2. Basics

2.1 Reminder: QED and gauge invariance

* Gauge symmetry is a fundamental principle that determines the form of the Lagrangian.

- Consider \( \phi(x) \) (complex-valued Dirac field).

we now demand the theory to be invariant under local phase transformations: \[ \phi(x) \rightarrow e^{i\lambda(x)} \phi(x) \]

Q: Which Lagrangian terms can we construct that are invariant?
A1: Terms that are also invariant under global phases, e.g., \( \bar{\phi}(x) \phi(x) \) (recall: Dirac adjoint \( \bar{\phi} \equiv \gamma^0 \phi^* \))

A2: For terms with derivatives, we need some preparation:
(recall: derivative in eq. \( n^{-} \) direction defined as differential quotient)

\[ \frac{\phi(x + \varepsilon \eta) - \phi(x)}{\varepsilon} \rightarrow n^{-} \partial_{\eta} \phi(x) \]


deal completely different phase transformation! \( \varepsilon \)

\( \Rightarrow \) for meaningful comparison, introduce a compensating (scalar) phase factor transforming as \( U(y, x) \rightarrow e^{i\lambda(y)} U(y, x) e^{-i\lambda(x)} \)

\( \Rightarrow \) def. covariant derivative \( \partial_{\varepsilon} \phi(x) \)

for infinitesimal separation of \( y, x \), expand:
\[ U(x + \varepsilon \eta, x) \approx 1 - i\varepsilon \eta \partial_{\eta} \phi(x) + O(\varepsilon^2) \]

\( \Rightarrow \) cov. deriv.: \[ \partial_{\varepsilon} \phi(x) = \partial_{\eta} \phi(x) + i\varepsilon \eta \partial_{\eta} \phi(x) \]

where \( \partial_{\eta} \phi(x) \rightarrow \partial_{\eta} \phi(x) - \frac{i}{\varepsilon} \partial_{\eta} \phi(x) \) (consistent with \( U \)-transformation)

now: \[ \partial_{\varepsilon} \phi(x) \rightarrow (check!) = e^{i\lambda(x)} \partial_{\varepsilon} \phi(x) \]

transforms the same way as the field \( \phi(x) \) \( \checkmark \)

\( \Rightarrow \) \( \bar{\phi}(x) \partial_{\varepsilon} \phi(x) \) also invariant.
A summary:

1. Local phase relation symmetry

   ⇒ Def. of covariant derivative
   and existence of vector field $A_\mu$ (connection)

   ⇒ Transformation properties of $A_\mu$

   ⇒ All terms that are globally $q \to e^{i\chi}$, a unit, invariant
   are also locally invariant if we replace all $x \to \varDelta x$.

2. How about (locally covariant) kinetic terms for $A_\mu$?

   (a) Construction using $U(y, x)$

   $U$ is pure phase: $U(y, x) = e^{iu(y, x)}$, $u(y, x) \in \mathbb{R}$

   Assume $U(y, x) = \prod_{\mu} U_{\mu}(y, x) \equiv e^{i\sum_{\mu} A_{\mu}(y, x)}$

   ⇒ $A_{\mu}(y, x) = \frac{1}{i\hbar} \left( \frac{\partial}{\partial y^\mu} - \frac{\partial}{\partial x^\mu} \right)$ is odd under $y \leftrightarrow x$.

   ⇒ Can write $U(x + \varDelta x, x) = e^{i\varepsilon \nabla_{\mu} A_{\mu}(x + \varepsilon \hat{e}_{\mu})} + O(\varepsilon^3)$

   Now, use this for comparing phase products around a small square $\delta x \times \delta y$.

   $U(\delta x) = U(x, y) U(x + \delta y, x + \delta x + \delta y) U(x + \delta x + \delta y, x + \delta x) U(x + \delta x, x)$

   $= e^{i\varepsilon \left\{ -A_{\mu}(x) \delta x^\mu - \langle A_{\mu}(x) \rangle \delta x^\mu + \mu_0 A_{\mu}(x) \delta x^\mu + \frac{1}{2} \mu_0^2 A_{\mu}(x) \delta x^\mu \right\} + O(\varepsilon^3)}$

   $= 1 - i\varepsilon \left\{ \frac{1}{2} \nabla_{\mu} A_{\mu}(x) - \nabla_{\mu} A_{\mu}(x) - 2A_{\mu}(x) + \frac{1}{2} \mu_0 A_{\mu}(x) + \mu_0^2 \langle A_{\mu}(x) \rangle \right\} + O(\varepsilon^3)$

   (Area of square) $\frac{1}{\delta x} = F_\mu(x) \equiv$ electromagnetic field strength tensor

   But $U(x)$ is locally invariant by construction!

   ⇒ $F_\mu(x)$ is a locally covariant function of $A_\mu$!
(6) construction using $D_i$

Since (see above) $\gamma \to e^{i\theta} \gamma$, $D_i \gamma \to e^{i\theta} D_i \gamma$

it also follows that $\gamma \to e^{i\theta} \gamma$

or $[\partial_I, \partial_J] \gamma \to e^{i\theta} [\partial_I, \partial_J] \gamma$ *(#)*

Now, note that

$$[\partial_I, \partial_J] \gamma = [\partial_I, \partial_J] + i e \left( [\partial_I, A_I] + [A_I, \partial_J] \right) \gamma - e^2 [A_I, A_J] \gamma$$

$$= i e \left( \partial_I A_J - \partial_J A_I \right) \gamma$$

or $[\partial_I, \partial_J] \gamma$ has no derivative acting outside *(#)*

$$\Rightarrow [\partial_I, \partial_J] = i e F_{IJ}$$

$$\Rightarrow \text{in (##), } F_{IJ} \text{ is just a multiplicative factor, must be invariant.}$$

* can now write the most general locally invariant Lagrangian

(for the electron field $\gamma$ and its associated vector field $A_I$)

$$\mathcal{L}_{geo} = \bar{\gamma} \left( i \gamma^{\mu} D_{\mu} - m \right) \gamma - \frac{i}{4} F_{IJ} F^{IJ} - e \bar{\gamma} \gamma \gamma^5 \gamma F_{IJ} F^{IJ}$$

Remarks:

- used operators of dimension $\leq 4$ here

  in general, there are many additional gauge invariant ops, e.g.:

  $\mathcal{L}_5 = \bar{\gamma} \gamma^5 \gamma^5 \gamma F_{IJ} F^{IJ}$

  $\mathcal{L}_6 = \left( \bar{\gamma} \gamma \gamma^5 \gamma \right)^2 = \left( \bar{\gamma} \gamma \gamma^5 \gamma \right)^2$

  ... (see later)

  $\Rightarrow$ all these are *non-renormalizable* interactions

  $\Rightarrow$ irrelevant for physics, in Wilsonian sense

- the coefficient $c = 0$ if we postulate invariance under (discrete) $P, T$ symmetries

  $\Rightarrow$ then only 2 free parameters in $\mathcal{L}$: $m, e$ (hidden in $D$)

**Summary:**

local phase rotation symmetry of electron field $\gamma$

$\Rightarrow$ existence + transformation properties of em. vector potential

$\Rightarrow$ most general (4d, renormalizable, Tor invariant)

Lagrangian is unique: Maxwell-Dirac-Lagrangian!