Solving Feynman integrals by means of difference equations

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MLWG

Goal

- What happend so far: Talked a lot about reducting Feynman integrals: IBP relations, Laporta algorithm, Gröbner bases,...
- Today: Want to get an explicit expression for the remaining master integrals, either completely analytic or if not possible with arbitrary numerical accuracy.
- Here: Difference equation generated by IBP relations and the Laporta algorithm

Difference equations [S. Laporta, hep-ph/0102033]

Difference equation

- Let us consider a master integral of the form

$$B = \int \frac{d^D k1 \dots d^D k_{N_k}}{D_1 D_2 \dots D_{N_d}} ,$$

where N_k loops and N_d lines.

- We raise one denominator to power x:

$$U_{D_1}(x) = \int \frac{d^D k1 \dots d^D k_{N_k}}{D_1^x D_2 \dots D_{N_d}} .$$

- Using integration-by-parts relations we find difference equations of R order

$$\sum_{j=0}^{R} p_j(x) U_{D_1}(x+j) = F(x) , \tag{1}$$

where p_j are polynomials and F(x) are some known functions.

Factorial series

- Factorial series play for difference equations the same role that power series play for differential equations. The series

$$\sum_{s=0}^{\infty} \frac{a_s \Gamma(x+1)}{\Gamma(x-K+s+1)} = \frac{\Gamma(x+1)}{\Gamma(x-K+1)} \left(a_0 + \frac{a_1}{x-K+1} + \dots \right)$$

  is called a factorial series of first kind. The series is similar to an asymptotic expansion in $1/n$ but with the advantage to be more convergent.
- Example: Let us consider the function $\Psi'(n) = \partial_n^2 \ln \Gamma(n)$. It satisfies the nonhomogeneous first order diff. eq.

$$\Psi'(n+1) - \Psi'(n) = -\frac{1}{n^2}$$

Factorial series:Convergence properties

- Expanding $\Psi'(n)$ in an asymptotic series, we get

$$\Psi'(n) = \frac{1}{n} + \frac{1}{2n^2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{n^{2k+1}} \,,$$

  where B_{2k} are the Bernoulli's numbers, convergent for $n < 1$.

- However, if one expands $\Psi'(n)$ in factorial series, one obtains

$$\Psi'(n) = \sum_{k=1}^{\infty} \frac{\Gamma(s)}{s} \frac{\Gamma(n)}{\Gamma(n+s)} \,,$$

  convergent for $n > 0$.

Operators π and ρ

- The idea is to rewrite the factorial expansion by using the π and ρ operators in an expansion in powers of ρ^{-1}.
- $\rightarrow$ Getting solutions for difference eq. in factorial series in the same manner as power series solutions of differential equations are obtained.
- Define the operator ρ, π as follows

$$\rho^m U(x) = \Gamma(x+1)/\Gamma(x-m+1)U(x-m)$$
$$\pi U(x) = x\,(U(x) - U(x-1))$$

where m is an arbitrary integer and $U(x)$ the solution of the homogeneous system.

Rewriting the difference equation

- Using some properties of π and ρ gives

$$\left[f_0(\pi,\mu) + f_1(\pi,\mu)\rho + \cdots + f_{m+1}(\pi,\mu)\rho^{m+1}\right] V(x) = 0 \quad (2)$$

where $V(x) = \mu^x U(x)$ and f_i are polynomials in π and μ.

- In case of difference equation it turns out that

$$f_{m+1}(\pi,\mu) = f_{m+1}(\mu) = 0\,,$$

the characteristic equation with R solutions different from zero.

- For each of these values $\mu = \mu_i$, the first canonical form Eq. (2) takes the form

$$\left[f_0(\pi) + f_1(\pi)\rho + \cdots + f_m(\pi)\rho^m\right] V(x) = 0\,. \quad (3)$$

Rewriting the difference equation

- Now let us try to satisfy the equation in V with the factorial series

$$V(x) = \sum_{s=0}^{\infty} \frac{a_s \Gamma(x+1)}{\Gamma(x-K+s+1)} = \sum_{s=0}^{\infty} a_s \rho^{K-s}$$

- Putting this in Eq. (3) we get

$$\begin{aligned}
&a_0 f_m(K+m) = 0,\\
&a_1 f_m(K+m-1) + a_0 f_{m-1}(K+m-1) = 0,\\
&\ldots\\
&a_s f_m(K+m-s) + \cdots + a_{s-m} f_0(K+m-s) = 0 \; (s \geq m)\,.
\end{aligned}$$

supposing $a_0 \neq 0 \Rightarrow f_m(K+m) = 0$.

Example: Massive tadpole

- To see how this works in practice we consider the massive tadpole
$$J(x) = \pi^{-D/2} \int \frac{d^D k}{(k^2 + m^2)^x} \,.$$
- We want to calculate the master integral $J(1)$. The IBP relation gives
$$(x-1)J(x) - (x-1-D/2)J(x-1) = 0 \,, \qquad (4)$$
which is a homogeneous first-order difference equation with polynomial coefficients.
- We look for a solution in form of a factorial series
$$J(x) = \mu^x V(x) = \mu^x \sum_{s=0}^{\infty} a_s \rho^{K-s} \,.$$

Example: Massive tadpole

- Using once again the properties of ρ and π we get
$$((\mu m^2-1)\rho^2+((2\mu m^2-1)(\pi-1)+D/2)\rho+\mu m^2\pi(\pi-1))V(x)=0\,.$$
- Now we can immediatly read of the polynomials f_i:
$$f_2(\pi,\mu)=\mu m^2-1 \Rightarrow \mu = 1/m^2\,,$$
$$f_1(\pi,\mu)=(2\mu m^2-1)(\pi-1)+D/2\,,$$
$$f_0(\pi,\mu)=\mu m^2\pi(\pi-1)\,.$$
- Therefore the difference equation becomes
$$(\underbrace{(\pi-1+D/2)}_{f_1(\pi)}\rho+\underbrace{\pi(\pi-1)}_{f_0(\pi)})V(x)=0\,.$$
- First recurrence relation says
$$a_0 f_{m=1}(\pi=1+K)=0\Rightarrow K+1-1+D/2=0\Rightarrow K=-D/2\,.$$

Example: Massive tadpole

- The other one gives us a recurrence for the coefficients a_s:

$$a_s f_1(K+1-s) + a_{s-1} f_0(K+1-s) = 0$$

- Hence $a_s = \prod_{i=1}^{s} \frac{1}{i} \left(i + \frac{D}{2}\right)\left(i + \frac{D}{2} - 1\right) a_0$. Performing the product we end up with

$$a_s = \frac{\Gamma(D/2+1+s)\Gamma(D/2+s)}{\Gamma(D/2+1)\Gamma(D/2)\Gamma(s+1)} a_0 \,.$$

- The coefficient a_0 is still unknown and can be obtained by comparing the large-x behaviours of $J(x)$ and its factorial series

$$\pi^{-D/2} \left(m^2\right)^{D/2+x} \int \frac{d^D k}{(k^2+1)^x} \approx \ldots \int_D e^{-xk^2} = \left(m^2\right)^{D/2+x} x^{-D/2}$$

$$J(x) = \mu^x \sum_{s=0}^{\infty} \frac{a_s \Gamma(x+1)}{\Gamma(x-K+s+1)} \approx a_0 x^K \mu^x \,.$$

Example: Massive tadpole

- Using the values for μ and K results in

$$a_0 = \left(m^2\right)^{D/2} .$$

- The final result for $J(x)$ becomes

$$\left(m^2\right)^{D/2-x} \sum_{s=0}^{\infty} \frac{\Gamma(D/2+1+s)\Gamma(D/2+s)}{\Gamma(D/2+1)\Gamma(D/2)\Gamma(s+1)} \frac{\Gamma(x+1)}{\Gamma(x+D/2+s+1)} ,$$

and after performing the summation we recover the well-known result

$$J(x) = \left(m^2\right)^{D/2-x} \frac{\Gamma(x-D/2)}{\Gamma(x)} .$$

Backup: Convergence of J(x)

- For $s \to \infty$ the term of the factorial series behaves as

$$\frac{a_s}{\Gamma(x + D/2 + s + 1)} \approx \frac{s!\, s^{D-1}}{s!\, s^{D/2+n}} = s^{-1+D/2-x}\,.$$

- Means the series is convergent for $x > D/2$ and diverges for $x \leq D/2$.
- However, in order to compute $J(1)$ we just compute $J(1+i)$ for some large integer i and use repeatedly the recurrence relation Eq. (4).