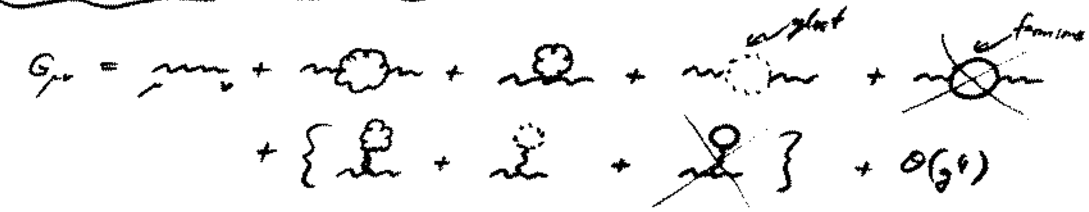
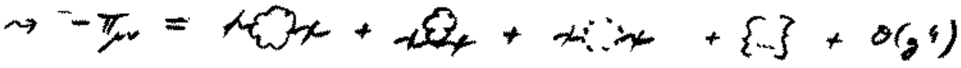
((in fact, due to a Ward-identity $Q^\mu G_{\mu\nu} Q^\nu = \alpha$,
so $\Delta_L(Q) = \frac{\alpha}{Q^2}$ exactly, $2(Q^2+\Pi_E)\Pi_L + \Pi_E^2 = 0$ $\rightsquigarrow$ 4Π's $\to$ 3Π's))

9) QCD; collective excitations, plasma frequency

calc. Π to lowest order: setup



omit fermion loops here.



tasks: • treat Lorentz-structure ✓ A..D
• show $\{\dots\} = 0$ $\leftarrow$ via $SU(N)$-rel's
• e.g. ghost diag

ex.:

$\equiv -\Pi^{ab}_{\mu\nu}(Q)\Big|^{(1)}_{ghost}$

$= 1\cdot(-1)\cdot \sum_K \left(-ig K_\mu f^{dca}\right)\frac{-i/\xi}{K^2}\left(-ig(K-Q)_\nu f^{dcb}\right)\frac{-i/\xi}{(K-Q)^2}$

gh loop; from $f^{cda}f^{cdb} = C_A\delta^{ab}$

$= -g^2 C_A \delta^{ab} \sum_K \frac{K_\mu (K-Q)_\nu}{K^2 (K-Q)^2}$

note that denominator is symmetric under $K \to Q-K$

$\Rightarrow$ numerator $\to K_\mu K_\nu - \frac{1}{2}\left[K_\mu Q_\nu + (Q-K)_\mu Q_\nu\right] = K_\mu K_\nu - \frac{1}{2}Q_\mu Q_\nu$

$= -\frac{g^2 C_A}{2}\delta^{ab}\sum_K \frac{2K_\mu K_\nu - Q_\mu Q_\nu}{K^2(K-Q)^2}$

philo: want to find poles of propagator $\Leftrightarrow$ $Q^2 = -\Pi_{44}(Q)$
since Π starts at g^2, Q will be (at least)
$\sim g$ at/near the poles.
$\Rightarrow$ split above integrand

$$= -g^2 C_A \delta^{ab}\Big\{ \underbrace{\sum_K \frac{K_\mu K_\nu}{K^2(K-Q)^2}}_{\text{"hard"}} - \underbrace{\tfrac{1}{2}Q_\mu Q_\nu \sum_K \frac{1}{K^2(K-Q)^2}}_{\text{"soft": } g^2\text{-suppressed if } Q\sim g} \Big\} \qquad (117)$$

add remaining two diags:

- sy factors $\frac{1}{2}$  $+\ \frac{1}{2}$ 
- more ~~tedious~~ Lorentz-structure due to gluon propagator

→ hw #18

result: $$-\Pi^{\mu\nu}_{1\,\text{loop, hard}}(Q) = -2g^2 C_A \sum_K \frac{K^2 g^{\mu\nu} - 2K^\mu K^\nu}{K^2 (K-Q)^2} \qquad (118)$$

⇒ α-independent! (but in $\mathcal{O}(g^2)$..)

decompose: take traces (cf. $FT^2/52$).

- $$Q_\mu \Pi^{\mu\nu} V_\nu \sim \sum_K \frac{K^2\cdot 0 - 2(KQ)(KV)}{K^2(-)^2} \quad , \quad 2(KQ) = K^2+Q^2-(-)^2$$
$$= \sum_K \Big\{ \underbrace{-\frac{(KV)}{(-)^2}}_{\substack{K\to Q-K \\ = -\cancel{\frac{KV}{K^2}}}} + \cancel{\frac{(KV)}{K^2}} - Q^2 \underbrace{\boxed{\frac{KV}{K^2(-)^2}}}_{= \frac{1}{2}\frac{KV+(Q-K)V}{K^2(-)^2} \overset{(QV=0)}{=} 0} \Big\}$$
$$= 0$$
$$\Rightarrow \Pi_c^{\text{hard}} = 0 \qquad (119a)$$
- $$Q_\mu \Pi^{\mu\nu} Q_\nu \sim \sum_K \frac{K^2Q^2 - 2(KQ)^2}{K^2(-)^2} = \overset{\text{same}}{\dots} = 0$$
$$\Rightarrow \Pi_d^{\text{hard}} = 0 \qquad (119b)$$
- $$\mathrm{Tr}\big((A+B)\Pi\big) = (A_{\mu\nu}+B_{\mu\nu})\Pi^{\mu\nu} = 2\Pi_t + \Pi_\ell$$
$$= (g_{\mu\nu} - D_{\mu\nu})\Pi^{\mu\nu} = +2g^2C_A \sum_K \frac{1}{K^2(-)^2}\Big(K^2 \underbrace{g_\mu^\mu}_{4} - 2K^2 \underbrace{- K^2 + 2\frac{(KQ)^2}{Q^2}}_{= 0 + \text{soft}^2 \ (*)}\Big)$$
$$\simeq +4g^2C_A \sum_K \frac{1}{K^2} = +4g^2C_A\left(-\frac{T^2}{12}\right)$$
$$\Rightarrow \boxed{2\Pi_t + \Pi_\ell = -\frac{1}{3}g^2C_AT^2} \qquad \text{(hard)} \qquad (119)$$

⤳ only one function left to calculate; choose Π_ℓ

(proof of (*): $$\sum_K \frac{1}{K^2(-)^2}\Big(K^2 - 2\frac{(KQ)^2}{Q^2}\Big) \quad , \text{use} \quad -2KQ = (-)^2 - K^2 - Q^2$$
$$= \sum_K \Big(\underbrace{\frac{1}{(-)^2}}_{K\to Q-K} + \frac{(KQ)}{Q^2}\Big[\underbrace{\frac{1}{K^2} - \frac{1}{(-)^2}}_{K\to Q-K} - \underbrace{\frac{Q^2}{K^2(-)^2}}_{\text{symmetrize under } K\to Q-K} \Big]\Big)$$
$$= \sum_K \Big(\cancel{\frac{1}{K^2}} + \frac{KQ}{Q^2K^2} - \frac{\cancel{Q^2} - KQ}{Q^2K^2} - \frac{1}{2}\frac{\cancel{KQ} + Q^2 - \cancel{KQ}}{K^2(-)^2}\Big)$$
$$= \sum_K \Big(\underbrace{\frac{2KQ}{Q^2K^2}}_{\text{odd in } K}{}^{\to 0} - \frac{1}{2}\frac{Q^2}{K^2(-)^2}\Big) = 0 + \text{soft}^2 \quad \text{qed} \Big)$$

- $$-\Pi_\ell^{hard} = \frac{1}{V^2} V_\mu \left(-\Pi^{\mu\nu}_{12,hard}\right) V_\nu$$

$$= -2g^2 C_A \sum_K \frac{K^2 - 2\frac{(KV)^2}{V^2}}{K^2(K-Q)^2} \qquad (120)$$

$$(VK)^2 = \left(Q^2 K_0 - Q_0 (KQ)\right)^2 = Q^2 \underbrace{Q^2}_{=Q_0^2-q^2} K_0^2 + \underbrace{Q_0^2}_{=Q^2+q^2}(KQ)^2 - 2K_0 Q_0 Q^2 (KQ)$$

$$= q^2 (KQ)^2 + Q^2\left[-q^2 \underbrace{K_0^2}_{=K^2+k^2} + \left(K_0 Q_0 - (KQ)\right)^2\right]$$

$$= \underbrace{-q^2 Q^2}_{\overset{!}{=} V^2}\left[-\frac{(KQ)^2}{Q^2} + K^2 + k^2 - \frac{(\vec{k}\vec{q})^2}{q^2}\right]$$

$$= -2g^2 C_A \sum_K \frac{+K^2 + 2\frac{(KQ)^2}{Q^2} - 2\left(k^2 - \frac{(\vec{k}\vec{q})^2}{q^2}\right)}{K^2 (K-Q)^2}$$

$> = 0 + \text{soft}^2$, see above

$$\simeq 4g^2 C_A \sum_K \frac{1}{K^2(K-Q)^2}\underbrace{\left(k^2 - \frac{(\vec{k}\vec{q})^2}{q^2}\right)}_{= k^2(1-\cos^2\theta)} \qquad (121)$$

calculate Π_ℓ^{hard}: (only sketched here)

a) perform frequency sum

$$T\sum_n \frac{1}{K_0+k}\,\frac{1}{K_0-k}\,\frac{1}{K_0-Q_0+\omega}\,\frac{1}{K_0-Q_0-\omega}, \quad K_0 = i\omega_n, \quad \omega = |\vec{k}-\vec{q}|$$

use contour int. formula;

simplify: $n(Q_0+\omega) = n(\omega)$, since $\beta Q_0 = \frac{1}{T} 2\pi n_0 T$

$n(-\omega) = -1 - n(\omega)$

b) decouple radial and angular k-integration (works in 'hard' approx.)

$$\omega = \sqrt{k^2 - 2\vec{k}\vec{q} + \vec{q}^2} \simeq k - \frac{\vec{k}\vec{q}}{k} + \underbrace{O(q^2)}_{\text{'soft'}}$$

$\rightsquigarrow$ denominators e.g. $Q_0 \pm (k-\omega) \simeq Q_0 \pm \frac{\vec{k}\vec{q}}{k} = Q_0 \pm q\cos\theta$ $\quad$ k-indep!

$$n(\omega) \simeq n(k) - q\cos\theta\, n'(k) \quad + \underbrace{O(q^2)}_{\text{'soft'}}$$

etc.

$$\int\frac{d^3k}{(2\pi)^3} = \frac{1}{(2\pi)^3}\int_0^\infty dk\, k^2 \int d\Omega \,, \qquad \int d\Omega = \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta) = 2\pi\int_{-1}^{1} d(\cos\theta)$$

e.g. $$\int_{-1}^{1} d(\cos\theta)\,\frac{\cos\theta}{Q_0 + q\cos\theta} = \frac{1}{q}\int_{-1}^{1} dx\, \frac{x + \frac{Q_0}{q} - \frac{Q_0}{q}}{x + \frac{Q_0}{q}} = \frac{1}{q}\left(x - \frac{Q_0}{q}\ln\left(x + \frac{Q_0}{q}\right)\right)_{x=-1}^{1} = \frac{2}{q}\left(1 - \frac{1}{2}\frac{Q_0}{q}\ln\left(\frac{\frac{Q_0}{q}+1}{\frac{Q_0}{q}-1}\right)\right)$$

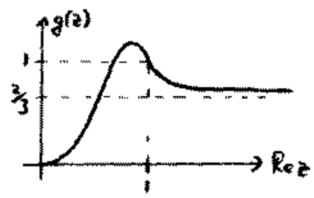
c) result is

$$-\Pi_\ell^{hard}(Q) \simeq \frac{g^2 C_A T^2}{3}\left(1 - g\left(\frac{Q_0}{q}\right)\right) \qquad (122)$$

where $g(z) = z^2 - \frac{z}{2}(z^2-1)\ln\left|\frac{z+1}{z-1}\right|$

$\xrightarrow{z\gg1} \frac{2}{3} + \frac{2}{15z^2} + \dots$

$\xrightarrow{z\ll1} 2z^2 + \dots$



$\Rightarrow$ get Π_t from (119): $$-\Pi_t^{hard}(Q) \simeq \frac{g^2 C_A T^2}{6}\, g\left(\frac{Q_0}{q}\right) \qquad (123)$$

$-\Pi_\ell(Q_0, q\to 0) = \frac{g^2 C_A T^2}{9} \equiv m^2$, $-\Pi_t(Q_0, q\to 0) = m^2$ ("static limit")

$-\Pi_\ell(Q_0\to 0, q) = \frac{1}{3}m^2$, $-\Pi_t(Q_0\to 0, q) = 0$

$\Rightarrow$ • limits $q\to 0$, $Q_0\to 0$ do <u>not</u> commute

• $\Pi_\ell \neq 0$ always ("<u>thermal</u> <u>mass</u>" for long. gluon, $m\sim gT$)

Π_t can vanish $\rightsquigarrow$ IR problems!

<u>plasma frequency</u>

((mini-review: $-\Pi_t(Q) = \frac{3}{2}m^2 g\left(\frac{Q_0}{q}\right)$, $-\Pi_\ell(Q) = 3m^2\left(1-g\left(\frac{Q_0}{q}\right)\right)$, $\Pi_c = \Pi_d = 0$, $m^2 = \frac{g^2 C_A T^2}{9}$))

from (115, 116): $$G_{\mu\nu}(Q) = \frac{A_{\mu\nu}(Q)}{Q^2 + \Pi_t(Q)} + \frac{B_{\mu\nu}(Q)}{Q^2 + \Pi_\ell(Q)} + \frac{\alpha D_{\mu\nu}(Q)}{Q^2} \qquad (124)$$

"frequency ω of transverse / longitudinal gluonic excitations is given by the poles of the propagator, after the analytic continuation $Q_0 = i\omega_n \to \omega + i\varepsilon$"

please believe. derivation: linear response $\rightsquigarrow$ §10

$$\rightsquigarrow \omega^2_{t/\ell}(q) = q^2 - \Pi_{t/\ell}(Q_0 \to \omega + i\varepsilon, q) \qquad (125)$$