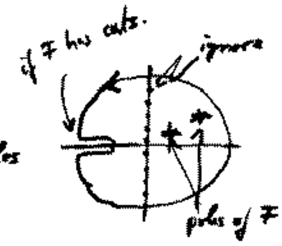
5. (close contour) $\quad$ if $F(z) \overset{|z|\gg 1}{\sim} z^{-\lambda}, \ \lambda > 1$

then no surface terms $\rightsquigarrow$ can close contour



(86)

version 3 $\rightarrow$ $$T\sum_n F(i\omega_n) = -\frac{\eta}{2\pi i}\oint dz\, F(z)\, n(z)\Big|_{\text{ignore poles of } n(z)}$$

version 4 $\rightarrow$ $$T\sum_n F(i\omega_n) = -\frac{\eta}{2\pi i}\int_D dz\, \big(F(z)+F(-z)\big)\left(\frac{\eta}{2}+n(z)\right)$$

example 1 $\quad \eta = 1, \quad F(z) = \frac{-1}{z^2-\omega^2} = F(-z), \quad \omega^2 = \vec{k}^2 + m^2$

$$\Rightarrow T\sum_n G_0(k) = T\sum_n \frac{-1}{k^2-m^2} \quad \left(= T\sum_n \frac{1}{\omega_n^2+\omega^2} \right)$$

$$= +\frac{1}{2\pi i}\int_D dz\; 2\,\frac{1}{|z|+\omega}\,\frac{1}{|z|-\omega}\left(\frac{1}{2}+n(z)\right)$$

$$= \frac{1}{\omega}\left(\frac{1}{2}+n(\omega)\right) \tag{87}$$

cf. $FT^2/33$ ✓✓

or $$= \frac{1}{2\pi i}\underbrace{\int_{-i\infty}^{i\infty} dz\, \frac{-1}{z^2-\omega^2}}_{z=-ik_4} + \frac{1}{2\pi i}\int_D dz\; 2\,\frac{1}{|z|+\omega}\,\frac{1}{|z|-\omega}\, n(z)$$

$$= \int_{-\infty}^{\infty}\frac{dk_4}{2\pi}\,\frac{1}{k_4^2+\omega^2} + \frac{n(\omega)}{\omega} \tag{88}$$

> also ✓

example 2 $$\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$= \lim_{\omega\to 0}\frac{(2\pi T)^2}{2}\left[\frac{1}{T}\, T\sum_{n=-\infty}^{\infty}\frac{1}{\omega_n^2+\omega^2} - \frac{1}{\omega^2}\right]$$

subtracts $n=0$ term

$$\overset{(87)}{=} \frac{1}{\omega}\left(\frac{1}{2}+n(\omega)\right)$$

$$= \frac{\pi^2}{6}$$

$$\zeta(4) = \sum_{n=1}^{\infty}\frac{1}{n^4} = \text{(same strategy)} = \frac{\pi^4}{90}$$

• self-Energy Π

$$\widetilde{G}_2(k_1,k_2) = [k_1+k_2]\, G_2(k_1)$$ ← no tilde ⇔ δ-fct stripped off

full propg. → $G_2(k) \equiv \dfrac{G_0(k)}{1+G_0(k)\Pi(k)} = \dfrac{-1}{k^2-m^2-\Pi(k)}$ ← bare propg.

$$\Leftrightarrow \quad \Pi(k) = \frac{1}{G_2(k)} - \frac{1}{G_0(k)} \qquad (89)$$

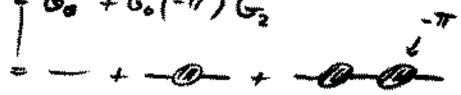
$G_2 = G_0 + 12$ $+ (144$  $+ 144$  $+ 96$  $) + O(\lambda^3)$

$\rightsquigarrow \Pi = -12$  $- (144$ $+ 96$  $) + O(\lambda^3)$ (external G_0's cut off) (90)

⇒ the diagram  has cancelled ("one-particle-reducible")

((this is a general feature: Π only contains 1PI diags. (one-part.-irreducible)
proof: see e.g. [R.H. Brandenberger, Rev Mod Phys 47 (1985) 1]))

((if one knows Π, then $G_2 = G_0 + G_0(-\Pi)G_2$

))

• evaluate $\Pi^{(1)}$

$$\Pi^{(1)} = -12\left(-\frac{\lambda}{4!}\right) \underbrace{\sum_Q G_0(Q)}_{=I^{vac}+I^{mat}}$$

(88) $\rightsquigarrow I^{mat} = \int_{\vec k} \dfrac{n(\omega)}{\omega} = \dfrac{T^2}{2\pi^2}\, \underbrace{\Gamma(3)\, h_3(y)}_{=\frac{\pi^2}{6}-\frac{\pi}{2}y+O(y^2)}$ high-T-expansion; $y = \frac{m}{T}$

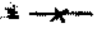
, $I^{vac} = \int_{k_E} \dfrac{1}{k_E^2+m^2}$ ← does not know about Temperature! ← Euclidean ($k_E^2 = \vec k^2 + k_4^2$)

'vacuum'-part is divergent!

$\rightsquigarrow$ regulate integral, e.g. by momentum cutoff Λ

$$I^{vac} = \frac{2\pi^2}{\Gamma(2)}\frac{1}{(2\pi)^4}\int_0^\Lambda dk\, \frac{k^3}{k^2+m^2} \qquad \text{used } \int d^n k = \frac{2\pi^{n/2}}{\Gamma(n/2)}\int_0^\infty dk\, k^{n-1} \qquad (91)$$

$$= \frac{1}{8\pi^2}\left(\frac{\Lambda^2}{2} - \frac{m^2}{2}\ln\frac{\Lambda^2}{m^2} - \frac{m^2}{2}\ln\left(1+\frac{m^2}{\Lambda^2}\right)\right)$$

(⇒ add counterterm to $\mathcal{L}$: $-\frac{1}{2}\delta m^2 \phi^2 =$ 

$\Pi \to \Pi^{ren} = \Pi + 2$  $= \Pi + \delta m^2$

choose counterterm such that $\Pi^{(1),ren}_{vac} = 0$) more details: ↓

have found that $T=0$ - piece of $\Pi^{(1)}$ is divergent.

→ digression: $(T=0)$ - renormalization (if needed ...)

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi_0)\partial^\mu \phi_0 - \tfrac{1}{2} m_0^2 \phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4$$

index 0: 'bare' quantities. correct, but not measurable.

mass of ϕ bosons = pole of propagator

(computed with bare theory)

$$G_2^{(0)}(k) = \frac{-1}{k^2 - m_0^2 - \Pi(k^2)} \equiv \frac{1}{F(k^2)}$$

$$\rightsquigarrow \; F(m^2) = 0 \quad , \text{ or } \quad m^2 = m_0^2 - \delta m^2 \quad , \quad \delta m^2 = -\Pi_{T=0}(m^2) \qquad (92)$$

Taylor-expand around $k^2 = m^2$:

$$F(k^2) = \underbrace{F(m^2)}_{=0} + (k^2 - m^2)\underbrace{F'(m^2)}_{= \Pi'(m^2) - 1} + (k^2 - m^2)^2 \underbrace{C(k^2)}_{= \frac{1}{2}F''(m^2) + (k^2 - m^2)\cdots + \cdots}$$

$$\rightsquigarrow \; G_2^{(0)} = \frac{1}{(m^2 - k^2)\left(1 - \Pi'(m^2) - (k^2 - m^2)C\right)}$$

$$= \frac{\frac{1}{1 - \Pi'(m^2)}}{(m^2 - k^2)\left(1 - \frac{k^2 - m^2}{1 - \Pi'(m^2)} C\right)}$$

was def'd as

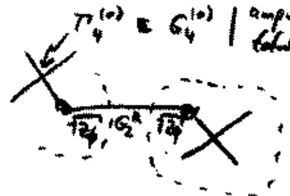
$$= FT\{\langle \hat{\phi}_0^H \hat{\phi}_0^H \rangle\} \equiv Z_\phi G_2^{(R)} \quad \text{(renormalized)}$$

$$\phi_0 = \sqrt{Z_\phi}\, \phi_R$$

$$Z_\phi = \frac{1}{1 - \Pi'(m^2)} = 1 + O(\lambda^2) \quad , \text{ since } \Pi \text{ indep. of } k \text{ in } O(\lambda) \qquad (93)$$

coupling $\quad G_2^{(R)}$ finite.

$\Gamma_4^{(0)} \equiv G_4^{(0)} \big|_{\text{amputated, totally conn.}}$



$= \Gamma_4^{(R)} \overset{!}{=}$ finite

$$\rightsquigarrow \text{ finite} \overset{!}{=} Z_\phi^2\, \Gamma_4^{(0)} \Big|_{\text{after } \lambda\text{-renorm.}}$$

$$\Gamma_4^{(0)} = \times + \times\!\!\times + \cdots$$

$$= \left(-\frac{\lambda_0}{4!}\right)\Big[\underbrace{1 - \frac{\lambda_0}{4!}\cdot \log \text{UV div} + \cdots}_{\equiv \frac{1}{Z_g}}\Big]$$

$$\rightsquigarrow \; -\frac{\lambda_R}{4!} \overset{!}{=} Z_\phi^2\left(-\frac{\lambda_0}{4!}\right)\frac{1}{Z_g} \qquad (94)$$

that was 'multiplicative renormalization'.

now: counterterms

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_0)\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4$$

$\phi_0 = \sqrt{Z_\phi}\,\phi$, $m_0^2 = m^2 + \delta m^2$, $\lambda_0 = \frac{Z_\lambda}{Z_\phi^2}\lambda$

((index 'R' omitted from now on))

$$= \frac{1}{2}(\partial_\mu\phi)\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \mathcal{L}_{int}$$

$$\mathcal{L}_{int} = -\frac{\lambda}{4!}\underbrace{Z_\lambda}_{1+O(\lambda)}\phi^4 - \frac{1}{2}\underbrace{\delta m^2}_{\lambda+O(\lambda^2)}\underbrace{Z_\phi}_{1+O(\lambda^2)}\phi^2 + \frac{1}{2}\underbrace{(Z_\phi-1)}_{O(\lambda^2)}\left((\partial\phi)^2 - m^2\phi^2\right) \quad (95)$$

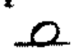
$$= -\frac{\lambda}{4!}\phi^4 - \frac{1}{2}\left(\delta m^2|_\lambda\right)\phi^2 + O(\lambda^2)$$

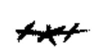
(end of digression)

→ this way, the (mult.) ren. can be incorporated into our diagrammatic expansions:

$\ln Z_{int} = 3$  $+$  $+ O(\lambda^2)$

renormalized quantities

$G_2 = G_0 + 12^{\frac{4\cdot 3}{2}\cdot 2\cdot\text{alter}}$  $+ 2^{\text{alter}}$  $+ O(\lambda^2)$

$\Pi = -12$  -2  $+ O(\lambda^2)$

$$= -12\left(-\frac{\lambda}{4!}\right)\underbrace{\sum_k G_0(k)}_{= I^{vac} + I^{mat}} - 2\left(-\frac{1}{2}\underbrace{\delta m^2|_\lambda}_{= -\Pi_{T=0}(m^2)|_\lambda = 12\left(-\frac{\lambda}{4!}\right)I^{vac}}\right) \quad (96)$$

$$= -12\left(-\frac{\lambda}{4!}\right)I^{mat}$$

now, we can finally

- evaluate $\ln Z^{(1)}$

$\ln Z^{(1)} = 3$  $+$ 

$$= 3\left(-\frac{\lambda}{4!}\right)\beta V\left(I^{vac} + I^{mat}\right)^2 + \left(-\frac{1}{2}\right)12\left(-\frac{\lambda}{4!}\right)I^{vac}\,\beta V\left(I^{vac} + I^{mat}\right)$$

$$= 3\left(-\frac{\lambda}{4!}\right)\beta V\left[(I^{mat})^2 - (I^{vac})^2\right] \quad (97)$$