that was 'multiplicative renormalization'.

now: counterterms

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi_0)\partial^\mu \phi_0 - \tfrac{1}{2} m_0^2 \phi_0^2 - \tfrac{\lambda_0}{4!}\phi_0^4$$

$$\phi_0 = \sqrt{Z_\phi}\,\phi \ , \quad m_0^2 = m^2 + \delta m^2 \ , \quad \lambda_0 = \frac{Z_g}{Z_\phi^2}\lambda$$

(index 'R' omitted from now on)

$$= \tfrac{1}{2}(\partial_\mu \phi)\partial^\mu \phi - \tfrac{1}{2} m^2 \phi^2 + \mathcal{L}_{int}$$

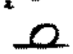
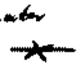
$$\left.\begin{aligned} \mathcal{L}_{int} &= -\frac{\lambda}{4!}\underbrace{Z_g}_{1+O(\lambda)}\phi^4 - \frac{1}{2}\underbrace{\delta m^2}_{\lambda+O(\lambda^2)}\underbrace{Z_\phi}_{1+O(\lambda^2)}\phi^2 + \frac{1}{2}\underbrace{(Z_\phi - 1)}_{O(\lambda^2)}\left((\partial\phi)^2 - m^2\phi^2\right) \\ &= -\frac{\lambda}{4!}\phi^4 - \frac{1}{2}\left(\delta m^2|_\lambda\right)\phi^2 + O(\lambda^2) \end{aligned}\right\} \quad (95)$$

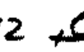
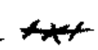
(end of digression)

→ this way, the (mult.) ren. can be incorporated into our diagrammatic expansions:

$$\ln Z_{int} = 3$$  $$+$$  $$+ O(\lambda^2)$$

renormalized quantities

$$G_2 = G_0 + \overset{\frac{4\cdot 3\cdot 2}{2}\text{ etc.}}{12}$$  $$+ \overset{\text{etc.}}{2}$$  $$+ O(\lambda^2)$$

$$\Pi = -12$$  $$- 2$$  $$+ O(\lambda^2)$$

$$= -12\left(-\frac{\lambda}{4!}\right)\underbrace{\sum_k G_0(k)}_{= I^{vac} + I^{mat}} - 2\Big(-\frac{1}{2}\underbrace{\delta m^2|_\lambda}_{= -\Pi_{T=0}(m^2)|_\lambda = 12\left(-\frac{\lambda}{4!}\right)I^{vac}}\Big) \qquad (96)$$

$$= -12\left(-\frac{\lambda}{4!}\right) I^{mat}$$

now, we can finally

- evaluate $\ln Z^{(1)}$

$$\ln Z^{(1)} = 3$$  $$+$$ 

$$= 3\left(-\frac{\lambda}{4!}\right)\beta V\left(I^{vac} + I^{mat}\right)^2 + \left(-\frac{1}{2}\right)12\left(-\frac{\lambda}{4!}\right)I^{vac}\,\beta V\left(I^{vac} + I^{mat}\right)$$

$$= 3\left(-\frac{\lambda}{4!}\right)\beta V\left[(I^{mat})^2 - (I^{vac})^2\right] \qquad (97)$$

$\rightsquigarrow$ in (97), the terms vac-mat have cancelled !!!

still, $\ln Z^{(1)}$ contains divergences : $(I^{vac})^2$, but is not physical.

physical quantities $F = -T\ln Z$, $p = T\partial_V \ln Z$?
$= p_0 + p_{int} = p_0 + p^{(1)} + \dots$

$$p^{(1)} = T\partial_V \ln Z^{(1)} = 3\left(-\frac{\lambda}{4!}\right)\left[(I^{mat})^2 - (I^{vac})^2\right] = p^{(1)}_{mat} + p^{(1)}_{vac}$$
(splits only because no vac-mat in $\ln Z$!)

one measures only differences (in energy, pressure, ...)

$\rightsquigarrow$ can choose to normalize $p_{vac} = 0$

$$\Rightarrow p^{(1)}_{phys} = p^{(1)}_{mat} = 3\left(-\frac{\lambda}{4!}\right)\underbrace{\left(\frac{T^2}{2\pi^2}\Gamma(3)\,h_3(y)\right)^2}_{\approx \frac{T^4}{144}(1+O(\frac{m}{T}))} \qquad \left(y = \frac{m}{T}\right)$$

$$\overset{m=0}{=} -\frac{\lambda}{4!}\frac{T^4}{48} \tag{98}$$

- evaluate $\ln Z^{(2)}$

$$\ln Z^{(2)} = 36\ \text{OOO} + 12\ \text{⦶} + \text{ct's}$$
(counterterms)

$$= 36\,\beta V\left(\frac{\Pi^{(1)}}{-12}\right)^2 \sum_{Q_1,Q_2}[Q_1+Q_2]\,G_0(Q_1)G_0(Q_2)$$
$$+12\,\beta V\left(-\frac{\lambda}{4!}\right)^2 \sum_{Q_1\dots 4}[Q_1+\dots+Q_4]\,G_0(Q_1)\dots G_0(Q_4) \quad + \text{ct's}$$

... heading for next term in pressure $\boxed{@\ m=0}$ ...

((could now proceed to calc above int's, renormalize (add ct's), and then be surprised that a (new type of) divergence occurs. So, to make the story shorter, investigate cure first!))

look at IR (infrared; small momentum) behaviour

2^{nd} diag, $n_1 = n_2 = n_3 (= n_4) = 0$ — from $(T\sum_n)^3$

$$\sim T^3\int d^3p_1\dots d^3p_4\ \frac{1}{p_1^2}\cdots\frac{1}{p_4^2}\,\delta(p_1+\dots+p_4)$$
$$= T^3\int d^3p_1\dots d^3p_3\ \frac{1}{p_1^2}\frac{1}{p_2^2}\frac{1}{p_3^2}\frac{1}{(p_1+p_2+p_3)^2}$$
$$\sim T^3\int_0^\infty dp_1\dots dp_3\ 1\cdot 1\cdot 1\cdot\frac{1}{(p_1+p_2+p_3)^2}$$
$$\sim T^3\, dp \qquad \text{IR convergent} \checkmark \quad \text{no danger.}$$

1^{st} diag $\sim \overset{\text{from } T\sum_n}{T} \int d^3p_1\, d^3p_2\, \frac{1}{p_1^2}\frac{1}{p_2^2}\,\delta(p_1+p_2)$

$= T\int d^3p_1\, \frac{1}{p_1^2}\frac{1}{p_1^2}$

$\sim T\int_0^\infty dp_1\; 1\cdot\frac{1}{p_1^2}$ IR divergent !

new $T\neq 0$ – effect ! vanishes at $T\to 0$

((physical reason for div: expanding with massless propagator; know that $\Pi_1^{ren} = \frac{\lambda}{4!}T^2$ is dynamically generated mass term))

IR div. gets worse in higher orders:

$\sim \beta V\,(\Pi_1)^N\; T\int d^3p\,\left(\frac{1}{p^2}\right)^N \sim \beta V\,(\Pi_1)^N\, T\int dp\,\left(\frac{1}{p^2}\right)^{N-1}$

simple comb. factor: $\underset{e^{a}}{\frac{1}{N!}}\;\underset{\text{connect}}{(3!}\cdot\underset{\text{connect to loop}}{2)^N}\;\underset{\text{\# of ways to order } 1..N \text{ on circle}}{\frac{(N-1)!}{2}}$ ((check: $N=2$, $\frac{1}{2!}(3!\cdot 2)^2\frac{1!}{2}=36$ ✓))

due to this simple structure, this class of diag's can actually be summed to all orders:

$$\text{oOo} + \ldots = \sum_{N=2}^\infty \frac{(3!\cdot 2)^N}{2N}\,\beta V\left(\frac{\Pi_1}{-12}\right)^N \sum_k \left(G_0(k)\right)^N$$

$$= \frac{1}{2}\beta V\sum_k\sum_{N=2}^\infty \frac{(-)^N}{N}\left(\Pi_1 G_0(k)\right)^N$$

$$= \frac{1}{2}\beta V\sum_k\Big(\underbrace{\sum_{N=1}^\infty \frac{(-)^N}{N}\left(\Pi_1 G_0\right)^N}_{\overset{!}{=}-\ln(1+\Pi_1 G_0)} + \Pi_1 G_0\Big) \qquad (99a)$$

'ring diagrams';

include mass counterterm $\Rightarrow \Pi_1\to\Pi_1^{ren}\left(= -12\left(-\frac{\lambda}{4!}\right)I^{mat}\big|_{m=0} = \frac{\lambda}{4!}T^2\right)$

(proof: see hw)

$\rightsquigarrow$ using $\Pi_1^{\,rm} = \frac{\lambda}{4!}T^2 \quad (m=0)$,

$$\text{ring diag's} = -\tfrac{1}{2}\beta V \sum_K \left(\ln\left(1+\frac{\frac{\lambda}{4!}T^2}{\omega_n^2+\vec{k}^2}\right) - \frac{\frac{\lambda}{4!}T^2}{\omega_n^2+\vec{k}^2}\right)$$

$$= -\tfrac{1}{2}\beta V T \int \frac{d^3k}{(2\pi)^3}\left(\ln\left(1+\frac{\frac{\lambda}{4!}T^2}{\vec{k}^2}\right) - \frac{\frac{\lambda}{4!}T^2}{\vec{k}^2}\right) \quad + \text{ nonstatic modes}$$

((n≠0 -modes; give higher corr. in λ))

static (n=0) modes only $\longrightarrow$

$$-\tfrac{1}{2}\beta V T \sqrt{\tfrac{\lambda}{4!}T^2}^{\,3}\,\frac{1}{2\pi^2}\int_0^\infty dk\, k^2\left(\ln\left(1+\frac{1}{k^2}\right) - \frac{1}{k^2}\right)$$

makes UV well behaved

finite integral ✓

(in particular, IR finite !!!)

$$\int_0^\infty dk\, k^2\left(\ln\left(1+\frac{1}{k^2}\right)-\frac{1}{k^2}\right) = -\frac{\pi}{3}$$

$$= \frac{\beta V}{12\pi}\left(\frac{\lambda}{4!}\right)^{3/2} T^4 \qquad (99b)$$

$\Rightarrow$ the 'NNLO' of $\ln Z$ (and $F, p, \ldots$) is actually of order $\lambda^{3/2}$, not λ^2 like in naive (no resummations) pert. expansion!

collecting from (65), (98) and (99b),
we get for the pressure of massless $\frac{\lambda}{4!}\phi^4$ – theory

$$\boxed{p_{(phys)} = T^4\left(\frac{\pi^2}{90} - \frac{1}{48}\frac{\lambda}{4!} + \frac{1}{12\pi}\left(\frac{\lambda}{4!}\right)^{3/2} + O(\lambda^2)\right)} \qquad (100)$$

... higher orders : see literature
e.g. [Parwani / Singh , PRD 51 (1995) 4518] $\leftarrow \ldots \lambda^{5/2}$
and ref's therein

convergence ... resummation ...