have used the following expansions:

(*): $y^{-\varepsilon} = 1 - \varepsilon \ln y + O(\varepsilon^2)$

$\frac{1}{2} B\left(\frac{1-\varepsilon}{2}, \frac{\varepsilon}{2}\right) = \frac{1}{\varepsilon} + \ln 2 + O(\varepsilon)$

(**): $\frac{1}{2} B\left(\frac{1-\varepsilon}{2}, \frac{1+\varepsilon}{2}\right) = \frac{\pi}{2} + O(\varepsilon^2)$

$\frac{1}{(2\pi)^\varepsilon} \zeta(1+\varepsilon) = \frac{1}{\varepsilon} + \gamma - \ln(2\pi) + O(\varepsilon)$

c) get $n=3$, $n=5$ -cases via integration

$$h_3(z) = \frac{\pi^2}{2}\left(\frac{1}{6} - z - z^2\left(\ln\frac{z}{2} + \gamma - \frac{1}{2}\right)\right) - \sum_{j=1}^{\infty} \frac{c_j}{j+1} \zeta(2j-1)\, z^{2j+2} \tag{76}$$

$$h_5(z) = \frac{\pi^4}{4!}\left(\frac{1}{15} - z^2 + 4z^3 + 3z^4\left(\ln\frac{z}{2} + \gamma - \frac{3}{4}\right)\right) + \sum_{j=1}^{\infty} \frac{6c_j}{(j+1)(j+2)} \zeta(2j-1)\, z^{2j+4} \tag{77}$$

etc...

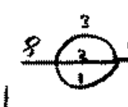
- 2nd order of pert. expansion (NNLO)

$$Z_{int} = 1 + 3\, \infty + \tfrac{1}{2}\left(9\, \infty\infty + 72\, \mathrm{ooo} + 24\, \oplus\right) + O(\lambda^3)$$

$$\ln Z_{int} = 3\, \infty + \left(36\, \mathrm{ooo} + 12\, \oplus\right) + O(\lambda^3) \tag{78}$$

(($\ln(1+x) = x - \frac{1}{2}x^2 + \dots$))

$\Rightarrow$ products of vacuum diag's cancelled!

ex:  $: 8\cdot 4\cdot 3\cdot 2\cdot 1$

(($\frac{1}{1+x} = 1 - x + x^2 + \dots$))

$$\tilde{G}_2 = \frac{1}{Z_{int}}\Big(— + \left(12\, \text{-o-} + 3\, \text{-}\infty\text{-}\right) + \tfrac{1}{2}\big(288\, \text{-8-} + 288\, \text{-oo-} + 192\, \text{-}\theta\text{-} + 72 + 9 + 72 + 24\big) + O(\lambda^3)\Big)$$

$$= — + 12\, \text{-o-} + \left(144\, \text{-8-} + 144\, \text{-oo-} + 96\, \text{-}\theta\text{-}\right) + O(\lambda^3) \tag{79}$$

$\Rightarrow$ vacuum subdiagrams cancelled!

these cancellations are a general feature of the perturbative expansion.

$\Rightarrow$ present a general (all-order) proof! $\rightsquigarrow$ $FT^2/36$, $FT^2/37$

• proof of vac-subdiag-cancellation

a) Z_{int}

$$\boxed{Z_{nt} = \left[e^{Q} W_0[j] \right]_{j=0}} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left[Q_1 \dots Q_n W_0 \right]_{j=0}$$

general structure: vertices Q (from $S^{(n)}_{int}$) contain δ-op's.

these can a) pull a $j \cdot G_0$ out of $W_0 = e^{\frac{1}{2} j \cdot G_0 j}$

b) kill a j downstairs

in the end, no j's or δ's left, $W_0[0]=1$

⇒ vertices connect in all possible ways.

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{m=1}^{\infty} \frac{1}{m!} \boxed{\sum_{\{r_i\}} \left[Q^{r_1} W_0 \right]_{con} \cdots \left[Q^{r_m} W_0 \right]_{con}} \equiv V_{n,m}$$

$$r_i \geq 1, \quad \sum_{i=1}^{m} r_i = n$$

groups

to distribute enumerated Q's over m numbered groups

m numerated groups overcount!

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{m=1}^{n} \frac{1}{m!} V_{n,m}$$

$$V_{n,1} = \left[Q^n W_0 \right]_{con}$$

$$V_{n,n} = n! \left(\left[Q W_0 \right]_{con} \right)^n$$

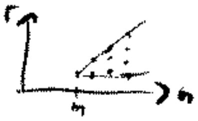
$$V_{n,m\geq 2} = \sum_{r=1}^{n-(m-1)} \binom{n}{r} \left[Q^r W_0 \right]_{con} V_{n-r,m-1} \qquad \text{(recursion)}$$

# of ways to pick r out of n Q's (and put in additional group)

$$\rightarrow \sum_{n=1}^{\infty} \sum_{m=1}^{n} = \sum_{m=1}^{\infty} \sum_{n=m}^{\infty}$$

$$= 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \underbrace{\boxed{\sum_{n=m}^{\infty} \frac{1}{n!} V_{n,m}}}_{\equiv S_m}, \quad S_1 = \sum_{n=1}^{\infty} \frac{1}{n!} V_{n,1} = \sum_{n=1}^{\infty} \frac{1}{n!} \left[Q^n W_0 \right]_{con} = \left[(e^{Q}-1) W_0 \right]_{con}$$

$$m \geq 2: \; S_m = \sum_{n=m}^{\infty} \frac{1}{\cancel{n!}} \sum_{r=1}^{n-(m-1)} \frac{\cancel{n!}}{r!(n-r)!} \left[Q^r W_0 \right]_{con} V_{n-r,m-1}$$



$$\rightarrow \sum_{n=m}^{\infty} \sum_{r=1}^{n-(m-1)} = \sum_{r=1}^{\infty} \sum_{n=(m-1)+r}^{\infty}$$

$$= \underbrace{\sum_{r=1}^{\infty} \frac{1}{r!} \left[Q^r W_0 \right]_{con}}_{2.\; = S_1} \; \underbrace{\sum_{n=m-1+r}^{\infty} \frac{1}{(n-r)!} V_{n-r,m-1}}_{1.\; = \sum_{n=m-1}^{\infty} \frac{1}{n!} V_{n,m-1} = S_{m-1}}$$

$$S_m = S_1 \cdot S_{m-1} \quad \rightsquigarrow \quad S_m = S_1^{\,m}$$

$$Z_{mf} = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} S_1^{\,m} = e^{S_1}$$

$$\Rightarrow \boxed{\ln Z_{mf} = S_1 = \left[(e^{Q}-1) W_0 \right]_{con}} \tag{80}$$

$\rightsquigarrow$ there are <u>no</u> products of clouds in $\ln Z_{mf}$!!!

((does this only apply to a <u>one-vertex-theory</u> ??))

---

6) $W[j]$

$$W[j] = \frac{1}{Z_{mf}} \; e^{Q} W_0[j] \qquad (\underline{\text{not}}\ j=0)\ ,\ W[0]=1$$

$$= \frac{1}{Z_{mf}} \left(W_0[j] + \sum_{n=1}^{\infty} \frac{1}{n!} \boxed{Q_1 \ldots Q_n W_0[j]} \right)$$

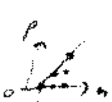
philo: p vertices are <u>not</u> part of vac-subdiag's (=clouds), the rest $(n-p)$ connects to clouds

$$= \sum_{p=0}^{n-1} \left(Q^p W_0 \right)_{\text{cloud free}} \binom{n}{p} \sum_{m=1}^{n-p} \frac{1}{m!} \underbrace{\sum_{\text{config's}} \left[Q^{r_1} W_0 \right]_{con} \cdots \left[Q^{r_m} W_0 \right]_{con}}_{= V_{n-p,m}}$$

$$\sum_i r_i = n-p \ , \quad r_i \geq 1$$

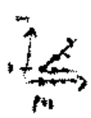
$$\overset{p=n\text{-term}}{+} \left(Q^n W_0 \right)_{cf} \binom{n}{n}_{=1}$$

$$= \frac{1}{Z_{mf}} \left(\left(e^{Q} W_0 \right)_{cf} + \sum_{n=1}^{\infty} \cancel{\frac{1}{n!}} \sum_{p=0}^{n-1} \left(Q^p W_0 \right) \frac{\cancel{n!}}{p!\,(n-p)!} \sum_{m=1}^{n-p} \frac{1}{m!} V_{n-p,m} \right)$$



$$= \sum_{p=0}^{\infty} \sum_{n=p+1}^{\infty}$$

$$= \sum_{m=1}^{\infty} \sum_{n=m+p}^{\infty}$$



$$= \frac{1}{Z_{mf}} \left(\left(e^{Q} W_0 \right)_{cf} + \sum_{p=0}^{\infty} \frac{1}{p!} \left(Q^p W_0 \right)_{cf} \sum_{m=1}^{\infty} \frac{1}{m!} \underbrace{\sum_{n=m+p}^{\infty} \frac{1}{(n-p)!} V_{n-p,m}}_{\overset{!}{=} S_m = S_1^{\,m}} \right)$$

$$= \frac{1}{Z_{mf}} \left(\cancel{\left(e^{Q} W_0 \right)_{cf}} + \left(e^{Q} W_0 \right)_{cf} \left(e^{S_1} \cancel{-1} \right) \right) \qquad S_1 = \left[(e^{Q}-1) W_0 \right]_{con}$$

$$= \frac{e^{S_1}}{Z_{mf}} \left(e^{Q} W_0 \right)_{cf}$$

$$\boxed{W[j] = \left(e^{Q} W_0 \right)_{\text{cloud-free}}} \tag{81}$$

one can do more:

- $\ln W$ generates <u>totally</u> con

((G_4 :

not tot con   tot con))

(one more) technical tool:

- frequency sums by contour integration

((have to handle, e.g., $T\sum_{n=-\infty}^{\infty}\frac{1}{K^2}$, $K=(i\omega_n,\vec{k})$, $\omega_n=\pi(2n+\{{}^0_1\})T$))
$K^2=(i\omega_n)^2-\vec{k}^2$ $\quad$ $\llcorner$ Fermi

more generally, $T\sum_{n=-\infty}^{\infty}F(i\omega_n)$ ← analytic fct (no poles on imag. axis, see below)

can trade $\sum_n$ for integral → easier ?!

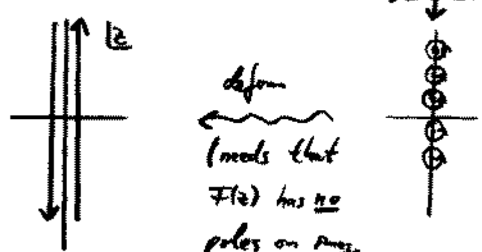
1. (make harder) $\quad F(i\omega_n)=\frac{1}{2\pi i}\oint_n dz\,\frac{F(z)}{z-i\omega_n}$ $\quad$ (82)

2. (play) $\quad \underbrace{\frac{T}{z-i\omega_n}}_{\text{inf. small!}}=\frac{1}{e^{\beta(z-i\omega_n)}-1}=\frac{1}{\boxed{\eta}e^{\beta z}-1}$

$\qquad\llcorner = e^{-i\beta(2n+\{{}^0_1\})T}=\pm 1 \quad\rightsquigarrow \eta^2=1$

$= \eta\, n(z)\quad,\quad n(z)=\frac{1}{e^{\beta z}-\eta}$ $\quad$ (83)

3. (deform contour) $\quad T\sum_{n=-\infty}^{\infty}F(i\omega_n)\overset{1.}{=}T\sum_n\frac{1}{2\pi i}\oint_n dz\,\frac{F(z)}{z-i\omega_n}\overset{2.}{=}\frac{\eta}{2\pi i}\sum_n\oint_n dz\,F(z)\,n(z)$



deform

(needs that $F(z)$ has no poles on imag. axis)

$$\Rightarrow T\sum_{n=-\infty}^{\infty}F(i\omega_n)=\frac{\eta}{2\pi i}\int_{\downarrow\uparrow}dz\,F(z)\,n(z) \qquad (84)$$

4. (rewrite) $\quad$ in left integral ($\int_\downarrow$), let $z\to -z$

$$\int_\downarrow dz\,F(z)n(z)=\int_{i\infty-\varepsilon}^{-i\infty-\varepsilon}dz\,F(z)n(z)=-\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon}dz\,F(-z)n(-z)=-\int_\uparrow dz\,F(-z)n(z)$$

note $\quad n(-z)=\frac{1}{e^{-\beta z}-\eta}=-\eta-\frac{1}{e^{\beta z}-\eta}=-\eta-n(z)$

$$\Rightarrow T\sum_{n=-\infty}^{\infty}F(i\omega_n)=\frac{1}{2\pi i}\int_\uparrow dz\,F(-z)+\frac{\eta}{2\pi i}\int_\uparrow dz\,\big(F(z)+F(-z)\big)\,n(z)$$

$$=\underbrace{\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dz\,\tfrac{1}{2}\big(F(z)+F(-z)\big)}_{\text{"}T=0\text{"-piece}}+\frac{\eta}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon}dz\,\big(F(z)+F(-z)\big)\,n(z) \qquad (85)$$

version of Kapusta (2.40)