{2}$$

- simple gluon exchange between $q$ and $\bar{q}$ in color singlet state:

$$(\prod)_{G_G} = \frac{1}{2} (G = G_S + G_G)$$

- consider eq. $6^\pm$, mult. x3

$$3 \left\{ \frac{1}{2} G_{G_S} + \frac{1}{2} G_{G_G} - \frac{1}{2} G_{G_G} \right\} \sim \frac{3}{2} \left( \frac{1}{2} - \frac{1}{2} \right)$$

- color force can be both repulsive and attractive.
1.4 Elements of group theory

- the color changes induced above can be treated much more rigorously.
- before (re)learning the rotation symmetry of charge from QED (cf. § 2.1), let us review some basic facts about the theory of continuous symmetry groups
- (much) more detail eg. in [H. Georgi: Lie Algebras in Particle Physics] or at [http://www.physik.uni-bielefeld.de/abteilungen/symmetrie/cover.html]

... can be rewritten in a different basis (just different linear combinations) of $3 \times 3$ Matrices (listed eg. rows by rows, columns by columns $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$).

- actually, $T^a = \frac{1}{2} \lambda^a$, where $\lambda^a$ are "Gell-Mann matrices", $a \in \{1, \ldots, 8\}$
- they form a possible representation of the "special unitary group" $SU(3)$, the fundamental representation

- some important properties (check?!): $[T^a, T^b] = if^{abc} T^c$

- $T^a T^b = \frac{1}{3} \delta_{ab} I + if^{abc} T^c$

- after we will need times

- could calculate $f^{abc}$ by multiplying Lie algebra with $T^a$, the taking traces:
  \[ f^{abc} = \frac{1}{2} \left( T^a (T^b T^c) - T^a (T^c T^b) \right) \]
  result (check?!): $f^{123} = 1$, $f^{145} = f^{246} = f^{347} = f^{158} = f^{268} = f^{378} = 0$
  rest by antisymmetry

- from a more general viewpoint, we have just seen one example of a broader mathematical concept: representations of Lie Groups

- a group contains abstract entities that obey certain algebraic rules
- interested in groups of unitary operators acting on vector space of states below
- interested in continuously generated groups
- certain elements arbitrarily close to identity
- can reach general group behavior by repeated action of infinitesimal one

- the set $T^a$ spans space of infinitesimal group transformations

- commutator is a linear combination of generators
  \[ [T^a, T^b] = i f^{abc} T^c \]
  vector space spanned by generators + commutator = Lie Algebra

- for us, symmetry $\equiv$ unitary transformation of a set of fields
- interested in Lie Groups with finite # of generators: "Compact"
- classification of Lie Algebras
  (group of phase rotations)
- of the $T^a$ commute with all others; Abelian subgroup, $\psi(D_0 + \psi)$
- if set of $T^a's$ cannot be divided into two mutually commuting sets: "Simple"
- general Lie Algebra = direct sum of non-Abelian simple components + additional Abelian generators

- $SU(n)$ $\{ U_{\alpha}, (\alpha = 1, \ldots, n^2 - 1) \}$, $SO(n)$ $\{ R_{\alpha\beta}, (\alpha = 1, \ldots, n^2 - 1) \}$, $Sp(n)$ $\{ i \tilde{Z}_{\alpha\beta}, (\alpha = 1, \ldots, n^2 - 1) \}$

- is equivalent set of compact simple Lie groups!
1.5 Notation and conventions

- **Natural units**
  \[ \hbar = c = k_B = 1 \]
  \[ m_e = m_u = 1 \text{ GeV}^{-1} \]

- **Vectors and Tensors**
  
  - **Indices**
    \[ \mu = 0, 1, 2, 3 \]  
    \[ \xi, \eta, \kappa, \lambda \]
  
  - **Tensors**
    \[ \gamma^\mu = \gamma^\nu = \text{diag} (1, -1, -1, -1) \]
    \[ x^\mu = (x^0, \vec{x}) \]
    \[ \gamma = \gamma^\nu = (\gamma_0, \vec{\gamma}) \]
  
- **Fully Symmetric Tensors**
  \[ \varepsilon^{0123} = 1 \]
  \[ \Rightarrow \varepsilon^{0132} = -1, \varepsilon^{123} = 1 \text{ etc.} \]

- **Matrices**
  
  - **Pauli Matrices**
    \[ \sigma^1 = i \gamma^1 \sigma^2 = i \gamma^2 \sigma \]
    \[ \sigma^0 = (0, 1), \sigma^1 = (0, i), \sigma^2 = (i, 0) \]
  
  - **Dirac Matrices**
    \[ \gamma^\mu = \left( \begin{array}{cc} 0 & a_{\mu 0} \\ a_{\mu 0} & 0 \end{array} \right), \gamma^\nu = \left( \begin{array}{cc} c & \sigma^\mu \\ -\sigma^\mu & -c \end{array} \right) \]
    \[ \gamma^\mu = i \gamma^\mu, \gamma^5 = \left( \begin{array}{cc} 1 & 0 \\ 0 & i \end{array} \right) \]

- **Bethe-Salpeter equation**

  - **External summation convention**
    \[ e \gamma^\nu x^\mu = \sum_{\mu} e \gamma^\nu x^{\mu} = e \gamma^\nu x_\mu + (\gamma^\nu x)_{\mu} = \gamma^\nu x_\mu - \gamma^\nu x_0 \]
    \[ e \gamma^\nu = \sum_{\mu} e \gamma^\nu = e \gamma^\nu x_0^{\mu} = e \gamma^\nu x_0 \]

2. Basics

2.1 Reminders: QED and gauge invariance

- **gauge invariance** is a fundamental principle that determines the form of the Lagrangian.

- Consider \( q(x) \) (unbroken Dirac field)
  
  - we now demand the theory to be invariant under local phase transformations:
    \[ q(x) \rightarrow e^{i \phi(x)} q(x) \]

  - Q: Which Lagrangian terms can we construct that are invariant?
    - All terms that are also invariant under global phases.
      
      - e.g. \( \overline{\psi}(x) \gamma^\mu \psi(x) \) (recall Dirac adjoint \( \overline{\psi} = \psi^\dagger \gamma^0 \))

- For terms with derivatives, we need some preparation:
  
  - Recall: derivative in e.g. \( x^\mu \)-direction defined as differential quotient
    \[ \frac{\overline{\psi}(x_0 + \Delta x^\mu) \gamma^\nu \psi(x_0)}{\Delta x^\mu} \]
    \[ \Rightarrow \text{ feel completely different phase transformation!} \]
    \[ \Rightarrow \text{ for meaningful computation, introduce a compensating (overall) phase factor transforming as} \]
    \[ (U_0, \eta \phi(x_0)) \rightarrow (U_0, \eta \phi(x_0) e^{-i \alpha}) \]
    \[ \Rightarrow \text{ def. covariant derivative} \]
    \[ \overline{\psi}(x_0) \gamma^\nu \psi(x_0) \rightarrow \overline{\psi}(x_0) \gamma^\nu \psi(x_0) \]

- For infinitesimal variation of \( x^\mu \), expand:
  
  - \( U(x^\mu, \epsilon) = 1 - i \epsilon \gamma^\mu \phi(x) \theta(x) \)
    
    - definition \( \phi \) new vector field! "connection"

  - cov. deriv.: \( \overline{D}_\mu \gamma^\nu \psi(x) = \overline{\gamma}^\mu \gamma^\nu \psi(x) + i (\gamma^\mu \phi(x) \gamma^\nu \psi(x)) \)
    
    - where \( \overline{D}_\mu \gamma^\nu \psi(x) = \overline{D}_\mu \gamma^\nu \psi(x) - i \frac{1}{2} \overline{D}_\mu \overline{D}_\nu \psi(x) \)
    
    - transforms the same way as the field \( \phi(x) \)
    
    - \( \overline{D}_\mu \gamma^\nu \psi(x) \) also covariant.
• Summary 1: Local phase rotation symmetry
  ⇒ def. of covariant derivative
  and existence of vector field $A_\mu$ (connection)
  and transformation properties of $A_\mu$

⇒ all terms that are globally \( \phi \cdot e^{i\theta} \) invariant
  are also locally invariant if we replace all \( A_\mu \to \mu A_\mu \).

• How about (locally invariant) kinetic terms for $A_\mu$?

(a) Construction using $U(x)$

\[ U \text{ is pure phase: } U(x) = e^{i\varphi(x)} \Rightarrow u(x) \in \mathbb{R} \]

\[ U(x) = \left[U(x)\right]^* \quad \Rightarrow \quad u(x) = -u(x) \]

\[ u(x) = \frac{\varphi}{\pi} \quad \text{(constant)} \quad \text{is odd under } \varphi \]

\[ \therefore \text{we can write } U(x) = e^{-i\varphi(x)}(x, x') + O(x^3) \]

new, use this for coupling phase product around a small space

\[ U(x) = e^{-i\varphi(x)}(x, x') \]

\[ = U(x, x') U(x' x') \]

\[ = e^{-i\varphi(x') - i\varphi(x') + i\varphi(x) - i\varphi(x)} \]

\[ = e^{-i\varphi(x') + i\varphi(x)} \]

\[ = 1 - i\varphi \left[ A_\mu(x') - A_\mu(x) \right] \]

\[ = 1 - i\varphi \left[ \frac{1}{2} A_\mu(x') - \frac{1}{2} A_\mu(x) \right] + O(x^3) \]

\[ \text{(now of space)} \]

\[ = \vec{F}(x) \equiv \text{electromagnetic field strength tensor} \]

\[ \text{but } U(x) \text{ is locally invariant by construction!} \]

⇒ $\mathcal{L}$ is a locally invariant function of $A_\mu$.

(b) Construction using $D_\mu$

\[ \rightarrow (\text{see above}) \quad \varphi(x) \to e^{i\phi(x)} D_\mu \quad \gamma(x) \to e^{i\phi(x)} D_\gamma \]

it also follows that $\bar{\varphi} A_\mu \gamma \to e^{i\phi(x)} \bar{D}_\mu D_\gamma$

\[ \Rightarrow \quad \mathcal{L}_3 \to e^{i\phi(x)} \bar{D}_\mu D_\gamma \]

now, note that

\[ [\bar{D}_\mu, D_\gamma] = i \left[ D_\mu \gamma + \bar{D}_\mu D_\gamma \right] \]

\[ = i \left( \frac{\nabla_\mu \gamma + \nabla_\mu \gamma}{2} - \frac{\gamma \nabla_\mu \gamma + \gamma \nabla_\mu \gamma}{2} - \frac{\gamma \nabla_\mu \gamma + \gamma \nabla_\mu \gamma}{2} \right) \]

\[ = i \left( \bar{D}_\mu \gamma - D_\mu \gamma \right) \quad \text{has no derivative acting inside } \gamma! \]

⇒ $[\bar{D}_\mu, D_\gamma] = i \vec{F}$

⇒ in (a), $\vec{F}$ is just a multiplicative factor, must be invariant.

⇒ can now write the most general locally invariant Lagrangian

\[ \mathcal{L} = \mathcal{L} (\varphi(x), D_\mu \gamma(x), \bar{D}_\mu D_\gamma(x)) \]

\[ \text{in the electron field } \varphi \text{ and its associated vector field } A_\mu \]

\[ \text{reminds: } \text{used operators of dimension } \leq 4 \text{ here} \]

⇒ in general, there are many additional gauge invariant ops, e.g.

\[ \mathcal{L} \equiv \mathcal{L} \left( \varphi, D_\mu \gamma, \bar{D}_\mu D_\gamma \right) \]

\[ \text{all these are nonrenormalizable interactions} \]

⇒ invariant for physics, in Wilson sense

⇒ the coefficients $c \equiv 0$ if we prohibit anomalous

\[ \Rightarrow \text{all fields commute} \quad \text{under (discrete) } P, T \text{ symmetries} \]

⇒ then only 2 free parameters in $\mathcal{L}$: $m, e$ (hiding $\lambda$)

• Summary 2: Local phase rotation symmetry of electron field $\varphi$

⇒ existence + transformation properties of an vector potential

⇒ most general (4th, renormalizable, $P, T$ invariant)

Lagrangian is unique: Thirring–Dine–Lagrange!
2.2 Generalization: Yang-Mills Lagrangian

Geometric construction can be generalized:
vertices under local phase rotations
→ vertices under any (continuous) symmetry group
here, we use 3d rotation group (O(3) or SU(2)) for brevity
→ in the end, single generalization to arbitrary local symmetry.

\[ \mathcal{Q}(x) = \left( \mathcal{Q}_1(x), \mathcal{Q}_2(x) \right) \]

\[ \text{(dublet of Dirac fields)} \]

\[ V(x) \]

desired invariance under local 3d rotations:
\[ \mathcal{Q}(x) \rightarrow \mathcal{Q}(x) e^{i \frac{\theta}{2} \mathbf{J}_{i j} \mathbf{a}^i} \]

(\text{where } \mathbf{a}^i \text{ is Pauli matrices} \{ \mathbf{1}, \mathbf{i} \mathbf{a}^1, \mathbf{i} \mathbf{a}^2, \mathbf{i} \mathbf{a}^3 \} ; \frac{\theta}{2} \text{ in radians})

Q: construct invariant Lagrangian?
→ need again a covariant derivative!
→ now, connection phase factor has to be a matrix,
with transformation: \[ U(x) \rightarrow U(x) V^T(x) \]

\[ U(x)^{-1} \text{ and } U \text{ is unitary} \]

→ can expand in terms of (hermitian: \( \mathbf{\sigma} \otimes \mathbf{1} \)) SU(2) - generators:
\[ U(x + i x) = \exp \left( i x \sum_i \mathbf{a}_i \mathbf{\sigma}^i \right) \]

\[ \text{canons. } \exp \text{ (anti-comm.) vector field} \]

→ covariant derivative:
\[ \nabla = \partial - i g \mathbf{A}^i \mathbf{\sigma}^i \]

\[ \text{where } \mathbf{A}^i(x) \mathbf{\sigma}^i \rightarrow V(x) \mathbf{A}^i(x) \frac{\mathbf{\sigma}^i}{2} + \frac{i}{2} \partial_x V(x) \]

(consistency)

(\text{U-deriv.})

now, infinitesimally,
\[ \mathcal{Q} \rightarrow (1 + i \mathbf{a}^i \mathbf{\sigma}^i \mathbf{\partial}_x^i) \mathcal{Q} \]

\[ \mathcal{Q} \rightarrow (\text{check!}) (1 + i \mathbf{a}^i \mathbf{\sigma}^i \mathbf{\partial}_x^i) \mathcal{Q} \]

→ again, \( \mathcal{Q}(x) \left( i \mathbf{D} \mathbf{\tau}^i \mathcal{Q} + \mathcal{Q} \mathbf{D} \mathbf{\tau}^i \right) \) is locally invariant.

→ gauge-invariant terms containing \( \mathbf{A}^i \) only?
→ here, we construct \( \chi \):
from above, we have \( [\mathbf{D}_x, \mathbf{D}_y] \mathcal{Q} = V(x)[\mathbf{D}_x, \mathbf{D}_y] \mathcal{Q}(x) \)

now, note that
\[ [\mathbf{D}_x, \mathbf{D}_y] \mathcal{Q} = \left( \mathbf{D}_x \mathcal{Q} \right) \mathbf{D}_y - \mathbf{D}_y \left( \mathbf{D}_x \mathcal{Q} \right) = \mathbf{D}^\alpha \mathbf{A}^\alpha \mathbf{\partial}_x \mathbf{\partial}_y \]

\[ \left( \mathbf{D}_x \mathcal{Q} \right) \mathbf{D}_y - \mathbf{D}_y \left( \mathbf{D}_x \mathcal{Q} \right) = \mathbf{D}^\alpha \mathbf{A}^\alpha \mathbf{\partial}_x \mathbf{\partial}_y \]

\[ \mathbf{D} \mathbf{\tau}^i \mathbf{A}^\alpha \mathbf{\partial}_x^i \mathbf{\partial}_y \]

\[ = -i g \left( \mathbf{D}_x \mathbf{A}^\alpha \mathbf{\partial}_y - \mathbf{D}_y \mathbf{A}^\alpha \mathbf{\partial}_x + \frac{i}{2} \mathbf{A}^\alpha \mathbf{\partial}_x \mathbf{\partial}_y \right) \mathbf{\tau}^i \]

\[ \text{does not vanish, as } \mathbf{A}^\alpha \mathbf{\partial}_x \mathbf{\partial}_y \text{ field strength tensor} \]

→ as before, \( [\mathbf{D}_x, \mathbf{D}_y] \) is not a derivative, but a current (charge)!
→ from above, the field strength is not invariant now,
but transforms as \( \mathbf{D}_x \mathbf{A} \mathbf{\partial}_y \rightarrow V(x) \mathbf{D}_x \mathbf{A} \mathbf{\partial}_y \mathbf{V}^T(x) \)

→ can construct the locally invariant terms from
\[ \text{tensors (even curl-free and } V^T \text{vector)} \]
\[ \mathcal{T}^\alpha \left( \mathbf{D}_x \mathbf{A} \mathbf{\partial}_y \mathbf{A} \mathbf{\partial}_x \mathbf{A} \mathbf{\partial}_y \right) = \frac{i}{2} \mathbf{D} \mathbf{\tau}^i \mathbf{A} \mathbf{\partial}_x \mathbf{\partial}_y \mathbf{A} \mathbf{\partial}_x \mathbf{\tau}^i \]

→ adding up:
\[ \mathcal{L} = \frac{1}{4} \left( i \mathbf{D} \mathbf{\tau}^i \mathcal{Q} - \frac{i}{2} \mathbf{\tau}^i \right)^2 \]

Yang-Mills Lagrangian

→ two parameters: \( m, g \)
→ variations → solutions of motion: \( \text{Dirac eqn.} + \text{eqn. for vector field} \)

→ generalize to other continuous symmetry groups:
\[ V \rightarrow \text{non unitary matrices; } \mathcal{Q}(x) \text{ is not } \mathbf{SU} \text{, } \mathcal{Q}(x) \rightarrow V(x) \mathcal{Q}(x) \]

\[ \text{expand } V(x) \rightarrow \mathbf{A} + i \mathbf{\tau}^i x^i \mathbf{\partial}_x + \mathbf{O}(x) \]

\[ \mathcal{L} \rightarrow \mathcal{T}^\alpha \mathbf{\partial}_x \mathbf{\tau}^\alpha \mathbf{\partial}_x \mathbf{A} \mathbf{\tau}^\alpha \]

→ completely arbitrary structure constants.
2.3 QCD and its symmetries

- Quantum Chromodynamics (QCD) is a Yang-Mills theory with a gauge group $SU(3)$.

- Matter fields (the $q$'s above) are quarks; they are in the fundamental representation of $SU(3)$, have spin 1/2, and there are six types (flavors) of quarks: $u$, $d$, $c$, $s$, $t$, $b$.

- The six $3^3 = 27$ vector fields (or gauge bosons) $A_T$, $a = 1, \ldots, 8$ are called gluons.

- The lagrangian for QCD:

$$\mathcal{L} = \frac{1}{4} F_{a\mu\nu}^{T} F^{a\mu\nu}$$

- Sometimes, it is useful to consider the generalizations:

$$SU(3) \rightarrow SU(N_f) \rightarrow \text{SU}(N_f) \rightarrow \text{SU}(N_f)$$

- QCD possesses not only the exact local $SU(3)$ color symmetry, but also an important approximate global symmetry:

$$\text{U}(1)_{\text{A}} \text{~(Baryon Number Conjugation)}$$

- Relations between different flavors under some of these are highly degenerate.

- QCD possesses a flavor symmetry, but it is not exact.
- Assume e.g. \( m_5 \approx m_6 \) \( \Rightarrow M = (m_5, m_6) \approx m_5 \Delta m_5 \)

with \( g = (g^1, g^2) \), where \( \Delta m_5 \) is the mass difference between the two states.

A symmetry under \( g \rightarrow e^{\frac{i}{\sqrt{2}}(\rho_5 + \rho_6)}(g^1 + \rho_5 g^2) \)

where \( \rho_5, \rho_6 \) are generators of SU(2),

- Symmetry \( \Delta m_5 \) is the same as in the standard model.

- Note: These symmetries are, in Nacul's theorem, associated with vector currents \( \partial \phi \cdot m_5 \), hence \( e^\gamma \).

- Note: The assumption \( m_5 \approx m_6 \) is a useful approximation.

- For massless flavors, the symmetry becomes even higher: \( M = (m, m) \).

- Use left- and right-handed particles.

- Decompose \( \phi = (\phi^- \phi^+ \phi^0) \).

- The symmetry becomes \( \mu = \phi^+ \phi^- \phi^0 \).

- Note: \( \Delta m_5 \) is the mass difference between the two states.

- Independent transformations \( \phi \rightarrow \mu \phi \).

- \( U \phi U^\dagger \) symmetry in \( \text{SU}(2) \).

- The symmetry \( \text{SU}(2)_L \otimes \text{SU}(2)_R \) is sometimes referred to as the \( \Delta m_5 \) effect.

- Using \( Q = (\bar{Q} \bar{Q}) \), \( X_{56} \rightarrow Q \bar{Q} \).

- Result: \( Q \rightarrow e^{\frac{i}{\sqrt{2}}(\rho_5 + \rho_6)}Q \) and \( \bar{Q} \rightarrow e^{\frac{i}{\sqrt{2}}(\rho_5 + \rho_6)} \bar{Q} \),

- \( \phi \rightarrow \text{SU}(2) \) symmetry of \( \Delta m_5 \) to show that

- The symmetry \( \Delta m_5 \) is\( \text{SU}(2)_L \otimes \text{SU}(2)_R \)

- Note: The symmetry \( \text{SU}(2)_L \otimes \text{SU}(2)_R \) is sometimes referred to as the \( \Delta m_5 \) effect.

- Using \( Q = (\bar{Q} \bar{Q}) \), \( X_{56} \rightarrow Q \bar{Q} \).

- Result: \( Q \rightarrow e^{\frac{i}{\sqrt{2}}(\rho_5 + \rho_6)}Q \) and \( \bar{Q} \rightarrow e^{\frac{i}{\sqrt{2}}(\rho_5 + \rho_6)} \bar{Q} \),

- \( \phi \rightarrow \text{SU}(2) \) symmetry of \( \Delta m_5 \) to show that

- The symmetry \( \Delta m_5 \) is\( \text{SU}(2)_L \otimes \text{SU}(2)_R \)
2.4 Quantization, path integral (reminds only)

- so far, have seen non-abelian gauge symmetry and must work on QCD
  - now, work out consequences for particle physics interactions
    - need rules for canceling Feynman diagrams
    - apply rules to compute amplitudes, cross sections

- local gauge symmetry (some Lagrangian shifts are unphysical)
  \[ \text{(this can be adjusted arbitrarily by gauge fixing)} \]
  cf. QED: a functional integral \( S[\phi] \propto e^{iS} \) the path part
  \[ \mathcal{S} = S[\phi] \propto -\frac{i}{\hbar} \int_0^\infty \text{ev.} \]

- for \( \mathcal{A}(\mathbf{x}) \) finite \( \mathcal{F} = 0 \Rightarrow \mathcal{S}[\phi] \propto e^{i\mathcal{S}} \) changes!
  - arbitrary scalar field
    \[ (\phi(x) \text{ has no inverse: cannot solve } \nabla^2 \phi(x) = 0 \text{ for } \mathcal{S}[\phi] \propto e^{i\mathcal{S}} \]
    - result: Adiabatic gauge
    - recall scalar gauge \( \mathcal{A}(\mathbf{x}) \rightarrow \mathcal{A}(\mathbf{x}) + \mathbf{a}(\mathbf{x}) \)
    - can choose \( \mathbf{a}(\mathbf{x}) \) any gauge-fixing
      \[ \text{Feynman gauge: } \mathcal{F} = 0 \]
    - can relate
      \[ \mathcal{A}(\mathbf{x}) \rightarrow \mathbf{a}(\mathbf{x}) \rightarrow \mathbf{g}(\mathbf{x}) \]
      \[ \mathcal{S}[\phi] \propto -\frac{i}{\hbar} \int_0^\infty \text{ev.} \]
      - gauge propagator

- propagator depends on arbitrary parameter \( \mathbf{k} \)?
  - physics does not: QED verifies \( S \) in such
    - due to Ward-Takahashi identities
  - similar structure in QED: \( S \)-correlations more complicated.

- we will make use of functional methods
  - most useful for interacting QFT's
    \[ \text{path integral method, relying on functional integration} \]
  - for (any) more details: [QFT lecture]
    \[ 1 \text{ lecture/Ch.} \]

- remember of functional derivatives:
  \[ \frac{\delta}{\delta \phi(x)} \delta^4(y - x) = \delta^4(x - y) \]
  \[ \text{can take functional derivatives as usual} \]
  \[ \frac{\delta}{\delta \phi(x)} e^{i \frac{i}{\hbar} \int \mathcal{L}[\phi(x)] dx} \]
  \[ \frac{\delta}{\delta \phi(x)} \frac{\delta}{\delta \phi(y)} e^{i \frac{i}{\hbar} \int \mathcal{L}[\phi(x)] dx} = -\frac{\delta^2}{\delta \phi(x) \delta \phi(y)} \quad \text{(after path integration)} \]

- remember of the generating functional of correlation functions
  \[ Z[J] = \int D\phi e^{i \frac{i}{\hbar} \int \mathcal{L}[\phi(x)] dx} \]
  \[ \text{source term} \]
  \[ \langle 0 | \hat{T} [\phi(x)] | 0 \rangle = \frac{i \delta}{\delta \phi(x)} \bigg|_{\phi=0} \quad \text{for any } \phi \]
  \[ \text{and that} \]
  \[ \langle 0 | \hat{T} [\phi(x)] | 0 \rangle = \frac{i \delta}{\delta \phi(x)} \bigg|_{\phi=0} = \left. \frac{\delta Z[J]}{\delta J(x)} \right|_{J=0} \quad \text{for any } J \]

- to see the absence of the \( Z[J] \) formulation,
  consider a free scalar field theory,
  \[ \mathcal{L} = \frac{i}{2} \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \phi^2 \]
  \[ \text{cf. eq. } 25 \text{a} (1962) \]
  \[ \mathcal{S}[\phi] = -\frac{i}{\hbar} \int_0^\infty \text{ev.} \]
  \[ \text{can relate } \]
  \[ \left. \frac{\delta Z[J]}{\delta J(x)} \right|_{J=0} = \left. \frac{\delta \mathcal{S}[\phi]}{\delta \phi(x)} \right|_{\phi=0} \quad \text{for any } \phi \]

- \( Z[J] = \int D\phi e^{i \frac{i}{\hbar} \int \mathcal{L}[\phi(x)] dx} \]
  \[ \text{two-point function} \quad \langle 0 | \hat{T} [\phi(x)] | 0 \rangle = \left. \frac{\delta}{\delta J(x)} \right|_{J=0} \]
  \[ \text{four-point function} \quad \langle 0 | \hat{T} [\phi(x) \phi(y)] | 0 \rangle = \left. \frac{\delta^2}{\delta J(x) \delta J(y)} \right|_{J=0} \]
  \[ \text{where } \left. \frac{\delta^2}{\delta J(x) \delta J(y)} \right|_{J=0} = \frac{\delta^2}{\delta \phi(x) \delta \phi(y)} = \left. \frac{\delta^2}{\delta \phi(x) \delta \phi(y)} \right|_{\phi=0} \]

- etc.
2.5 QCD Feynman rules

- recall for above reasons that for a perturbative treatment, we need to read if propagators (166.1) and vertices (166.2)
- from the topography $\lambda$.

- $\lambda = \left\{ \text{vertices} \right\}$
- $\lambda$ interactions $\leftrightarrow$ vertices
- between $n$ fields $\leftrightarrow$ propagators

- recall: $\lambda_{\text{vertex}} = \frac{-\i \lambda_{\mu}}{3!} \frac{1}{\alpha_s}$

- graph-theoretic relations

- $3 \to 1$ vertex: need to fix contractions

- in Fock space, $3 \to 1$ $\leftrightarrow$ $1 \to \i \lambda_{\mu}$

- symmetrise this with $A$: $3!$ possible permutations

- Feynman rules: $\i \lambda_{\mu}$ form $6 \to 1$ possible permutations

- (sets of 4 are equal)

- this combination is actually genuine: (connected parts) $\leftrightarrow$ (disconnected parts)

- works for all higher correlation functions as well.
for the propagator, recall that
\[ S_0 = \int d^4x \{ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{\lambda}{2} \phi^2 \phi \} \]

- **main review:** anti-commuting (Gaussmann) numbers

\[ \{ \theta, \phi \} = 0 \Rightarrow \theta^2 = 0, \quad \text{Taylor } f(\theta) = a + b \theta \text{ vanishes!} \]

- integrals:
  \[ \int d\theta = 0, \quad \int d\theta \theta = 1 \]

- complex Gaussmann \#1:
  \[ \theta = i \phi, \quad \bar{\theta} = -i \bar{\phi}, \quad (\theta)^* = \bar{\theta}, \quad (\bar{\theta})^* = \theta \]

- complex Gauss \#2:
  \[ \int d\theta \theta e^{-i\theta\phi} = \int d\theta \bar{\theta} e^{i\bar{\theta}\phi} = \int d\theta \theta (1 + i\theta\phi) = \int d\theta \theta (1 - \theta^2) = 0 \]

- another one:
  \[ \int d\theta \theta e^{i\theta\phi} = \int d\bar{\theta} e^{\bar{\theta}\phi} = \int d\bar{\theta} \bar{\theta} \bar{\theta} = 1 \]

- higher order Gauss \#7:
  \[ \int d\theta \theta^2 e^{-i\theta\bar{\theta}} = (\int d\theta \theta) (\int d\theta \bar{\theta}) = \int d\theta \bar{\theta} \theta = 0, \quad \text{hermitian, diagonalize by unitary transform!} \]

- **general propagator:** consider see more below. \( \mathcal{K} \equiv \Theta \equiv f(x, y) \phi \sum_{\text{complex Gaussmann \#1}} \mathcal{K}_{\text{complex Gaussmann \#1}} \)

\[ \sum_{\text{complex Gaussmann \#1}} \mathcal{K}_{\text{complex Gaussmann \#1}} \]

- \( S_0 \equiv \frac{i}{2} \mathcal{K} \phi \partial_\mu \phi \partial^\mu + \frac{m}{2} \mathcal{K} \phi^2 \phi \)

\[ \mathcal{K} = \frac{i}{2} \mathcal{K} \phi \partial_\mu \phi \partial^\mu + \frac{m}{2} \mathcal{K} \phi^2 \phi \]

- **main review:** defining the fundamental action of a gauge theory

\[ \mathcal{S}_0[f(y)] \]

- gauge number:
  \[ G(\phi) = G(y), \quad \mathcal{S}_0 + \mathcal{S}_0 \mathcal{G}(f(y), \phi) \]

\[ \mathcal{G}(f(y), \phi) = \mathcal{G}(f(y), \phi) + \mathcal{G}(f(y), \phi) \]

\[ \mathcal{G}(f(y), \phi) \text{ is an arbitrary field} \]

- covariant gauge

\[ \delta \phi \equiv \delta (\phi, y) \]

\[ \text{note: } \delta (\phi, y) \text{ may be } \mathcal{S}_0 \text{ in its definition} \]

\[ \mathcal{S}_0 \] \[ \text{and gauge number of } \mathcal{S}_0 \text{, } \mathcal{G}(f(y), \phi) \]

\[ \text{and } \mathcal{S}_0 \text{, } \mathcal{G}(f(y), \phi) \]

\[ \text{is the variable that} \]

\[ \text{the "gauge" or "fundamental" field is defined in expectation value!} \]

- new average over \( \mathcal{G}(f(y), \phi) \)

\[ \mathcal{G}(f(y), \phi) = \mathcal{G}(f(y), \phi) \]

\[ \mathcal{G}(f(y), \phi) = \mathcal{G}(f(y), \phi) \]

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- now compute the "Faddeev-Popov determinant" \( \mathcal{A} \) from its definition:

\[ \mathcal{A}(f(y), \phi) \equiv \frac{d\mathcal{A}(f(y), \phi)}{d\mathcal{G}(f(y), \phi)} \]

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- **main review:** defining the fundamental action of a gauge theory

\[ \mathcal{S}_0 \equiv \frac{i}{2} \mathcal{K} \phi \partial_\mu \phi \partial^\mu + \frac{m}{2} \mathcal{K} \phi^2 \phi \]

- **main review:** defining the fundamental action of a gauge theory

\[ \mathcal{S}_0 \equiv \frac{i}{2} \mathcal{K} \phi \partial_\mu \phi \partial^\mu + \frac{m}{2} \mathcal{K} \phi^2 \phi \]

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- **main review:** defining the fundamental action of a gauge theory

\[ \mathcal{S}_0 \equiv \frac{i}{2} \mathcal{K} \phi \partial_\mu \phi \partial^\mu + \frac{m}{2} \mathcal{K} \phi^2 \phi \]
• ghost propagator

collection form above, \( G(\frac{1}{2}) G(\frac{1}{2}) = e^{\frac{i}{\hbar} \gamma_\mu A^\mu} e^{\frac{i}{\hbar} \gamma_\nu A^\nu} \)

\[
\begin{align*}
\Sigma_0 &= \delta^{\mu\nu} - \frac{1}{2} \sigma^\mu_\nu A^\mu \gamma_\nu + \frac{i}{2} \bar{\psi} \gamma^\mu \gamma_\nu \psi \\
&= \left( \frac{1}{2} \gamma^\mu \gamma_\nu - \frac{1}{2} \sigma^\mu_\nu A^\mu \gamma_\nu \right) A^\mu A^\nu \\
\end{align*}
\]

\[
\hat{\Sigma}_0 \approx \frac{i}{\hbar} \left( \gamma^\mu \gamma_\nu - \left( 1 - \frac{1}{2} \gamma^\mu \gamma_\nu \right) A^\mu A^\nu \right)
\]

(see also \( \Sigma_0 = \hat{\Sigma}_0 + \Sigma_1 \))


• Field-theor refer ghost fields

have to take care of det \( T(\theta) \) factor (on bottom of pg 28)

using Grassmann numbers again (see Grass integral on pg 25), rewrite

\[
\begin{align*}
\det T(\theta) &= \det \left( \frac{1}{2} \theta [ \phi^+ \phi + \phi \phi^+] \right) \\
&= \int D\phi \Delta \phi e^{i \phi \phi^+ \Delta \phi} \left( \sum_{\phi} \phi \phi^+ \right)
\end{align*}
\]

where the "\( \phi \) fields" are anticommuting fields
(but there are no \( \gamma \) matrices \( \Rightarrow \text{spin 0} \) !)

→ ghost propagator

\[
\begin{array}{c|c}
\Sigma_0 & \frac{i}{\hbar} \gamma_\mu A^\mu \\
\hline
\Sigma_1 & -\frac{i}{\hbar} \gamma_\mu A^\mu + \frac{i}{2} \bar{\psi} \gamma^\mu \gamma_\nu \psi \\
\end{array}
\]

(for physical interpretation of ghosts, see e.g. Peskin/Schroeder §16.3)

→ now, bow all propagators and vertices of QCD,

so we can again (as in § 2.4, see § 2.5) do perturbative expansions via generating functional.

• short summary:

from pg. 18, 26, 27 we have now

\[
\begin{align*}
\Sigma_{q_0} &= \sum_{i=1}^{N} \left( \gamma \cdot p - m \right) q_i - \frac{i}{\hbar} \frac{e}{2} \left( \gamma \cdot p - m \right) q_i \Delta \lambda + \frac{i}{2} \left( \gamma \cdot p - m \right) q_i \Delta \lambda^2 \\
\end{align*}
\]

where \( \Delta \lambda = \left( \gamma \cdot p - m \right) q_i \) is the "current derivative on the adjoint representation".

→ this expression is still invariant, but not under a local

→ gauge transformation as in § 2.2; the relevant transformation

→ now the ghost fields \( \xi \) is an essential way and is
called "\( \mathcal{BRST} \) transformation"


(a symmetry with constants but unpropagating parameters)

(inc: QFT lecture; e.g. Peskin/Schroeder § 5)

→ the found treatment of pg. 26 had implicitly assumed

that the gauge condition \( \left( \psi \right) \left( \frac{\partial}{\partial \theta} \right) \) selects

(via the Sezdi et al.) one unique representation \( \phi \) (\( \Delta \theta \))

for each "gauge orbit" \( \phi \).

Hence, Gelfand, C. I. [Gelf, Math. Phys. 3 (1937) 1]

has demonstrated that for non-Abelian theories,

this would always be guaranteed.

In practice, this fact has little relevance.

• set of "Feynman rules" as usual

→ see QFT lecture; QFT physics lecture; ...

draw diagrams - for symmetry reasons - just Feynman rules for

→ propagator vertices - perform traces and locate algebra -

regularize divergent integrals - locate relations - evaluate Feynman integrals - ...
3. Fundamentals

- Our \( L_\alpha \) contains operators of dimension \( \leq 4 \)
  - Theory is renormalizable: all divergences can be canceled by a finite number of counterterms
  - 'Nuclear rogue' some divergences in QCD
  - Important physical consequence: asymptotic freedom!

- Mini-review: renormalization (See e.g. Peskin\textit{}/Schroeder \S10)
  - Loops \( \to S_L \Rightarrow \) ultraviolet (UV) divergences: treat large \( \ell \) (or small \( k \))
    - Common and natural in QCD
  - Counting of UV diverges (no IR diverges): supercritical degree of divergence
  - Idea: 'hide' diverges in \( \lambda' \) by rescaling fields + coupling
  - \( \lambda' \Rightarrow 2 \lambda' \to \ldots \Rightarrow 2 \lambda' + \Theta(\lambda') \)

- Need intermediate regularization of divergent loop integrals among possibilities (dimensional regularization, cutoff, 
  - Highly problematic: introduce \( \epsilon \forder \)
  - \( \lambda' \rightarrow 0 \) analytic cont.

- Ultraviolet regularization
  - Dimensional regularization: \( \int \frac{d^n k}{(2\pi)^n} \Rightarrow \int \frac{d^\epsilon k}{(2\pi)^\epsilon} \)
  - Analytic cont. \( \lambda' \rightarrow 0 \), \( \lambda' \rightarrow \lambda \)

- Introduce ultraviolet regularization \( \lambda' \rightarrow \lambda' + \epsilon \appa \rightarrow \lambda' \rightarrow 0 \)

- The renormalization group (RG) equations:
  - Describe \( \lambda' \)-dependence of parameters / Green's functions

- Mini-review: dimensional regularization
  - \( \int \frac{d^n k}{(2\pi)^n} = \int \frac{d^\epsilon k}{(2\pi)^\epsilon} \)
  - \( d^n k = (2\pi)^n \delta^n(k) \Rightarrow \int \frac{d^n k}{(2\pi)^n} = \frac{\Gamma(n/2)}{\Gamma(n/2) \Gamma(\epsilon/2)} \)
  - \( \delta^n k = \delta^\epsilon k \Rightarrow \int \frac{d^n k}{(2\pi)^n} = \frac{\Gamma(n/2)}{\Gamma(n/2) \Gamma(\epsilon/2)} \)
  - \( \delta^n k_0 = \delta^\epsilon k_0 \Rightarrow \int \frac{d^n k_0}{(2\pi)^n} = \frac{\Gamma(n/2)}{\Gamma(n/2) \Gamma(\epsilon/2)} \)
  - \( \delta^n k_0 = \delta^\epsilon k_0 \Rightarrow \int \frac{d^n k_0}{(2\pi)^n} = \frac{\Gamma(n/2)}{\Gamma(n/2) \Gamma(\epsilon/2)} \)
  - \( \delta^n k_0 = \delta^\epsilon k_0 \Rightarrow \int \frac{d^n k_0}{(2\pi)^n} = \frac{\Gamma(n/2)}{\Gamma(n/2) \Gamma(\epsilon/2)} \)

- Important integrals
  - \( \int \delta^n k \sim \frac{1}{n!} \delta^n k \Rightarrow \int \frac{d^n k}{(2\pi)^n} = \frac{\Gamma(n/2)}{\Gamma(n/2) \Gamma(\epsilon/2)} \)
  - \( \int \frac{d^n k}{(2\pi)^n} = \frac{\Gamma(n/2)}{\Gamma(n/2) \Gamma(\epsilon/2)} \)
  - \( \int \frac{d^n k}{(2\pi)^n} = \frac{\Gamma(n/2)}{\Gamma(n/2) \Gamma(\epsilon/2)} \)
  - \( \int \frac{d^n k}{(2\pi)^n} = \frac{\Gamma(n/2)}{\Gamma(n/2) \Gamma(\epsilon/2)} \)

3.1 One-loop divergences in QCD

- Goal: evaluate 1-loop gluon self-energy diagrams

- Use dimensional regularization, \( d = 4 - \epsilon \)

- Feynman gauge \( \xi = 1 \) for gluon propagator (for simplicity)

- New diagrams:
  - Consider new gluon diagram, \( \Rightarrow \xi = 0 \) in the end

- \begin{align*}
  \frac{\partial}{\partial (2\pi)^{d-2\epsilon}} \int \frac{d^n k}{(2\pi)^n} & \left[ \left( \frac{\partial f}{\partial k_a} \right)^2 \right] \frac{d^\epsilon k}{(2\pi)^\epsilon} \\
  & \left[ \left( \frac{\partial f}{\partial k_a} \right)^2 \right] \frac{d^\epsilon k}{(2\pi)^\epsilon} \\
  & \left[ \left( \frac{\partial f}{\partial k_a} \right)^2 \right] \frac{d^\epsilon k}{(2\pi)^\epsilon} \\
  & \left[ \left( \frac{\partial f}{\partial k_a} \right)^2 \right] \frac{d^\epsilon k}{(2\pi)^\epsilon} \\
  & \left[ \left( \frac{\partial f}{\partial k_a} \right)^2 \right] \frac{d^\epsilon k}{(2\pi)^\epsilon} \end{align*}

- \begin{align*}
  \frac{\partial}{\partial (2\pi)^{d-2\epsilon}} \int \frac{d^n k}{(2\pi)^n} & \left[ \left( \frac{\partial f}{\partial k_a} \right)^2 \right] \frac{d^\epsilon k}{(2\pi)^\epsilon} \\
  & \left[ \left( \frac{\partial f}{\partial k_a} \right)^2 \right] \frac{d^\epsilon k}{(2\pi)^\epsilon} \end{align*}
\[ \begin{align*}
\text{sum diagrams } & 2364 \\
& \left( \frac{\alpha_s^2 f_{11b}^2}{2 f_{11b}^2} \right) \int_{-\infty}^{0} \left[ 2 \left( x - y \right)^2 - \frac{x^2}{x^2 + y^2} \right] \frac{dx}{dy} \\
& - \frac{\alpha_s^2 f_{11b}^2}{2 f_{11b}^2} \int_{-\infty}^{0} \left[ 2 \left( x - y \right)^2 - \frac{x^2}{x^2 + y^2} \right] dy \\
& \text{with } \alpha_s = \text{LO renormalization} \\
& \left( \frac{\alpha_s^2 f_{11b}^2}{2 f_{11b}^2} \right) \int_{-\infty}^{0} \left[ 2 \left( x - y \right)^2 - \frac{x^2}{x^2 + y^2} \right] \frac{dx}{dy} \\
& - \frac{\alpha_s^2 f_{11b}^2}{2 f_{11b}^2} \int_{-\infty}^{0} \left[ 2 \left( x - y \right)^2 - \frac{x^2}{x^2 + y^2} \right] dy \\
& \text{note that first line is summed under } x \rightarrow 1-x \\
& \left\{ \begin{array}{l}
\int_{-\infty}^{0} \left[ 2 \left( x - y \right)^2 - \frac{x^2}{x^2 + y^2} \right] \frac{dx}{dy} = 0 \\
\int_{-\infty}^{0} \left[ 2 \left( x - y \right)^2 - \frac{x^2}{x^2 + y^2} \right] dy = 0
\end{array} \right.
\end{align*} \]

3.2 more 1-loop diagrams in QCD

\[ \begin{align*}
\text{goal: evaluate 1-loop diagrams (again, in described Feynman gauge)} \\
\int_{-\infty}^{\infty} & \left( \frac{\alpha_s^2 f_{11b}^2}{2 f_{11b}^2} \right) \int_{-\infty}^{0} \left[ 2 \left( x - y \right)^2 - \frac{x^2}{x^2 + y^2} \right] \frac{dx}{dy} \\
& - \frac{\alpha_s^2 f_{11b}^2}{2 f_{11b}^2} \int_{-\infty}^{0} \left[ 2 \left( x - y \right)^2 - \frac{x^2}{x^2 + y^2} \right] dy
\end{align*} \]
\[ \int \frac{d\theta}{(2\pi)^n} \frac{1}{(\theta^2 + k^2)^m} \frac{1}{(\theta^2 + \mu^2)^n} \frac{1}{(\theta^2 + \nu^2)^p} = \frac{\Gamma(m+n+p-1)}{\Gamma(m+n+p+1)\Gamma(2m+2n+2p+1)} \]

The integral is evaluated over the variables \( \theta \), with\( k, \mu, \nu \) being constants. The Gamma function \( \Gamma \) is used to simplify the expression.

\[ \Gamma(m+n+p-1) \]

\[ \Gamma(m+n+p+1) \]

\[ \Gamma(2m+2n+2p+1) \]

For the general case, the integral becomes:

\[ \int \frac{d\theta}{(2\pi)^n} \frac{1}{(\theta^2 + k^2)^m} \frac{1}{(\theta^2 + \mu^2)^n} \frac{1}{(\theta^2 + \nu^2)^p} = \frac{\Gamma(m+n+p-1)}{\Gamma(m+n+p+1)\Gamma(2m+2n+2p+1)} \]

This integral is typical in quantum field theory and statistical mechanics, involving the distribution of momenta in phase space.
3.3 one-loop counterterms in QCD

Now, renormalize the theory.

use freedom of redefining fields, parameters/couplings
schematically:
\[ \phi = \sqrt{Z_\phi} \phi, \quad \lambda = \sqrt{Z_\lambda} \lambda \]

\( Z = \{ 1, 2, \ldots \} \) for "bare"
\( R = \{ m, p, \ldots \} \) for renormalized

where the multiplicative renormalization factors \( Z_i \)
depend on the renormalized parameters (not the defined),
and are taken to be dimensionless, \( Z_i = 1 + \delta Z_i \approx \delta \gamma_i \) (see last)

\[ Z_\lambda = \gamma \]

\[ \gamma = \left( \frac{g^2}{g^2 - \mu^2} \right) \gamma_0 - \frac{\delta \gamma_0}{2 \pi} \gamma_0 - \frac{\delta (\gamma_2)}{2 \pi} + \frac{\epsilon}{2 \pi} (\gamma_2'^2) \epsilon_2' \]

\( \lambda = \gamma \lambda_0 \)

\( \lambda_0 = \gamma \lambda_0 \)

(see last line, all \( \gamma, \epsilon, \epsilon_2', \epsilon_2'' \) should have an index \( R \))

\[ \frac{2}{3} \gamma_0 \rightarrow \gamma_0 \quad \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \]

\[ \frac{2}{3} \gamma_0 \rightarrow \gamma_0 \quad \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \]

\[ \gamma = \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \]

\[ \gamma = \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \]

\[ \gamma = \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \]

\( \text{(note, with } \gamma \text{, we have } R \text{, with } \lambda \text{, we have } R \text{ for all } \gamma, \lambda, \gamma_2, \gamma_2' \text{)} \)

\[ Z_\lambda + \text{counterterms} \]

\[ \frac{1}{2 \pi} \lambda_2 \rightarrow \lambda_2 \rightarrow \lambda_2 \]

\[ \frac{1}{2 \pi} \lambda_2 \rightarrow \lambda_2 \rightarrow \lambda_2 \]

\[ \lambda = \gamma \lambda_0 \]

\( \lambda_0 = \gamma \lambda_0 \)

\( \gamma = \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \)

\( \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \)

\( \gamma = \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \)

\( \text{(note, all indices are } R \text{, avoid index) } \)

\[ \text{etc. as additional interactions.} \]

\[ \text{get additional Feynman rules} \]

\( \text{values are easy } (\text{have the same form as before}) \]

\[ \lambda = \gamma \lambda_0 \]

\( \lambda_0 = \gamma \lambda_0 \)

\( \gamma = \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \)

\( \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \)

\( \gamma = \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \)

\( \text{(note, all indices are } R \text{, avoid index) } \)

\[ \text{etc. as additional interactions.} \]

\[ \text{get additional Feynman rules} \]

\( \text{values are easy } (\text{have the same form as before}) \]

\[ \lambda = \gamma \lambda_0 \]

\( \lambda_0 = \gamma \lambda_0 \)

\( \gamma = \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \)

\( \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \)

\( \gamma = \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \)

\( \text{(note, all indices are } R \text{, avoid index) } \)

\[ \text{whence put } \alpha_1 = a \text{ a number of choice,} \]

\[ \text{often used: "minimal subtraction (MS) scheme" put } 0 \text{ here} \]

\[ \text{many other schemes possible,} \]

\[ \text{e.g. modified \( \overline{\text{MS}} \) (O \( \overline{\text{MS}} \)), see below} \]

\[ \lambda = \gamma \lambda_0 \]

\( \lambda_0 = \gamma \lambda_0 \)

\( \gamma = \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \)

\( \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \)

\( \gamma = \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \)

\( \text{(note, all indices are } R \text{, avoid index) } \)

\[ \text{\( \lambda = \gamma \lambda_0 \) as additional interactions.} \]

\[ \text{get additional Feynman rules} \]

\( \text{values are easy } (\text{have the same form as before}) \]

\[ \lambda = \gamma \lambda_0 \]

\( \lambda_0 = \gamma \lambda_0 \)

\( \gamma = \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \)

\( \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \)

\( \gamma = \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \)

\( \text{(note, all indices are } R \text{, avoid index) } \)

\[ \text{etc. as additional interactions.} \]

\[ \text{get additional Feynman rules} \]

\( \text{values are easy } (\text{have the same form as before}) \]

\[ \lambda = \gamma \lambda_0 \]

\( \lambda_0 = \gamma \lambda_0 \)

\( \gamma = \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \)

\( \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \)

\( \gamma = \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \rightarrow \frac{2}{3} \gamma_0 \)

\( \text{(note, all indices are } R \text{, avoid index) } \)
\[
\text{finite} \quad \sum \left[ (\frac{2}{N_c^2}) - \frac{1}{2} \left( \frac{3z}{N_c} - \frac{1}{4} \right) \right] + O(N_c^{-4})
\]
We had defined $\Delta^2 = \frac{\partial^2}{\partial \eta^2}$, $\Delta^4 = \frac{\partial^4}{\partial \eta^4}$, etc.

For $\eta = 0$:

$$\Delta^2 \eta = \frac{\partial^2}{\partial \eta^2} \eta = \frac{1}{\Delta^2}$$

$\Delta^4 \eta = \frac{\partial^4}{\partial \eta^4} \eta = \frac{1}{\Delta^4}$

In four dimensions, arbitrary mass scale, $[\mu] = 1$

$$\frac{\partial}{\partial \mu} \frac{\partial}{\partial \mu} = \frac{\partial}{\partial \mu} - \frac{\partial}{\partial \mu} = 0$$

In fact, all our expressions above $q^2 \to q^2$ was understood as

$$2 - 1 + \frac{\partial^2}{\partial \eta^2} \eta = 1 + \frac{\partial^2}{\partial \eta^2} \eta + \frac{\partial^4}{\partial \eta^4} \eta, \quad 2 \text{ is } \mu \text{-independent}$$

**QCD Beta Function**

Immediate consequence: $\gamma_{\beta}^2 = \text{a function of } \eta$

$$\gamma_{\beta}^2 = \frac{\partial^2}{\partial \eta^2} \eta^2 = \text{a function of } \eta$$

$$\beta(\gamma) = \gamma_{\beta}^2 = \frac{\gamma^2}{1 + \gamma^2(2g^2 + \frac{\gamma^2}{2})}$$

In 4 dimensions, $[g^2]$:

$$2g = 1 + \frac{\partial^2}{\partial x^2} + O(g^2)$$

where $a = \frac{1}{16\pi} \left( \frac{1}{2} m - \frac{1}{2} g \right)$

$$\Rightarrow a = \frac{\gamma^2}{16\pi} / \frac{1}{2} m^2 + O(g^2)

\Rightarrow \gamma^2 = \text{a function of } a$$

$$\beta(\gamma) = \gamma_{\beta}^2 = \frac{\gamma^2}{1 + \gamma^2(2g^2 + \frac{\gamma^2}{2})}$$

More convenient to use $h = 2g - \frac{\gamma^2}{2}$:

$$\beta(h) = \frac{\gamma_{\beta}^2}{1 + \gamma^2(2g^2 + \frac{\gamma^2}{2})} = -\gamma_{\beta} h^2 - \gamma_{\beta}^2 h^4 - \gamma_{\beta}^4 h^6 + O(h^8)$$

**Running Coupling**

Solve the differential equation $\gamma''(\mu) = -\lambda \gamma^2$

$$\gamma(\mu) = e^{-\lambda \int_{\mu_0}^\mu d\mu' \lambda(\mu')}$$

$$\gamma_{\mu_0} = \frac{1}{\lambda} \ln \frac{\mu}{\mu_0}$$

for $\mu > \mu_0$

**One often needs $\mu$-expansions of Gamma functions.**

**Recall**: $\gamma(n) = \sum_{k=0}^{\infty} \frac{n}{(n+k)_n}$

**Useful**: $\gamma(n) = \sum_{k=0}^{\infty} \frac{n}{(n+k)_n}$

**Euler-Mascheroni**: $\gamma = -\ln(\mu) + \mu - \ln(\mu)$

$\ln(\mu) = 0.5772...$
4. QCD in $e^+e^-$ annihilation

- need to compare basic properties of (perturbative) QCD with experiment.

- consider $e^+e^- \rightarrow$ hadrons
  - total cross section $\sigma \sim \frac{\alpha_s^n}{n!}$
    - calculate $\alpha_s$ corrections, $\sigma \rightarrow \sigma(1+\alpha_s)$
    - renormalization scheme dependence arises at $\alpha_s^3$
  - inclusive cross sections $(1+\alpha_s+\alpha_s^2+\alpha_s^3)$ known, high precision QCD result!
  - non-perturbative corrections expected to be small
- used as one of the most precise measurements of $\alpha_s$.
  - QCD predicts "jet" structure for final-state hadrons
    - define "jet" structure
    - calculate this, compare with experiment
    - can also be used to measure $\alpha_s$, and $\xi$ in high gluon region.

4.1 $e^+e^- \rightarrow$ hadrons at leading order

- represent $e^+e^-$ annihilation in QCD.
- detector measures on
  - $d\sigma \sim \propto d\Omega d^3p$
  - spherical coords.: $S(d\sigma) = \int d\Omega d^3p$
  - total cross section $\sigma = \int S(d\sigma)$ < differential cross section

5. Fermi's golden rule

\[
\frac{d\sigma}{d\Omega} \sim \frac{1}{\pi} \int \left| \mathcal{M} \right|^2 d\Omega \quad \text{amplitude, eq. from Fermi's golden rule}\\
\]

\[
\phi \sim \left( \frac{d\sigma}{d\Omega} \right) (\text{state}) \sim \left( \frac{d\sigma}{d\Omega} \right) (\text{state})
\]

\[
\text{where} \quad \phi \sim \begin{cases} \frac{1}{\pi} \left( \frac{d\sigma}{d\Omega} \right) (\text{state}) & \text{state} \\ \frac{1}{\pi} \left( \frac{d\sigma}{d\Omega} \right) (\text{pair}) & \text{pair} \end{cases}
\]

- $e^+e^- \rightarrow \mu^+\mu^-$ [see, e.g., "hep-th/9902135"]

\[
e^+e^- \rightarrow \mu^+\mu^- = \mathcal{F}(\gamma_{\mu\nu})(\gamma_{\mu\nu}) \mathcal{F}(\gamma_{\mu\nu})(\gamma_{\mu\nu})\mu^+\mu^-
\]

\[
\text{spin sum and energy product}
\]

\[
\text{spin sum over spin states } e_1^+, e_2^- \equiv \frac{1}{e_1^+} \frac{1}{e_2^-} \text{ detector does not measure spin of final state}
\]

\[
\text{sum over spins } e_1^+, e_2^-
\]

\[
\frac{1}{e_1^+} \frac{1}{e_2^-} \left| \mathcal{M} \right|^2 \left| \mathcal{M} \right|^2 = \frac{1}{e_1^+} \frac{1}{e_2^-} \left| \mathcal{M} \right|^2 \left| \mathcal{M} \right|^2
\]

\[
\frac{1}{e_1^+} \frac{1}{e_2^-} \left| \mathcal{M} \right|^2 \left| \mathcal{M} \right|^2
\]

\[
\text{CNS: } \tilde{p}_1 \rightarrow \tilde{p}_1, \tilde{p}_2 \rightarrow \tilde{p}_2, \text{ E-const.} \quad \text{E} = E_1 + E_2,
\]

\[
\text{E} = E_1 + E_2 \equiv E_1 \equiv E_2 \equiv E_1 + E_2 - \text{m} \quad \text{E-const.}
\]

\[
\frac{1}{e_1^+} \frac{1}{e_2^-} \left( E_1 + m_1 E_1 + m_1 E_1 + (E_1+m_1)(E_2+m_2) \text{ cos} \theta \right) \left( \tilde{p}_1 \tilde{p}_2 \right)
\]
6 41. Plots of cross sections and related quantities

ψ and ω in e+e− Collisions

\[ \sigma_{ee} = \frac{1}{(2\pi)^{2}} \frac{1}{|s^{\frac{1}{2}}|} \left( \frac{m_{e}^{2} - m_{\pi}^{2}}{s^{\frac{1}{2}}} \right) \cdot \theta(s^{\frac{1}{2}}) \]

\[ \sigma_{\mu\mu} = \frac{1}{(2\pi)^{2}} \frac{1}{|s^{\frac{1}{2}}|} \left( \frac{m_{\mu}^{2} - m_{\pi}^{2}}{s^{\frac{1}{2}}} \right) \cdot \theta(s^{\frac{1}{2}}) \]

\[ R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(\mu^+\mu^- \to hadrons)} = \frac{N_{\text{ee}}}{N_{\text{\mu\mu}}} \cdot \frac{\theta(s^{\frac{1}{2}})}{\theta(s^{\frac{1}{2}})} \]

\[ \frac{\sigma}{\sqrt{s}} \quad [\text{GeV}] \]

Figure 64.4: World data on the total cross section of e+e− → hadrons. The ratio R = \sigma(e^+e^- → hadrons)/\sigma(\mu^+\mu^- → hadrons) and R = \sigma(e^+e^- → \mu^+\mu^-)/\sigma(\mu^+\mu^- → \mu^+\mu^-) are shown. Data points are from below 2 GeV and statistical above 2 GeV. The curves are an educated guess of the hadronic case (green) and a quark-parton model prediction, and the solid one (red) is the QCD prediction (see "Quantum Chromodynamics" section of this Review). For further details, see Ref. 6944. (See also Ref. 6934, 6433). Brown-Hepburn parameterizations of \sigma(e^+e^- → hadrons) and (s^{\frac{1}{2}}) \in [1, 20000] are also shown. The full list of references to the original data and the details of the R ratio extraction from them can be found in Ref. 6911. Very accurate data files are available at http://hep.cb.u.nist.gov/current/xsect/ (Courtesy of the COMPASS Detector and HERAIB-DAFNAE (Durham) Groups, May 2004). See full-color version on color pages at end of book.)
41. Plots of cross sections and related quantities

---

### $R$ in Light-Flavor, Charm, and Beauty Threshold Regions

**u, d, s**

- **3 loop pQCD**
- **Naive quark model**

**Sum of exclusive measurements**

**Inclusive measurements**

---

**J/ψ, ψ(2S), ψ(3S), ψ(4S)**

- Model
- Model + LGW
- MarkII
- FIUETO
- DASP
- CrystalBall
- BES

---

**$\sqrt{s}$ [GeV]**

---

**4.2 The 2-pole R(κ)**

- In §4.1, we identified tree level:

\[
\frac{\text{d}^2 \sigma_{\text{excl}}}{\text{d} t \text{d} \cos \theta} = \frac{\lambda_2}{16\pi} \left[ (1+\cos^2 \theta) \left( Q_2^2 - 2Q_2 t \nu_{\text{excl}} \frac{1}{(s+m_2^2)^2 + \Gamma_2^2} \right) \right]^{1/2} \]

use $\lambda_2 = \frac{\nu_{\text{excl}}}{\Gamma_2}$

\[
\frac{\lambda_2}{16\pi} \left( \frac{t}{s+m^2} \right)^{1/2} \left[ 1 + \left( \frac{t}{s+m^2} \right)^2 + \left( 1 - \frac{t}{s+m^2} \right) \cos^2 \theta \right] \]

- Generalization of Standard Model:

\[
\lambda_2 = \frac{\nu_{\text{excl}}}{\Gamma_2} + \frac{\nu_{\text{excl}}}{\Gamma_2} \]

**Use SMErenner rules:**

\[
\frac{3}{2} \left( \frac{t}{s+m^2} \right)^{1/2} \left( 3 \lambda_2 - 2Q_2 \lambda_2 \nu_{\text{excl}} \right)
\]

where $\nu_{\text{excl}} = 0.23$ a tree level mixing angle $\lambda_2 = 0.5$, $Q_2 = f \left[ \frac{1}{3} \right]$, and $\lambda_2$ is the real and complex $V_2 = e^{i\phi} \nu_{\text{excl}}$

---

** supplement of references and tables on color page at end of book.**
4.3 QCD corrections & R(\epsilon)

\[ R_{\text{ew}} = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} \]

Only 5 graphs. The 2 (\(\epsilon = 127.6\%\))

\[ \frac{3}{\epsilon} + \left( \frac{3}{\epsilon} - 1 \right) \frac{1}{\epsilon} + \frac{3}{\epsilon} \left( \frac{1}{\epsilon - 1} \right) \]

\[ \approx 20.095 \]

\[ \text{adding the } \epsilon \text{ channel, value changes to } 19.984 \]

So our result would look like this: step function from \(\epsilon \to \infty\) + broad peak for 2 channels.

Comparison with experiment:

\[ \text{LEP measured } R_{\text{ew}} = 20.374 \pm 0.025 \]

Which is 3.5% higher than the above lowest-order prediction.

Discrepancy is mostly due to higher-order QCD corrections!

\[ \Rightarrow \text{compute } \sigma(e^+e^- \rightarrow \text{hadrons}) \]

Then use experiment to determine \(\sigma_{\text{ew}}\).
\[
\mathcal{M} = \frac{-i e \alpha g q^2}{q^3} \mathcal{O}^{\alpha} T^\alpha \mathcal{O}^\beta T^\beta \sum_{y} \left[ \sum_{\nu} J^{y \nu} \frac{\mathcal{O}^{\nu}}{m^2} \right] \mathcal{O}^\delta T^\delta
\]

\[
\langle 1/M^2 \rangle = \frac{M^2}{\mathcal{M}^2} \mathcal{M} \mathcal{M}^\dagger
\]

\[
\sum_{\nu} J^{y \nu} \frac{\mathcal{O}^{\nu}}{m^2} \mathcal{O}^\delta T^\delta
\]

\[
\mathcal{O}^\alpha T^\alpha \mathcal{O}^\beta T^\beta
\]

\[
\sum_{\nu} J^{y \nu} \frac{\mathcal{O}^{\nu}}{m^2} \mathcal{O}^\delta T^\delta
\]

\[
\langle 1/M^2 \rangle = \frac{M^2}{\mathcal{M}^2} \mathcal{M} \mathcal{M}^\dagger
\]

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\langle 1/M^2 \rangle = \frac{M^2}{\mathcal{M}^2} \mathcal{M} \mathcal{M}^\dagger
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\langle 1/M^2 \rangle = \frac{M^2}{\mathcal{M}^2} \mathcal{M} \mathcal{M}^\dagger
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\langle 1/M^2 \rangle = \frac{M^2}{\mathcal{M}^2} \mathcal{M} \mathcal{M}^\dagger
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\langle 1/M^2 \rangle = \frac{M^2}{\mathcal{M}^2} \mathcal{M} \mathcal{M}^\dagger
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\langle 1/M^2 \rangle = \frac{M^2}{\mathcal{M}^2} \mathcal{M} \mathcal{M}^\dagger
\]

\[
\langle 1/M^2 \rangle = \frac{M^2}{\mathcal{M}^2} \mathcal{M} \mathcal{M}^\dagger
\]
4.3.2 virtual corrections: $\mathcal{O}_{\text{vir}}$ at $O(\alpha_s)$

In dimensional regularization:

$$\left(\alpha_s^2 - \mathcal{O}(\alpha_s^3)\right)^2 \approx \abs{\alpha_s^2}^2 + \alpha_s \left(\mathcal{O}(\alpha_s^3)\right)^2 \approx \mathcal{O}(\alpha_s^3)$$

$$\Rightarrow \mathcal{O}_{\text{vir}} \approx \mathcal{O}_{\text{vir}}^0 \cdot \frac{\alpha_s}{\pi} \cdot \frac{1}{\epsilon} \cdot \frac{1}{\epsilon^2}$$

Note that $\epsilon$ is dimensional for $E \to 0$.

Not a detector (see p. 47): $\text{kin}$ (real and virtual case) should exist.

We will be complete next.

Physical reasons of divergences:

In a 'naive' calculation ($\epsilon = 0$),

$$\mathcal{I} = \int \frac{d^4 k}{2^3 \pi^2} \frac{k^2 \delta^4}{(k^2 - m^2)^2}$$

no divergences come from $M$ and $N$.

$$\begin{align*}
1 - k^2_E &= 1 - \frac{2 \epsilon k^2}{\epsilon^2} + \frac{(2 \epsilon)^2}{\epsilon^4} - \frac{2 \epsilon k^2 + \epsilon^2 k^2}{\epsilon^4} \\
&\Rightarrow \epsilon^2 k^2 \to 0 \quad \Rightarrow k \to E_1 (1, E_1)
\end{align*}$$

$$\mathcal{I} = \frac{2}{2 \epsilon_1} \mathcal{I}(\epsilon) \frac{k^2}{\epsilon^2}$$

Divergences originate from $E_1$.

$$\frac{1}{(\epsilon-1)} \mathcal{I}(\epsilon) \frac{k^2}{\epsilon^2}$$

Divergences originate from diverging propagators.

Collider limit: $\epsilon \to 0$.

Soft limit: $E_1 \to 0$.

In the present formulation,

$$\frac{1}{2 \epsilon} = \frac{1}{2 \epsilon_1} \frac{1}{\epsilon} \approx \frac{1}{\epsilon}$$

$$\Rightarrow$$ Divergences proportional to $\mathcal{I}(\epsilon)$.

4.3.2 virtual corrections: $\mathcal{O}_{\text{vir}}$ at $O(\alpha_s)$

Structure of cross section calculation:

$$\left| H_0^+ + H_1^+ + O(\alpha_s^2) \right|^2 \approx \left| H_0^2 \right|^2 + \alpha_s \left( H_0 H_1 + H_1 H_0^* + O(\alpha_s) \right)$$

$$\Rightarrow$$ need to compute interference term $H_0 H_1$ only.

Recall $H_0 = s^{1/2}$

$$\Rightarrow H_1 = t^{1/2} + u^{1/2} + \text{other terms}$$

(a) (b) (c)

$$\Rightarrow$$ in dimensional regularization,

$$\mathcal{I}(\epsilon) \approx \int \frac{d^4 k}{(2\pi)^4} \frac{k^2}{(k^2 - m^2)^2} \approx \mathcal{O}(\epsilon)$$

with $\text{dim} \left[ \mathcal{I}(\epsilon) \right] = \epsilon^2 - 4$

but $\epsilon \to 0$ (due to overall condition), $\epsilon \to 0$ scalars $\to 0$

$$\Rightarrow H_0 H_1 = 0 \quad \text{in dim. reg.}$$

For diagram (c), need to do some computation.

$$\mathcal{I}_{\text{vir}} \approx \frac{1}{2 \epsilon} \left( \frac{d^4 k}{(2\pi)^4} \frac{k^2}{(k^2 - m^2)^2} \right) \approx \epsilon^{\epsilon - 4} \mathcal{O}(\epsilon^2) \quad \Rightarrow \mathcal{O}(\epsilon^2)$$

Structure of $\alpha \mathcal{M}_1(\epsilon)$ is similar to $H_0$.

Recall (18 40) $H_0 = \mathcal{I}(\epsilon) \approx \frac{e^2 Q_1}{4 \pi} \frac{1}{\epsilon^0} u_{\alpha} \bar{u}_{\alpha} \frac{1}{\epsilon^0} v_2$

$$\Rightarrow \mathcal{I}(\epsilon) \approx \frac{e^2 Q_1}{4 \pi} \frac{1}{\epsilon^0} \left( \frac{1}{\epsilon^0} + \frac{1}{\epsilon^2} \right)$$

$$\Rightarrow \mathcal{I}(\epsilon) \approx \frac{e^2 Q_1}{4 \pi} \frac{1}{\epsilon^0} \left( \frac{1}{\epsilon^0} + \frac{1}{\epsilon^2} \right)$$

...
\[
T^2 \tau = T^2 \tau = \frac{1}{2} \left( T_{r1} S_{r1} - T_{r2} S_{r2} \right) + \frac{1}{2} \left( T_{r2} S_{r1} - T_{r1} S_{r2} \right) \frac{d^2 C}{dt^2} + \frac{1}{2} \left( T_{r1} S_{r1} - T_{r2} S_{r2} \right) \frac{d^2 C}{dt^2}.
\]

\[
\Rightarrow <n_1(n_1, n_2, +1) \mid \n_1(n_1, n_2) > = \frac{1}{4} \left( \delta^{\lambda_1 \lambda_2} - \delta^{\lambda_1 0} \right) (\delta^{\lambda_2 1} - \delta^{\lambda_2 2}) (n_1(n_1, n_2) + c.e.)
\]

Note that the contribution of \( A_\mu \) inside the 2nd trace can be reduced to \( A_\mu = n_1(n_1, n_2) C_\mu \phi(n_1) \).

Then:
\[
< n_1(n_1, n_2, +2) > = < n_1(n_1, n_2) > \cdot C_\mu \Re [\phi(n_1)] + c.e.
\]

The idea is to profit from the overall conditions.

We can use the identity:
\[
\delta^{\lambda_1 \lambda_2} - \delta^{\lambda_1 0} \mid \phi(n_1) = 0
\]

\[
A_\mu = \phi(n_1, n_2) c_\mu \left( \frac{d^2 C}{dt^2} \right)^2 \left( \frac{d^2 C}{dt^2} \right)^2
\]

\[
\Rightarrow \left( T_{r1} S_{r1} - T_{r2} S_{r2} \right) \frac{d^2 C}{dt^2} = \left( T_{r1} S_{r1} - T_{r2} S_{r2} \right) \frac{d^2 C}{dt^2}
\]

\[
0 = \left( T_{r1} S_{r1} - T_{r2} S_{r2} \right) \frac{d^2 C}{dt^2}
\]

Solve the loop integral \( \int^\infty \) using Feynman parameters (see p.30).

\[
\int^\infty \frac{d^4 \tau}{(2\pi)^4} \frac{1}{(n_1^n + n_2^n + n_3^n + n_4^n)^2}
\]

\[
\text{denominator} \quad [\Delta] = \left( (n_1^n + n_2^n + n_3^n)^2 + n_4^n \left( n_1^n + n_2^n + n_3^n \right)^2 \right) ^2
\]

\[
\Rightarrow \int^\infty \frac{d^4 \tau}{(2\pi)^4} \left( \frac{1}{n_4^n} \right)^2 \left( \frac{1}{n_1^n + n_2^n + n_3^n} \right)^2
\]

\[
\Rightarrow \phi(n_1, n_2) c_\mu \left( \frac{d^2 C}{dt^2} \right)^2 \left( \frac{d^2 C}{dt^2} \right)^2
\]

\[
\text{collecting, we finally have}
\]

\[
\phi(n_1, n_2) c_\mu \left( \frac{d^2 C}{dt^2} \right)^2 \left( \frac{d^2 C}{dt^2} \right)^2
\]

\[
\Rightarrow \left( \frac{d^2 C}{dt^2} \right)^2 \left( \frac{d^2 C}{dt^2} \right)^2
\]

\[
\phi(n_1, n_2) c_\mu \left( \frac{d^2 C}{dt^2} \right)^2 \left( \frac{d^2 C}{dt^2} \right)^2
\]

\[
\Rightarrow \phi(n_1, n_2) c_\mu \left( \frac{d^2 C}{dt^2} \right)^2 \left( \frac{d^2 C}{dt^2} \right)^2
\]

\[
\Rightarrow \phi(n_1, n_2) c_\mu \left( \frac{d^2 C}{dt^2} \right)^2 \left( \frac{d^2 C}{dt^2} \right)^2
\]
\[ R_{\text{M } 1} = \frac{\sigma(e^+ \to \mu^+ \nu\bar{\nu})}{\sigma(e^+ \to \ell^+ \nu\bar{\nu})} \left|_{\Sigma = \frac{1}{2}} \right. = \frac{1.934}{1.934} \left( 1 + \frac{0.5}{0.5} \right) \]

Note that NLO correction is positive.

- Comparing with the (correctly scaled) experimental result
  \[ R_{\text{exp}}(\text{LO}) = 20.767 \pm 0.025 \quad (\text{see eq. 32}) \]
  \[ \text{our first measurement of } R_{\text{exp}} = 20.3 \pm 0.004 \]
  (compare with "world average" from PDG (2010): 20.24 \pm 0.002)

- As another determination of \( R_{\text{exp}} \), let us compare to data taken by SHERPA (UEC), at \( \Sigma = 34 \text{ GeV} \):
  \[ R(34 \text{ GeV}) = 3.22 \pm 0.03 \]
  \[ \text{we would predict (ad hoc - i.e. very low)} \]
  \[ R(34 \text{ GeV}) = 3 \left( 2(3) + 2(3) \right) \left( 1 + \frac{0.5}{0.5} \right) \]
  \[ \frac{2}{2} = 3.3 \]

\[ \text{or, including } Z, \quad R(34 \text{ GeV}) = 3.3 \left( 1 + \frac{0.5}{0.5} \right) \]

- Our second measurement of \( R_{\text{exp}} \):
  \[ \chi_2(34 \text{ GeV}) = 0.162 \pm 0.026 \]

- Recall (eq. 14, p. 49) that \( \chi_2 \) is non-zero!
  \[ \chi_2(\mu) = \frac{h}{h} (\frac{1}{h}) \]
  \[ \chi_2(\mu) = \frac{h}{h} (\frac{1}{h}) \]

- \[ \chi_2(\mu) - \frac{1}{\chi_2(\mu)} = \frac{1}{h} \left( \frac{1}{h} \right) \]

- 2nd measurement from \( \chi_2 \):
  \[ \chi_2(\mu = 92.2 \text{ GeV}) = 0.135 \pm 0.018 \]
4.3.4 Higher-order QCD corrections to $R(s)$

\[ R(s) = \frac{3}{\xi} \left( \frac{4}{3} \xi^2 + \sum_{n=1}^{\infty} C_n(\frac{\lambda}{\Lambda}) \right) \]

where $C_{\text{had}} = 1 + 1 \cdot \frac{\xi}{\xi_{\Lambda}} + \sum_{n=1}^{\infty} \frac{C_1(\lambda)}{n}$.

- $C_1(\lambda)$ follow from higher-order computations.
- $C_2$ from
  \[ \text{two-loop corrections} \]
  \[ \text{one-loop corrections} \]

... etc...

- Note: In our computation, there were no UV divergences. (In fact, there is $x^2 + x^4$ in our exact result, but we did not need to renormalize, because our coefficient did not depend on the renormalization scale $\mu$: $C_1(\lambda) = 1$.)

- At higher order, we will encounter UV divergences, hence $C_n$ are renormalization scale dependent.

...
5. Deep inelastic scattering (DIS)

We now (temporarily) go back to treed-level phenomenology.

Q. are quark and gluon constituents of hadrons,
or put a mathematical framework for describing the hadron world?

5.1 Structure functions, Bjorken scaling, parton distribution fits

\[ e^- e^+ \rightarrow e^- e^+ \]

\[ Q^2 = -p^2 \]

\[ Q^2 \xi \rightarrow \text{elastic p scattering} \]

\[ Q^2 \rho \rightarrow \text{elastic p scattering} \]

we are interested in deep (Q^2 \gg \Lambda^2) inclusive (pp \rightarrow p\bar{p}) scattering

\[ \frac{d^2 \sigma}{dQ^2 dx} = \frac{\alpha_s^2}{16\pi^2} \frac{1}{L^{(2)}} \frac{1}{\lambda} \frac{d^2 \sigma}{dQ^2 dx} \]

for a inclusive process (do not measure X \rightarrow sum over gluons),

\[ \frac{d^2 \sigma}{dQ^2 dx} = \frac{1}{2\pi} \left( \frac{Q^2}{\Lambda^2} \right)^2 \]

\[ H_{\nu} \]

\[ \text{consider } H_{\nu} \text{ with summed and integrated all } X \text{ dependence,} \]

so \( H_{\nu} \) (q, p), must be symmetric in \( q, p \) (parity cons. in \( q, p \) space)

\[ H_{\nu} \rightarrow H_{\nu} + H_{\bar{\nu}} \]

where \( H_{\nu} \) are scalar fields, \( H_{\bar{\nu}} \)

\[ H_{\nu} (q^2 = 0, \rho = 0, \lambda = 0) \]

\[ \text{of course, } E^2 = p^2 = m^2 \]

\[ Q^2 < 0, \rho = 0, \lambda = 0 \]

\[ \text{kinematic limits: } Q^2 < 0, \rho = 0, \lambda = 0 \]

\[ H_{\nu} \text{ exchange, } \ldots + H_{\bar{\nu}} \text{ exchange} \]

\[ \text{(directly)} \]

\[ L^{(2)} H_{\nu} \rightarrow \delta (p_\perp) H_{\nu} + \delta (q_\perp) \frac{p_\perp}{q_\perp} H_{\nu} + 0 \cdot H_{\nu} + 0 \cdot H_{\bar{\nu}} \]

\[ Q^2 < -\frac{\Lambda^2}{4}, \rho = 0, \lambda = 0 \]

\[ \text{and } \delta (p_\perp) \text{ is negligible} \]

\[ \text{now } Q^2 = q^2 - (E - E')^2 = 2\rho^2 - 2\lambda^2 \]

\[ 5 = (p_\perp)^2 = 2\rho^2 + 2\lambda^2 \]

\[ 0 \leq \rho = \lambda = 0 \]

\[ L^{(2)} H_{\nu} = 2 \delta (p_\perp) \delta (q_\perp) \]

\[ \text{and } \delta (p_\perp) \text{ is negligible} \]

\[ \text{now } Q^2 = q^2 - 2\rho^2 - 2\lambda^2 \]

\[ \frac{p_\perp}{q_\perp} \]

\[ \frac{p_\perp}{q_\perp} = \frac{p_\perp}{q_\perp} \]

\[ \frac{p_\perp}{q_\perp} = \frac{p_\perp}{q_\perp} \]

\[ \frac{p_\perp}{q_\perp} = \frac{p_\perp}{q_\perp} \]

\[ = 4 \Lambda^2 H_{\nu} + 2 \frac{\Lambda^2}{4} (1/4) H_{\bar{\nu}} \]
5.2 Parton distribution functions

- Description of process in LLA

- Proton model: must satisfy
  
  \[ \frac{d^2\sigma}{d^2t} = \frac{1}{4} \frac{Q^2}{s} \left( \frac{1}{4} \ln^2 \frac{Q^2}{s} + \frac{2}{4} \ln \frac{Q^2}{s} \right) \quad \text{if} \quad t \ll \frac{Q^2}{s} \]

\[ \left( \text{valid for } s \gg m^2 \right) \]

\[ \rightarrow \text{somewhere} \] with 

- Structure functions of the proton
  
  \[ F_2(x, Q^2) = \frac{2}{3} F_1(x, Q^2) \]

- Transverse

- Longitudinal

\[ F_1(x, Q^2) = F_2(x, Q^2) - 2 x F_1(x, Q^2) \]

- Useful since (for large virtualities) \[ y \times 1 \]

\[ \frac{d^2\sigma}{d^2t} = \frac{2 x^2}{20} \left( \left( 1 + \frac{1}{y^2} \right) F_2(x, Q^2) + 2 \left( 1 - \frac{1}{y^2} \right) F_1(x, Q^2) \right) \]

\[ \left( \text{Bjorken scaling} \right) \]

\[ \rightarrow \text{exponentially, this is true (a pretty good approximation);} \]

\[ \rightarrow \text{perturbative results are valid, so have the interaction possible of the partons} \]

\[ \rightarrow \text{perturbative model} \]

\[ \text{Note:} \]

- For massless partons,

\[ (s + m^2)^2 = s + 2 m^2 \iff s \gg m^2 \]

\[ \rightarrow \text{intermediate} \]

\[ F_2 = 0 \]

\[ \text{Collins-Glashow relation} \]

\[ \text{Description of process in LLA} \]

\[ \text{Proton model: must satisfy} \]

\[ \text{valid for } s \gg m^2 \]

\[ \rightarrow \text{somewhere} \] with

- Structure functions of the proton
  
  \[ F_2(x, Q^2) = \frac{2}{3} F_1(x, Q^2) \]

- Transverse

- Longitudinal

\[ F_1(x, Q^2) = F_2(x, Q^2) - 2 x F_1(x, Q^2) \]

- Useful since (for large virtualities) \[ y \times 1 \]

\[ \frac{d^2\sigma}{d^2t} = \frac{2 x^2}{20} \left( \left( 1 + \frac{1}{y^2} \right) F_2(x, Q^2) + 2 \left( 1 - \frac{1}{y^2} \right) F_1(x, Q^2) \right) \]

\[ \left( \text{Bjorken scaling} \right) \]

\[ \rightarrow \text{exponentially, this is true (a pretty good approximation);} \]

\[ \rightarrow \text{perturbative results are valid, so have the interaction possible of the partons} \]

\[ \rightarrow \text{perturbative model} \]

\[ \text{Note:} \]

- For massless partons,

\[ (s + m^2)^2 = s + 2 m^2 \iff s \gg m^2 \]

\[ \rightarrow \text{intermediate} \]

\[ F_2 = 0 \]

\[ \text{Collins-Glashow relation} \]
to obtain the parton model prediction for structure functions, need to calculate parton cross sections.

\[ \frac{\alpha_s}{2\pi} \frac{N_c}{N_f} \left| \frac{e}{\pi} \right|^2 \]

and replace \( q \rightarrow q^* \) by "crossing symmetry" from \( q \rightarrow q^* \)

\[ \frac{1}{(2\pi)^3} \int d^3 p \frac{1}{2m} \frac{N_c}{N_f} \left( \frac{e}{\pi} \right)^2 \]

\[ \left| \frac{e}{\pi} \right|^2 \]

(see pg. 44)

\[ \left( \frac{e}{\pi} \right)^2 \left[ \left( \frac{e}{\pi} \right)^2 + \left( \frac{e}{\pi} \right)^2 \right] \]

\[ \frac{g^2 \alpha_s}{(2\pi)^3} \frac{N_c}{N_f} \left( \frac{e}{\pi} \right)^2 \left[ \left( \frac{e}{\pi} \right)^2 + \left( \frac{e}{\pi} \right)^2 \right] \]

\[ \left[ \left( \frac{e}{\pi} \right)^2 + \left( \frac{e}{\pi} \right)^2 \right] = \left( \frac{e}{\pi} \right)^2 \left[ 1 + \left( \frac{e}{\pi} \right)^2 \right] \]

\[ \frac{\alpha_s}{2\pi} \frac{N_c}{N_f} \left| \frac{e}{\pi} \right|^2 \frac{1 + \left( \frac{e}{\pi} \right)^2}{4 \left( \frac{e}{\pi} \right)^2} \]

\[ \rightarrow \text{phase space integration, see pg. 60} \]

\[ \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2m} \frac{N_c}{N_f} \left( \frac{e}{\pi} \right)^2 \text{ beam } X \text{ events of } \tau \text{ massive parton} \]

\[ \int \frac{d^3 p}{(2\pi)^3} \left( \frac{e}{\pi} \right)^2 \delta\left(p^2 - M^2\right) \]

\[ \frac{1}{2\pi^2} \frac{N_c}{N_f} \left| \frac{e}{\pi} \right|^2 \left| \frac{e}{\pi} \right|^2 \]

\[ \frac{1}{2\pi^2} \frac{N_c}{N_f} \left| \frac{e}{\pi} \right|^2 \left| \frac{e}{\pi} \right|^2 \]

\[ \frac{g^2 \alpha_s}{(2\pi)^3} \frac{N_c}{N_f} \left( \frac{e}{\pi} \right)^2 \left| 1 + \left( \frac{e}{\pi} \right)^2 \right| \]

\[ \frac{\alpha_s}{2\pi} \frac{N_c}{N_f} \left| \frac{e}{\pi} \right|^2 \frac{1 + \left( \frac{e}{\pi} \right)^2}{4 \left( \frac{e}{\pi} \right)^2} \]

\[ \rightarrow \text{parton cross section} \]

\[ \sigma(q^2,\mu^2) = \frac{1}{2\pi^2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2m} \frac{N_c}{N_f} \left| \frac{e}{\pi} \right|^2 \left| \frac{e}{\pi} \right|^2 \]

5.3 QCD corrections in DIS

\[ \rightarrow \text{ is not small, so one avoids LO treatment of DIS; might get important corrections.} \]

\[ \rightarrow \text{ how does the parton model change for QCD?} \]

\[ \rightarrow \text{ structure functions (theory) logarithmically depend on } \alpha_s, \]

\[ \text{leading to violation of Bjorken scaling.} \]

\[ \text{ how to compute NLO corrections to DIS?} \]

\[ \text{ divergence, splitting functions, evolution (DGLAP) evolution} \]

\[ \text{ data}
\]
5.3.1 DIS @ NLO: $e g \to e g$

- 2 diagrams, and $<H_W>$ by "crossing" (see §4.2.1)

\[
<\langle\tilde{H}_W\rangle> = \frac{\alpha_s^2}{4\pi} \frac{8C_F^2 \alpha_s^2 \beta_0^2}{(Q^2)^4 (\mu FR)} \left( (Q^2)^2 + (\tau \theta)^2 + (\tau \theta)^2 \right)
\]

\[
\rightarrow \text{ phase space: } \frac{d^2\sigma}{d^2x} = \frac{d^2x}{4x_e x_h} \frac{d^2x}{4x_e x_h} \frac{d^2x}{4x_e x_h} \text{ for } \lambda < 2 \mu_F^2
\]

where $(\tau_\theta)$ refer to direction of $p_\perp$ in CMS system of $\gamma p p_g$

alternatively, use finite-mass variable $z = \frac{p_T}{x} \equiv \frac{1}{2} (1 - \cos \theta)$

\[
\rightarrow \text{ partonic cross section: } \frac{d^2\sigma}{d^2x} = \frac{1}{2\pi} \frac{\alpha_s^2}{4x^2 x_h^2} \int \frac{dx}{4x_e x_h} \frac{r^2}{4x_e x_h} \left( \frac{2C_F \alpha_s^2 \beta_0^2}{(Q^2)^4 (\mu FR)} \right) \left[ \text{finite } z \right]
\]

\[
\text{ rewrite scalar products in terms of kinematic variables,}
\]

\[
\text{ from } \gamma \rightarrow \text{ quarkonium, use } x_\gamma = \frac{z}{2}
\]

\[
\text{will give a non-zero } z \rightarrow \frac{z}{2} \text{ due to } \frac{d^2x}{d^2x}
\]

Figure 1: The structure function $F_2$ as a function of $x$ for various $Q^2$ values, exhibiting Bjorken scaling, taken from [Ellis/Stirling/Webber].

Figure 2: Parton distribution function set A from the Martin-Roberts-Stirling group, taken from [Ellis/Stirling/Webber]. Note that this uses the common notation of defining valence quark distributions, $f_{vl} = f_v - f_{\bar{v}}, f_{vl} = f_d - f_{\bar{d}}$. 

\[
\begin{align*}
\text{Figure 1} & : \text{structure function } F_2 \text{ as a function of } x \text{ for various } Q^2 \text{ values, exhibiting Bjorken scaling, taken from [Ellis/Stirling/Webber].} \\
\text{Figure 2} & : \text{Parton distribution function set A from the Martin-Roberts-Stirling group, taken from [Ellis/Stirling/Webber]. Note that this uses the common notation of defining valence quark distributions, } f_v = f_v - f_{\bar{v}}, f_d = f_d - f_{\bar{d}}. 
\end{align*}
\]
Figure 3: Results of a recent compilation of $\alpha_s$ values, see [arXiv:0903.0979 [hep-ph], arXiv:hep-ex/0006035]. The scale dependence shows excellent agreement with the predictions of perturbative QCD over a wide energy range. When translated into measurements of $\alpha_s(M_Z)$, the separate measurements cluster strongly around the average value, $\alpha_s(M_Z) = 0.1204 \pm 0.0009$.

Figure 4: Fit to the $F_2$ data over a wide range of $Q^2$ values, exhibiting violation of Bjorken scaling.
5.3.2 DIS = NC0: Heap cq → q

diagram: $\chi \Gamma \sim \chi \delta$

in $dFH^2$, only need subchannel term with two hard diagrams

$| \chi \Gamma + \delta(z') |^2 = | \chi \Gamma |^2 + (\chi \chi \chi + \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \chi \cha
(a) known limits: note that \( \frac{1}{\mu^2} \to \frac{1}{\mu^2} \) (cf. p. 69);

- strong phase \( S(x) \) and gluon \( g(x) \) in perturbation

\[ 0 \notin (1 + \alpha)^2 = (\beta - 1)^2 = \frac{1}{\beta^2} + \frac{2}{\beta} (\gamma \cdot \frac{1}{\beta^2}) + \frac{2}{\beta} (\gamma \cdot \frac{1}{\beta^2}) + \frac{2}{\beta} (\gamma \cdot \frac{1}{\beta^2}) \]

- since pdf's contain physics below \( \mu \) only, they depend on \( \mu \)

\[ \tilde{f}_i(x, \mu^2) = \sum \frac{1}{\beta} \int \left( \frac{d}{\sqrt{2}} f_i \left( \frac{x}{\sqrt{2}}, \mu^2 \right) \right)^2 \left\{ \delta(1-x) + \frac{1}{\beta} \left[ \left( \gamma \cdot \frac{1}{\beta^2} \right) + \delta(1-x) \right] \right\} \]

- strong \( \beta \) depends on \( \mu^2 \) now, unlike Bjorken scaling!

- \( \mu^2 \) dependence? (see literature)

- physical cross sections cannot depend on \( \mu^2 \)

\[ \tilde{\sigma}_J \tilde{f}_i(x, \mu^2) \neq 0 \]

- at least in our calculation, \( \mu^2 \tilde{f}_i(x, \mu^2) = 0 \)

\[ \Rightarrow \int d^2 \tilde{f}_i(x, \mu^2) = \frac{1}{\sqrt{2}} \left( \frac{1}{\beta} \int \left( f_i \left( \frac{x}{\sqrt{2}}, \mu^2 \right) \right)^2 \right) \left\{ \delta(1-x) + \frac{1}{\beta} \left( \gamma \cdot \frac{1}{\beta^2} \right) + \delta(1-x) \right\} \]

Dobaczewski, G. - Lister, S. - Althoff, P.

- by to understand physical content of DGLAP eqs.

\[ \left( \frac{1}{\mu^2} \right) \frac{1}{\beta} \int \left( f_i \left( \frac{x}{\sqrt{2}}, \mu^2 \right) \right)^2 \left\{ \delta(1-x) + \frac{1}{\beta} \left( \gamma \cdot \frac{1}{\beta^2} \right) + \delta(1-x) \right\} \]

- now, \( \int \tilde{f}_i(x, \mu^2) = \frac{1}{\sqrt{2}} \left( \frac{1}{\beta} \int \left( f_i \left( \frac{x}{\sqrt{2}}, \mu^2 \right) \right)^2 \right) \left\{ \delta(1-x) + \frac{1}{\beta} \left( \gamma \cdot \frac{1}{\beta^2} \right) + \delta(1-x) \right\} \]

- new, \( \tilde{f}_i(x, \mu^2) = \frac{1}{\sqrt{2}} \left( \frac{1}{\beta} \int \left( f_i \left( \frac{x}{\sqrt{2}}, \mu^2 \right) \right)^2 \right) \left\{ \delta(1-x) + \frac{1}{\beta} \left( \gamma \cdot \frac{1}{\beta^2} \right) + \delta(1-x) \right\} \]

- new, \( \tilde{f}_i(x, \mu^2) = \frac{1}{\sqrt{2}} \left( \frac{1}{\beta} \int \left( f_i \left( \frac{x}{\sqrt{2}}, \mu^2 \right) \right)^2 \right) \left\{ \delta(1-x) + \frac{1}{\beta} \left( \gamma \cdot \frac{1}{\beta^2} \right) + \delta(1-x) \right\} \]

- pdf's change very little since they cannot depend on \( \mu^2 \)

- so at given \( \mu \), pdf's remain fixed (i.e., process and product)

- for \( \mu^2 \), evolution of pdf's, DGLAP eq is used

- \( f_i \) also depends on choice of \( \mu \), "scheme dependence"

- for any physical quantity, all \( \mu \) and \( \mu^2 \)-dependent sums

- scheme and scale-dependent pdf's \( f_i(x, \mu^2) \) are universal

- common choices:

\[ \begin{align*}
\alpha_S \text{ scheme } & (k_t < \mu) \\
\text{BS scheme} & (k_t > \mu)
\end{align*} \]
6. "Anomalies" [Rabin/Schramm §19; De S W.7]

\[ \phi \rightarrow \begin{array}{l}
\text{Q: symmetry of classical physics \ iff \ symmetry of quantum physics?} \\
\text{action } S(\phi) \text{ invariant, path integral } \int_{\gamma} e^{iS(\phi)} \text{ invariant} \\
\text{not always; measure } \Delta \phi \text{ may or may not be invariant.}
\end{array} \]

\[ \text{example: rotational invariance of QED would be strange if quantum fluctuations prevented specific direction!} \]

\[ \text{but: quantum fluctuation can break (time) classical symmetries;} \]

\[ \text{this phenomenon is called "anomaly"; exceptionally clear: change of integration variables \Rightarrow hard to find \ does important self-cannibal QFT \Rightarrow many ways oflooking at it \Rightarrow see a class. symmetry violating, by explicit Feynman loop calc.} \]

\[ \text{theory of one massless fermion: } \begin{pmatrix}
\chi_1 \\
\bar{\chi}_2
\end{pmatrix} \rightarrow \begin{pmatrix}
\chi_1 \\
\bar{\chi}_2
\end{pmatrix} e^{i\theta_y} \text{ and } \theta_y \Rightarrow \text{conserved current } \partial_0 \chi + \partial_y \bar{\chi} \text{ and } \partial_0 \bar{\chi} + \partial_y \chi \text{ (no charge) } \]

\[ \begin{pmatrix}
\theta_y = 0 \\
\theta_y \neq 0
\end{pmatrix} \text{ for class. calc. of vacuum } \psi = \psi \text{ and } \psi = \psi e^{i\theta_y} \text{ (no charge) } \]

\[ \text{calculate (Feynman transform) amplitude } \langle 0 | T \bar{\chi}(x_1) \chi(x_2) | 0 \rangle \]

\[ \Delta^{\mu\nu}(x_1, x_2) = \int \frac{d^4k}{(2\pi)^4} \frac{\delta^{4}(p)}{p^{0} - \epsilon} \text{ such that } \]

\[ \begin{pmatrix}
\frac{1}{2} & \frac{i}{2} \\
\frac{i}{2} & \frac{1}{2}
\end{pmatrix} \text{ relates } \begin{pmatrix}
\chi_1 \\
\bar{\chi}_2
\end{pmatrix} \text{ to } \begin{pmatrix}
\phi \\
\bar{\phi}
\end{pmatrix} \]

\[ \text{Q}^{\text{field}} \text{ dependence of } \frac{1}{2} \text{ is entirely driven by } \phi^{\text{field}} \text{ dependence of pdf, which is canceled by DSE&P evolution eqs; } \]

\[ \text{structure fits data over a wide range of } Q^2 \text{ provides a stringent test of perturbative QCD \Rightarrow Figure 4} \]
6.1 Vector current conservation

- would non-conservation of cellon current be a disaster?
- \( \mathcal{J} \): charge \( Q = \int d^3x \mathcal{A} \text{ should not change} \)
  - would it be difficult to interpret if not conserved?
  - could plotion g, pion hole emerging uith metr \( \mu \)
    - would have prop \( \mathcal{J} \) \( \mathcal{A} \)
    - gausy dephinen falls out if \( f_{\mu} \mathcal{J}^{\mu} = 0 \)

- \( \mathcal{J} \): who cares if main charge \( \mathcal{J} = \int d^3x \mathcal{A} \) changes in time?

- more calculation

\[
\mathcal{J}_\mu \mathcal{J}^{\mu}(x, y) = \int \left( \frac{d^3p}{(2\pi)^3} \right) \left( \frac{1}{2} \frac{1}{p^2} \right) \left( \frac{1}{2} \frac{1}{p^2} \right) \left( \frac{1}{2} \frac{1}{p^2} \right) \left( \frac{1}{2} \frac{1}{p^2} \right)
\]

- more careful calculation

when \( \Delta \mathcal{J} \) is to shift integration variables?

\[
\Delta \mathcal{J} = \int [\mathcal{J}(p) - \mathcal{J}(-p)] = \int \left[ \mathcal{A}(p) + \mathcal{A}(-p) \right] \left[ f(p) - f(-p) \right] \mathcal{G}
\]

\[
\Delta \mathcal{J} = \int \left[ \mathcal{A}(p) f(p) - \mathcal{A}(-p) f(-p) \right] = \left[ \frac{1}{2} \frac{1}{p^2} \right] f_p(2p) \mathcal{G}(2p)
\]

\( \mathcal{G} \) is "infinite" of integration variables, minus \( \frac{1}{2} \)

for our 4 dim momentum integral, for each orbit

\[
\Delta \mathcal{J} = \int \left[ \mathcal{A}(p) f(p) - \mathcal{A}(-p) f(-p) \right] = \left[ \frac{1}{2} \frac{1}{p^2} \right] f_p(2p) \mathcal{G}(2p)
\]

we get, for \( \Delta \mathcal{J} = 0 \)

\( \mathcal{A} \) is conserved

- note: Feynman rules are not sufficient to compute \( \mathcal{J}(t) \)

- note: \( \mathcal{J}^{\mu}(x, y) \) is defined by \( \mathcal{A}^{\mu}(x, y) - \mathcal{A}^{\mu}(x, y) \)
6.2 Axial current non-conservation

\[ q_3 D_{\mu} A_{\nu}^{(a)} = q_3 D_{\mu} A_{\nu}^{(a)}(b, c, \xi, \eta) = \frac{i}{\xi \eta} \varepsilon^{a b c} \varepsilon_{\mu \nu} \xi \eta \]

\[ \Rightarrow \] 

Axial current is not conserved!

This is known as 'axial anomaly':

Q-model fluctuations destroy the (classical) axial current conservation.

\[ \text{Consequences/Remarks (w/o derivation)} \]

\[ \text{gauge our theory } \chi \quad : \quad \chi^{(\mu)} = \frac{1}{\xi_{\mu}(b, c, \xi, \eta)} \]
• can one work with QCD in the regime where
  the strong coupling is actually strong?
  
  → by open question: confinement
  
  would like to have the complete force between quarks for these
  weak coupling limit: Coulomb potential
  → strong coupling limit: fermionic potential, confab model
    ("string"?")
  
  → derivation model new tools:

• [Wilson, Phys. Rev. D 10 (1974) 2445]: lattice gauge theory
  
  while continuum invariance and quark invariance
  formulate theory on 4d Euclidean space; lattice
  
  perform continuum limit a → 0 in this end, to recover
  4d rotational invariance and (after that etc.) lattice invariance.
  
  fundamental variables: four elements \( U_i \), \( V_i \), \( U_i \) = unity, \( V_i \) make
  lattice commutative:

  - four elements:

    \[ U_i, V_i \] = \[ U_i, V_i \]

  - four invariance:

    \[ U_i \] = \[ U_i \]

  \[ x = \frac{2\pi a}{a} \]

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