Elements of group theory

- The color change introduced above can be treated much more rigorously.

- Before leaving the connection, let us review some basic facts about the theory of contravariant symmetry groups.

- (Much) more detail eg. in [H. Georgi: Lie Algebras in Particle Physics] or at [http://www.physik.uni-bielefeld.de/~klein/symmetrien/cover.html](http://www.physik.uni-bielefeld.de/~klein/symmetrien/cover.html)

- Our provisional color assignment to gluons (cf. §1.3) can be rewritten in a different basis (just different linear combinations) of $3 \times 3$ matrices (labeled eg. rows by colors, columns by the indices $i, j, k$)

\[
T^1 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \hspace{1cm} T^2 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & -i & 0 \\ 1 & 0 & 0 \end{pmatrix} \hspace{1cm} T^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\]

- Actually, $T^a = \lambda^a$, where $\lambda^a$ are "Gell-Mann matrices," $a = 1, 2, 3, 8$

- They form a possible representation of the infinitesimal generators of the "special unitary group" $SU(3)$, the fundamental representation.

- Some important properties (check?!

\[ [T^a, T^b] = i \epsilon^{abc} T^c \]

\[ \epsilon^{abc} \text{ antisymmetric structure constants} \]

\[ \{ T^a, T^b \} = \frac{1}{2} \delta^{ab} [4_{abc} + \epsilon^{abc} T^c] \]

\[ \epsilon^{abc} \text{ symmetric structure constants} \]

\[ T^a T^b = \frac{1}{2} \left( \delta^{ab} [4_{abc} + \epsilon^{abc} T^c] \right) \]

\[ T^a T^b = \frac{1}{2} \left( \delta^{ab} - \frac{1}{2} \delta^{ij} \delta_{kl} \right) \]

- The identity

\[ \text{Tr} (T^a) = 0 \]

\[ \text{Tr} (T^a T^b) = \frac{1}{2} \delta^{ab} \]

- Often we will need traces

\[ \text{etc.} \]
→ could calculate \( f^{abc} \) by multiplying Lie algebra with \( T^a \), then taking traces:
\[
f^{abc} = \frac{2}{i} \left( \text{Tr}(T^a T^b T^c) - \text{Tr}(T^b T^a T^c) \right)
\]
result (check?!) : \( f^{123} = f^{132} = f^{213} = f^{231} = f^{312} = f^{321} = \frac{1}{2} \)
\( f^{458} = f^{485} = \frac{1}{2} \) ; rest by antisymmetry

• from a more general viewpoint, we have just seen one example of a broader mathematical concept: representations of Lie Groups

well : group contains abstract entities that obey certain algebraic rules
well : interested in groups of unitary operators, acting in vector space of states
here : interested in continuously generated groups
contain elements arbitrarily close to identity

can reach general group element by repeated action of infinitesimal ones
\( g(x) = 1 + ix^a T^a + O(x^2) \)

→ Hermitian ops; "generators" of unitary group

→ a group with this structure is called a "Lie group"

• the set \( T^a \) spans space of infinitesimal group transformations

→ commutator is linear combination of generators
\([ T^a, T^b ] = i f^{abc} T^c \)

→ vector space spanned by generators + commutator = Lie Algebra

→ \([ T^a, [ T^b, T^c ] ] + [ T^b, [ T^c, T^a ] ] + [ T^c, [ T^a, T^b ] ] = 0 \) \( \text{Jacobi identity} \)

→ \( f^{abc} f^{def} + f^{ade} f^{bef} + f^{ade} f^{cfb} = 0 \)

• for us, symmetry is unitary transformation of a set of fields

→ interested in Lie groups with finite # of generators: "compact"

• classification of Lie Algebras

→ (group of phase rotations)

→ if one \( T^a \) commutes with all others : Abelian subgroup, \( g = e^{i \theta_a T_a} \)

→ if set of \( T^a \)'s cannot be divided into two mutually commuting sets: "Simple"

→ general Lie algebra \( \cong \) direct sum of non-Abelian simple components

→ \( SU(N) \) (\( (U^a, det(U) = 1) \)), \( SO(N) \) (\( (R^T I, det R = 1) \)), \( Sp(N) \); \( G_2, F_4, E_6, E_7, E_8 \)

→ is complete set of compact simple Lie groups!
1.5 Notation and conventions

- **Natural units**
  \( \hbar = c = \ell_P = 1 \)
  \[ \text{[length]} = \text{[time]} = \text{[energy]}^{-1} = \text{[mass]}^{-1} = \text{GeV}^{-1} \]

- **Vectors + Tensors**
  - indices \( \mu = 0, 1, 2, 3 \) or \( t, x, y, z \)
  - metric tensor \( g_{\mu \nu} = g^{\mu \nu} = \text{diag} (1, -1, -1, -1) \)
  - four vectors \( x^\mu = (x^0, \hat{x}) \), \( \hat{x} = x_\mu = (0, \vec{x}) \)
  - totally antisymmetric tensor \( \varepsilon_{0123} = 1 \) \( (\Rightarrow \varepsilon_{0123} = -1, \varepsilon^{0123} = 1 \text{ etc.)} \)

- **Matrices**
  - Pauli: \( \sigma^i \sigma^j = i \varepsilon^{ijk} \sigma^k \)
  - \( \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)

- **Dirac**
  - \( \{ \gamma^\mu, \gamma^\nu \} = 2 \gamma^{\mu \nu} \)
  - standard basis: \( \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \)

- **Einstein summation convention**
  - \( \rho^\mu x^\nu = \sum_{\mu=0}^{3} \rho^\mu x^\nu = \rho^0 x^0 + (-\vec{p}) \cdot \vec{x} = \rho x^0 - \vec{p} \cdot \vec{x} \)
  - \( A^\mu T^\nu = \sum_{\mu=0}^{3} A^\mu T^\nu \)