8) QCD ; setup , Fy rules

Lagrangian (cf. (22))

$$\left.\begin{aligned}
\mathcal{L} &= \bar\psi(i\not{D}-m)\psi - \tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}\\
\text{with}\quad \not{D} &= \gamma^\mu(\partial_\mu - g A_\mu)\\
F_{\mu\nu} &= \tfrac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] = F^a_{\mu\nu}T^a\\
A_\mu &= A^a_\mu T^a
\end{aligned}\right\}\quad (101)$$

generators of $SU(N)$, $a=1..N^2-1$ ($N=3$ colors $\leadsto$ 8 gluons)

aha! ?? step back ...

- ψ_α ($\alpha = 1..N$) ; quarks, N colors ; directions in color-space unphysical (cannot see/measure)

  $\to$ invariance (of action / eqns of motion) under

  $\psi_\alpha \to U_{\alpha\beta}\psi_\beta$, $U^\dagger U = 1$, U global transformation (x-indep ; for the moment)

  can parametrize $U = e^{i\Lambda}$, $\Lambda = \Lambda^\dagger$ hermitian

  Λ : $\frac{1}{2}(2N^2)$ real param's

  $\Lambda = (-g)\underset{\text{real}}{\underbrace{\lambda^a}}\underset{N\times N}{T^a}$, $a = 1..N^2$, $T^{a\dagger} = T^a$

  choose $T^{N^2} = \mathbb{1}_{N\times N}$

  assume $\operatorname{tr}(T') = s$, def $T'_{new} = T' - \frac{s}{N}\mathbb{1}$, $\operatorname{tr}T'_{new} = s - \frac{s}{N}N = 0$

  $\leadsto$ can construct all T^a ($a = 1..N^2-1$) traceless .

  $\Rightarrow U = e^{-ig\lambda^a T^a}e^{-ig\lambda^{N^2}}$

  (λ^a: real ; T^a: hermitian, traceless)

- local gauge invariance ((on the moon, same physics measured, but may have chosen different λ's ...))

  $\lambda \to \lambda(x)$

$$\psi \to \underset{\substack{U(1)\\ \to\text{photon}}}{\underbrace{e^{-ig\lambda^1(x)}}}\;\underset{\substack{SU(N)\\ \to\text{gluon}}}{\underbrace{e^{-ig\lambda^a(x)T^a}}}\,\psi \qquad (102)$$

$SU(N)$ $\leftarrow$ properties : later

• the gauge principle in action
(here: strip off $U(1)$ factor $\rightsquigarrow$ $U = e^{-ig\alpha^a(x)T^a}$, $a=1..N^2-1$))

$$\underbrace{\bar\psi U^{-1}}_{(\bar\psi)^U}\left[i\gamma^\mu\left(\partial_\mu + \mathcal{W}_\mu^U\right) - m\right]\underbrace{U\psi}_{\stackrel{!}{=}\psi^U} = 0$$

((upper index U: gauge-transformed objects))

$$= \bar\psi\left[i\gamma^\mu\left(\partial_\mu + \underbrace{U^{-1}(\partial_\mu U) + U^{-1}\mathcal{W}_\mu^U U}_{\stackrel{!}{=}\ \mathcal{W}_\mu}\right) - m\right]\psi$$

$\hookrightarrow$ (x-dep.) $N\times N$ - matrix. traceless, because..

a) $\partial_\mu e^M = \int_0^1 ds\, e^{sM}(\partial_\mu M)\, e^{(1-s)M}$ (103)

proof: hw #14

$$\rightsquigarrow \mathrm{tr}\left(e^{-M}\partial_\mu e^M\right) = \int_0^1 ds\, \mathrm{tr}\left(e^{-M}e^{sM}(\partial_\mu M)\, e^{(1-s)M}\right)$$
$$= 1\cdot \mathrm{tr}\left((\partial_\mu M)\right)$$
$$\stackrel{!}{=} 0 \quad (T^a \text{ traceless})$$

$$\Rightarrow \mathrm{tr}\left(U^{-1}\partial_\mu U\right) = 0$$

b) can always decompose $\mathcal{W}_\mu = \underbrace{\frac{1}{N}\mathrm{tr}(\mathcal{W}_\mu)\mathbb{1}}_{\text{photon. not here.}} + \mathcal{W}_\mu$ ← traceless part

$$\Rightarrow \mathrm{tr}\left(U^{-1}\mathcal{W}_\mu^U U\right) = \mathrm{tr}\left(\mathcal{W}_\mu^U\right) = 0 \quad \text{here.}$$

$\rightsquigarrow$ can write $\mathcal{W}_\mu$ in terms of T's

$$\mathcal{W}_\mu = -ig\, A_\mu^a(x)T^a = -ig\, A_\mu(x)$$

↑ 8 gluons (N^2-1)

$$\Rightarrow (i\not{D} - m)\psi = 0\,, \quad D_\mu = \partial_\mu - igA_\mu$$

$$\left.\begin{aligned} A_\mu^U &= U A_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \\ D_\mu^U &= U D_\mu U^{-1} \end{aligned}\right\} \text{ seen from above.} \qquad (104)$$

• now, ez to check gauge invariance (g.i.) of F^2: (L-scalar, $1^{\le 4}$ ren!)

$$F_{\mu\nu}^U = \frac{i}{g}\left[D_\mu^U, D_\nu^U\right] = U F_{\mu\nu} U^{-1}$$

$$\rightsquigarrow \mathrm{tr}\left(F_{\mu\nu}^U F^{\mu\nu U}\right) = \mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) \quad \text{g.i. ✓}$$
$$\stackrel{!}{=} F_{\mu\nu}^a F^{\mu\nu b}\, \underbrace{\mathrm{tr}\left(T^aT^b\right)}_{\text{some \# } (=\frac{1}{2})}$$

$\mathrm{tr}(T^a)=0 \leadsto \det U = e^{\mathrm{tr}\ln U} = 1$

- a few $SU(N)$ relations

$$\left.\begin{array}{l} \text{generators } T^a,\quad a=1..N^2-1,\quad T^a=T^{a\dagger},\quad \mathrm{tr}(T^a)=0 \\ \text{orthonorm.:}\quad \mathrm{tr}(T^aT^b)=\frac{1}{2}\delta^{ab}\quad (\text{in fund. rep.}) \\ \text{structure const:}\quad [T^a,T^b] = i f^{abc}T^c \\ \quad \hookrightarrow \text{totally antisymm.} \\ (\!(\text{more: see e.g. Cheng \& Li, §4 ; or copy in lect notes})\!) \\ (\!(\Rightarrow F^a_{\mu\nu} \underset{\substack{\uparrow \\ \text{from (101)}}}{=} \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu)\!) \end{array}\right\} \qquad (105)$$

Faddeev - Popov

write partition fct for QCD, manifestly g.i.:

$Z_{ill} = \mathcal{N}\int\mathcal{D}[\bar\psi,\psi,A]\; e^{\int^\beta \mathcal{L}}$

but ill-defined: overcounts the A-integration!

have to consider only one representative of each equiv. class!

$\leadsto$ "fix" a gauge

$$\text{procedure:}\quad 1 = \Delta(A,B)\int\mathcal{D}U \prod_{a,x}\delta(\mathcal{F}^a[A^U]-B^a) \qquad (106)$$

B^a: indep. of A

$\mathcal{F}^a$: "gauge condition", e.g. $\mathcal{F}^a = \partial^\mu A^a_\mu$ (cov. gauge)

$\int\mathcal{D}U$: integrate over gauge group

note that Δ is g.i.: a gauge transformation of its argument A can be shifted away in integral

multiply this factor of one into Z.

then, shift $[A^a_\mu]^U \to A^a_\mu$ under $\int\mathcal{D}A$ ((and note that $\mathcal{L}$ is g.i.))

$$\Rightarrow Z_{ill} = \mathcal{N}\int\mathcal{D}[\bar\psi,\psi,A]\; e^{\int^\beta\mathcal{L}}\;\Delta(A,B)\underbrace{\int\mathcal{D}U}\prod_{a,x}\delta(\mathcal{F}^a[A]-B^a)$$

can be factored out now!

(∞) "volume" of gauge group.

next, note that Z is B-independent

$\rightsquigarrow$ can integrate over B, with some weight $e^{\int^\beta G[B]}$ $\left(\text{e.g. } G = -\frac{1}{2\alpha} B^2 \right)$

$$\Rightarrow Z = \frac{\mathcal{N}}{\underbrace{\int DB\, e^{\int^\beta G[B]}}_{Z_B}} \int D[\bar\psi, \psi, A]\; \Delta(A, \mathcal{F})\, e^{\int^\beta (\mathcal{L} + G[\mathcal{F}])} \tag{107}$$

this factor guarantees that Z stays independent of any constants in G (e.g. α). If absorbed into $\mathcal{N}$, then write e.g. $\mathcal{N}(\alpha)$.

<u>ghosts</u> goal: $\Delta(A, \mathcal{F}) = e^{\int^\beta \cdots}$

- can write Δ formally as determinant

from (106), $\Delta^{-1} = \int DU \prod_{a,x} \delta\left(\mathcal{F}^a[A^U] - B^a \right)$

go (along gauge orbit) near δ-point; then, <u>infinitesimal</u> gauge transformation suffices to cross δ-point.

$\hookrightarrow U_{xy} = 1 - ig\lambda + \mathcal{O}(g^2)$, $\lambda = \lambda^a T^a$

$\rightsquigarrow (A_\mu^U)_{xy} = \left(A_\mu^a - D_\mu^{ab} \lambda^b \right) T^a$

with $D_\mu^{ab} = \partial_\mu \delta^{ab} - g f^{abc} A_\mu^c$

$$= \int D\lambda^a \prod_{a,x} \delta\big(\underbrace{\mathcal{F}^a[A_\mu - D_\mu \lambda] - B^a}_{\equiv\, \phi^a} \big)$$

$$= \int \frac{D\phi^a}{\left| \det \frac{\delta\phi^a}{\delta\lambda^b} \right|} \prod_{a,x} \delta(\phi^a)$$

$$\Rightarrow \Delta = \left| \det M^{ab} \right| \quad , \quad M^{ab} = \frac{\delta \mathcal{F}^a[A]}{\delta A_\mu^c}\, D_\mu^{cb} \tag{108}$$

$$= \int D[\bar c c]\; e^{\int^\beta i g \bar c^a M^{ab} c^b}$$

phase; can <u>choose</u> (as seen on $\mathcal{F}_\beta$ rules).

$\bar c, c$ (<u>ghosts</u>) are Grassmann-valued fields (<u>anti</u>commuting #s).

$$\left.\begin{aligned} &\Rightarrow Z = \frac{\mathcal{N}}{Z_B} \int \mathcal{D}[\bar\psi \psi A \bar c c]\; e^{\int^\beta (\mathcal{L} + \mathcal{L}_{GF} + \mathcal{L}_{FP})} \\ &\text{with} \quad \mathcal{L} = \bar\psi (i\not{D} - m)\psi - \tfrac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} \\ &\qquad \mathcal{L}_{GF} = G[\mathcal{F}] \quad , \quad \mathcal{L}_{FP} = i g \bar c^a M^{ab} c^b \end{aligned}\right\} \tag{109}$$

choose a class of gauges

the gauge-condition $\mathcal{F}[A]$ is, up to now, an arbitrary functional of A, such that the operator M^{ab} be invertible (otherwise $\det M = 0$).

covariant gauge(s) : choose $\mathcal{F}^a[A] = \partial^\mu A_\mu^a$

and $G[B] = -\frac{1}{2\alpha} B^2$

$$\Rightarrow \quad \mathcal{L}_{GF} = -\frac{1}{2\alpha}\left(\partial^\mu A_\mu^a\right)^2 \qquad (110)$$

$$\mathcal{L}_{FP} = ig\, \bar{c}^a \partial^\mu \mathcal{D}_\mu^{ab} c^b$$

↳ some places; to compare with other authors...

Feynman rules → hw # 15

(know procedure from ϕ^4: write $\int^\beta \mathcal{L}$ in Fourier space, read off propag's, vertices)

(($f(x) = \sum_k e^{-ikx} f(k) \;\rightsquigarrow\; \partial_\mu \to -ik_\mu$))

$$\int^\beta \mathcal{L}_0 = \sum_k \left\{ \bar{\psi}_- \underbrace{(\not{K} - m)}_{(1)} \psi - \frac{1}{2} \underbrace{\left(k^2 g_{\mu\nu} - k_\mu k_\nu + \frac{1}{\alpha} k_\mu k_\nu\right)}_{(3)} A_-^\nu A^\mu \underbrace{- ig K^2}_{(2)} \bar{c}_- c \right\}$$

$$(111) \begin{cases} \Rightarrow \text{quark propagator} \overset{(1)}{\rightsquigarrow} \dfrac{-1}{\not{K} - m} \\ \text{ghost propagator} \overset{(2)}{\rightsquigarrow} \dfrac{-i/g}{K^2} \quad (\cdot \delta^{ab} \text{ : color-diagonal}) \equiv D(k)\, (\cdot \delta^{ab}) \\ \text{gluon propagator} \overset{(3)}{\rightsquigarrow} \dfrac{1}{K^2}\left(g_{\mu\nu} + (\alpha - 1)\dfrac{k_\mu k_\nu}{K^2}\right) \quad (\cdot \delta^{ab}) \equiv G_{\mu\nu}(k)\, (\cdot \delta^{ab}) \end{cases}$$

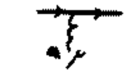
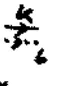
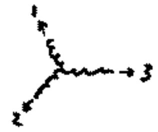
((since then $(G^{-1}(k))_{\mu\nu}\, G^{\nu\rho}(k) = g_\mu{}^\rho$ ✓))

$$\int^\beta \mathcal{L}_{int} = \sum_{K_{123}} [k_1 + k_2 + k_3] \left\{ -g g (k_3)^\mu f^{abc}\, \bar{c}^a(k_1) A_\mu^c(k_2)\, c^b(k_3) + \bar{\psi}(k_1)\, g \gamma^\mu T^a A_\mu^a(k_2)\, \psi(k_3) \right.$$

$$\left. + 3ig f^{312} \left((k_3)_1 g_{23} - (k_3)_2 g_{13}\right) \frac{1}{3!} A_1^1(k_1) A_2^2(k_2) A_3^3(k_3) \right\}$$

$$+ \sum_{K_{1234}} [k_1 + \ldots + k_4] \left\{ -6 g^2 f^{a12} f^{a34} g_{13} g_{24}\, \frac{1}{4!} A_1^1(k_1) \ldots A_4^4(k_4) \right\}$$

$$\left.\begin{array}{lll}
\Rightarrow \text{quark-gluon-vertex} & g T^a \gamma_\mu & : \\
\text{ghost-gluon-vertex} & -\xi g k_\mu f^{abc} & : \\
\text{3-gluon-vertex} & -i g f^{123}\left((k_1-k_2)_3 g_{12} + \text{cycl } 123\right) & : \\
\text{4-gluon-vertex} & -g^2 f^{a12} f^{a34}\left(g_{13}g_{24} - g_{14}g_{23}\right) + \text{cycl } 234 & :
\end{array}\right\} \quad (112)$$


((for 3- & 4-gluon-vertex, symmetry in labels 123 / 1234 was used))

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all this was like QCD @ $T=0$. Fy rules look the same (except for prefactors). What is ~~different~~ different @ $T\neq 0$ is

- $T\sum_n$ in closed loops, $k_0 = i\omega_n$, $\omega_n = \pi T \begin{cases} 2n & \text{Bose} \\ 2n+1 & \text{Fermi} \end{cases}$ ✓ learned @ ϕ^4
- Lorentz structure (new)

$\hookrightarrow$ consider the (full) 2-pt fct $G_{\mu\nu}(Q)$. ← symmetric matrix $\mu\nu$

(($T=0$: available Lorentz-structures are Q_μ, $g_{\mu\nu}$

$\rightsquigarrow$ build transverse / longitudinal projectors ← $P_T^2 = P_T$, $P_T P_L = 0$, $Q P_T = 0$, $Q P_L = Q$

$$P_T^{\mu\nu}(Q) = g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}, \quad P_L^{\mu\nu}(Q) = \frac{Q^\mu Q^\nu}{Q^2} \quad))$$

available L-structures are $g_{\mu\nu}$, Q_μ, U_μ ← 4-velocity of heat bath; rest frame: $U = (1, \vec{0})$, $U^2 = 1$

$\rightsquigarrow$ can build $\underline{\underline{4}}$ symmetric tensors, e.g.

$$g_{\mu\nu}, \; U_\mu U_\nu, \; U_\mu Q_\nu + Q_\mu U_\nu, \; Q_\mu Q_\nu$$

$\rightsquigarrow$ more useful: set of orthogonal tensors.

def. $$V_\mu = Q^2 U_\mu - (Q^\nu U_\nu) Q_\mu, \quad V^2 = -Q^2 q^2 \qquad (113a)$$

$$(\Rightarrow V \cdot Q = Q^2 (U \cdot Q) - (Q \cdot U) Q^2 = 0 \;\rightarrow V \text{ is projector})$$

choose
$$\left.\begin{aligned}
A_{\mu\nu} &= g_{\mu\nu} - B_{\mu\nu} - D_{\mu\nu} \\
B_{\mu\nu} &= \frac{1}{V^2} V_\mu V_\nu \\
C_{\mu\nu} &= \frac{1}{\sqrt{2}\, Q^2 q} \left(Q_\mu V_\nu + V_\mu Q_\nu\right) \qquad (q = |\vec{Q}|) \\
D_{\mu\nu} &= \frac{1}{Q^2} Q_\mu Q_\nu
\end{aligned}\right\} \quad (113)$$