calculate next-to leading order (NLO) - terms ← the fun begins. tame combinatorics → invent diagrams

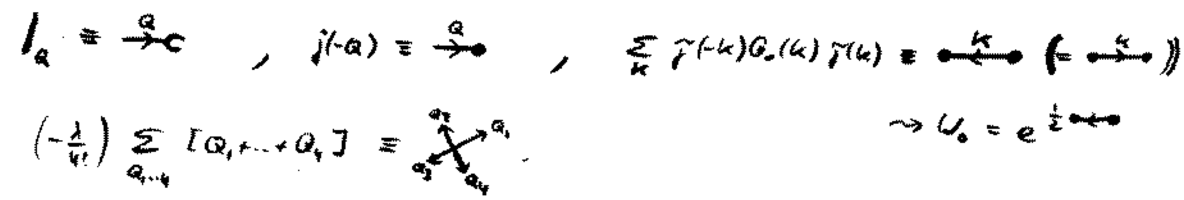
$$Z_{int} = 1 + QW_0|_{\tilde{j}=0} + O(\lambda^2) \equiv 1 + Z^{(1)}_{int} + O(\lambda^2)$$

$$Z^{(1)}_{int} = \left(-\frac{\lambda}{4!}\right) \sum_{Q_1 \dots 4} [Q_1 + \dots + Q_4] \, I_{Q_1} \dots I_{Q_4} \, e^{\sum \frac{1}{2} \tilde{j} \cdot G_0 \tilde{j}}$$

→ recursively apply the two relations

$$I_Q W_0[\tilde{j}] = \tilde{j}(-Q) G_0(Q) W_0[\tilde{j}] \quad (a); \qquad I_K \tilde{j}(-Q) = [K+Q] \quad (b)$$

to do so, can use some primitive pictures:

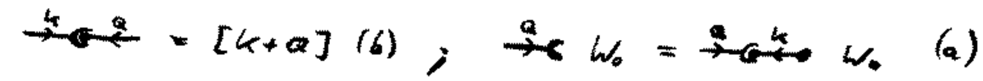


$$I_Q \equiv \; , \quad \tilde{j}(-Q) \equiv \; , \quad \sum_K \tilde{j}(-K) G_0(K) \tilde{j}(K) \equiv$$

$$\left(-\frac{\lambda}{4!}\right) \sum_{Q_1 \dots 4} [Q_1 + \dots + Q_4] \equiv$$

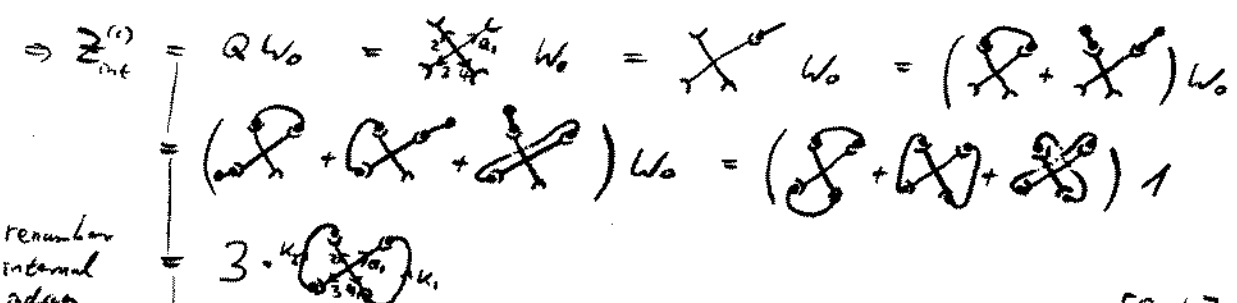
$\rightarrow W_0 = e^{\frac{1}{2} \cdots}$

⇒ the two relations take the form



$$= [K+Q] \; (b) \; ; \qquad W_0 = \; W_0 \quad (a)$$

→ "W_0 is endless source of ⸺'s"



$$\Rightarrow Z^{(1)}_{int} = QW_0 = \dots W_0 = \dots W_0 = (\dots + \dots) W_0 = (\dots + \dots + \dots) W_0 = (\dots + \dots + \dots) 1$$

renumber internal indices

$$= 3 \cdot \dots$$

$$= 3 \cdot \left(-\frac{\lambda}{4!}\right) \sum_{Q_1 \dots 4} [Q_1 + \dots + Q_4] \sum_{K_1} G_0(K_1) \sum_{K_2} G_0(K_2) \, [Q_1 + K_1][Q_2 + K_2][Q_3 - K_2] \, [Q_4 - K_1]$$

$$= 3 \cdot \left(-\frac{\lambda}{4!}\right) [0] \left(\sum_K G_0(K)\right)^2 \qquad (70)$$

$$\tilde{G}_2(K_1, K_2) = \frac{1}{Z_{int}} \left(I_{-K_1} I_{-K_2} W_0 + I_{-K_1} I_{-K_2} QW_0 + O(\lambda^2) \right)$$

$$= \underbrace{[K_1 + K_2] \, G_0(K_1)}_{\equiv \tilde{G}^{(0)}_2} \; \underbrace{- Z^{(1)}_{int} \tilde{G}^{(0)}_2 + I_{-K_1} I_{-K_2} QW_0}_{\equiv \tilde{G}^{(1)}_2} + O(\lambda^2)$$

like above, derive second part of $\tilde{G}^{(1)}_2$:

$$= \dots \cdot 3 \dots + 2 \cdot 6 \dots$$

$$= \tilde{G}^{(0)}_2 \cdot Z^{(1)}_{int} + 12 \cdot \left(-\frac{\lambda}{4!}\right) [K_1 + K_2] \left(G_0(K_1)\right)^2 \sum_P G_0(P)$$

$$\Rightarrow \quad \widetilde{G}_2^{(1)} = 12\cdot\left(-\frac{\lambda}{4!}\right)[k_1 \circ k_2]\left(G_0(k_1)\right)^2 \sum_P G_0(P) \qquad (71)$$

now it's simple to go to a more advanced level of diagrams:

$\longrightarrow \equiv G_0(K)$, $\times \equiv \left(-\frac{\lambda}{4!}\right)$

$\Rightarrow$ draw all possible diagrams;
put momenta on lines, obey E-p-conservation at each vertex, $\sum_Q$ for each loop;
put overall E-p conserving δ-function;
derive combinatoric factor. (71)

$\rightsquigarrow \; Z_{int}^{(1)} = \infty$ ← 3 via comb's

12 = 2·6 via comb's

$$\widetilde{G}_2 = \frac{1}{1+\infty}\left(\;—\; + \;\underline{\infty}\; + \;\underline{\Omega}\; + \ldots\right) = \left(\;—\; + \;\underline{\Omega}\; + \ldots\right)$$

7) $\lambda\phi^4$, pressure to NNLO

- perform sum-integrals $\sum_K G_0(K)$
- discover IR div. → resummation

already seen a first example of $\sum_K$ ($\rightsquigarrow FT^2/29$) (needed for $\ln Z_0$)
collecting from there:

$$\sum_K \ln\left(\beta^2(m^2-K^2)\right) = T\int\frac{d^3k}{(2\pi)^3}\left(\mathrm{const}_{\beta\omega} + \beta\omega + 2\ln\left(1-e^{-\beta\omega}\right)\right) \qquad \omega^2 \equiv k^2+m^2$$

$$= T\int\frac{d^3k}{(2\pi)^3}\left(\mathrm{const}_{\beta\omega}+\beta\omega\right) - \frac{T^4}{3\pi^2}\underbrace{\Gamma(5)\,h_5(y)}_{\approx \frac{\pi^4}{15}-\frac{\pi^2}{4}y^2+\frac{\pi}{2}y^3+\ldots} \qquad y\equiv\beta m$$

$$h_n(y) = \frac{1}{\Gamma(n)}\int_0^\infty dx\,\frac{x^{n-1}}{r}\,\frac{1}{e^{r}-1}\;, \quad r\equiv\sqrt{x^2+y^2} \qquad (72)$$

"typical Bose integrals" (Kapusta, App. A)

((the $T\to\infty$ - terms of $h_n(y)$ are easy to derive:

$$h_n(0) = \frac{1}{\Gamma(n)}\int_0^\infty dx\, x^{n-2}\frac{1}{e^x-1} = \frac{1}{\Gamma(n)}\int_0^\infty dx\, x^{n-2}\sum_{j=1}^\infty e^{-jx} \quad \text{(geom. series)}$$

$$\Rightarrow \Gamma(3)\,h_3(0) = \sum_j\int dx\, x\, e^{-jx} = \sum_{j=1}^\infty\frac{1}{j^2} = \zeta(2) = \frac{\pi^2}{6}$$))

etc.

<u>second example of $\sum_K$</u> (need for $Z^{(1)}_{int}$, $\tilde{G}_2^{(1)}$)

$$\sum_K G_0(K) = T\sum_{n=-\infty}^{\infty}\int\frac{d^3k}{(2\pi)^3}\boxed{\frac{-1}{K^2-m^2}} \quad = \frac{1}{(2\pi n T)^2+\vec{k}^2+m^2} = \frac{\beta^2}{(2\pi n)^2+(\beta\omega)^2}$$

note $$\sum_n \frac{1}{(2\pi n)^2+(\beta\omega)^2} = \frac{1}{2\beta\omega}\,\partial_{\beta\omega}\underbrace{\sum_n \ln\left((2\pi n)^2+(\beta\omega)^2\right)}_{=\,\text{const}_{\beta\omega}+\beta\omega+2\ln(1-e^{-\beta\omega})}$$
$$= \frac{1}{\beta\omega}\left(\frac{1}{2}+\frac{1}{e^{\beta\omega}-1}\right)$$

$$= \int\frac{d^3k}{(2\pi)^3}\,\frac{1}{\omega}\left(\frac{1}{2}+\frac{1}{e^{\beta\omega}-1}\right)$$

$$= \int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega} + \frac{T^2}{2\pi^2}\underbrace{\int_0^\infty dx\,\frac{x^2}{\Gamma}\,\frac{1}{e^{\Gamma}-1}}_{=\,\Gamma(3)h_3(y)\,\approx\,\frac{\pi^2}{6}-\frac{\pi}{2}y+\mathcal{O}(y^2)} \quad , \quad \Gamma\equiv\beta\omega=\sqrt{x^2+y^2} \;,\; y=\beta m \qquad (73)$$

((check: note $G_0(K)=\frac{1}{m^2-K^2}=\underbrace{\boxed{\partial_{m^2}}}_{=\frac{1}{2m}\partial_m=\frac{\beta^2}{2y}\partial_y}\ln\left(\beta^2(m^2-K^2)\right)$

$$\to \sum_K G_0(K) = \int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega} - \frac{T^2}{6\pi^2}\,\frac{1}{y}\partial_y\,\Gamma(5)h_5(y)$$
$$= \cancel{} \quad + \frac{T^2}{2\pi^2}\left(\frac{\pi^2}{6}-\frac{\pi}{2}y+\mathcal{O}(y^2)\right) \quad ✓ \quad))$$

$\to$ general feature: $(T=0)$ + $(T\neq 0)$

frequency sum does this split.

<u>relation between Bose-integrals</u>

$$\frac{1}{y}\partial_y h_n(y) = \frac{1}{y}\partial_y\,\frac{1}{\Gamma(n)}\int_0^\infty dx\; x^{n-1}\,\frac{1}{\Gamma}\,\frac{1}{e^{\Gamma}-1} \quad , \quad \text{there, } \frac{1}{y}\partial_y \overset{=\frac{1}{y}(\partial_y\Gamma)(\partial_\Gamma x)\partial_x}{=} \frac{1}{x}\partial_x$$

$$= -\frac{n-2}{\Gamma(n)}\int_0^\infty dx\,\frac{x^{n-3}}{\Gamma}\,\frac{1}{e^{\Gamma}-1} + \frac{1}{\Gamma(n)}\,\frac{x^{n-2}}{\Gamma}\,\frac{1}{e^{\Gamma}-1}\Bigg|_{x=0}^{\infty}$$

$$= -\frac{1}{\Gamma(n)}(n-2)\Gamma(n-2)\,h_{n-2}(y) - \lim_{x\to 0}\frac{1}{\Gamma(n)}\,\frac{x^{n-2}}{\Gamma}\,\frac{1}{e^{\Gamma}-1} \qquad (74)$$

((surface term $= 0$ for $\begin{cases} y\neq 0, & n>2 \\ y=0, & n>4 \end{cases}$

$= \text{finite}$ for $\begin{cases} y\neq 0, & n=2 \\ y=0, & n=4 \end{cases} = \begin{cases} -\frac{1}{y}\frac{1}{e^y-1} \\ -\frac{1}{\Gamma(4)} \end{cases}$

$= \infty$ for $\begin{cases} y\neq 0, & n<2 \\ y=0, & n<4 \end{cases}$))

collecting results

$$\mathcal{F} = -T\ln Z = -T\ln Z_0 - T\ln Z_{int} \equiv \mathcal{F}_0 + \mathcal{F}_{int}$$

$$p = -\partial_V \mathcal{F} = -\partial_V \mathcal{F}_0 - \partial_V \mathcal{F}_{int} \equiv p_0 + p_{int}$$

$$\mathcal{F}_0 \overset{hw\#7}{=} const_{m,\lambda} - \frac{V}{2}\sum_Q \ln\left(T^2 G_0(Q)\right)$$

$$= const + \frac{V}{2}\left[T\int\frac{d^3k}{(2\pi)^3}\left(const_{\beta\omega} + \beta\omega\right) - \frac{T^4}{3\pi^2}\underbrace{\Gamma(5)\,h_5(\beta m)}_{\approx \frac{\pi^4}{15} + O\left(\frac{m^2}{T^2}\right)}\right]$$

$$\mathcal{F}_0^R \overset{T\to\infty}{=} -VT^4\frac{\pi^2}{90}$$ (regularization + renormalization (T=0))
→ see $FT^2/\dots$

$$p_0^R = T^4\frac{\pi^2}{90}$$

$$\mathcal{F}_{int} = -T\ln\left(1 + Z_{int}^{(1)} + O(\lambda^2)\right) = -T Z_{int}^{(1)} + O(\lambda^2)$$

$$= -T\,3\cdot\left(-\frac{\lambda}{4!}\right)\underbrace{[0]}_{=\beta V}\Big(\underbrace{\sum_K G_0(K)}_{=\dots+\frac{T^2}{2\pi^2}\Gamma(3)h_3(\beta m) = \dots + \left(\frac{T^2}{12} + O\left(\frac{m}{T}\right)\right)}\Big)^2 + O(\lambda^2)$$

$$\mathcal{F}_{int}^R = VT^4\left[\frac{1}{48}\frac{\lambda}{4!}\left(1 + O\left(\frac{m}{T}\right)\right) + O(\lambda^2)\right]$$

$$p_{int}^R = -T^4\left[\ \dots\ \right]$$

$$\Rightarrow p^R = T^4\left(\frac{\pi^2}{90} - \frac{1}{48}\frac{\lambda}{4!} + O(\lambda^2)\right)$$ ($m=0$, or $T\to\infty$)

• had def'd 'typical bose integrals' (72)

$$h_n(y) = \frac{1}{\Gamma(n)}\int_0^\infty dx\, \frac{x^{n-1}}{\Gamma}\,\frac{1}{e^{\Gamma}-1} \qquad , \Gamma \equiv \sqrt{x^2+y^2}$$

and derived the relation (74)

$$\frac{1}{y}\partial_y h_n(y) = -\frac{1}{n-1}\, h_{n-2}(y) \qquad \text{((valid for } y\neq 0,\ n>2\text{))}$$

$$\Rightarrow h_{n>2}(z) = h_n(0) - \frac{4\pi^2}{n-1}\int_0^z dz'\, z'\, h_{n-2}(z') \qquad , z \equiv \frac{y}{2\pi}$$

can be used for $h_1 \to h_3 \to h_5 \to \dots$ etc (a possible sol'n of hw#9)

<u>a)</u> calc. constant part $h_n(0)$

$$h_{n>2}(0) = \frac{1}{\Gamma(n)}\int_0^\infty dx\, \frac{x^{n-2}}{e^x-1}$$

use geom. series: $\frac{1}{e^x-1} = \frac{e^{-x}}{1-e^{-x}} = e^{-x}\sum_{j=0}^\infty e^{-xj}$

$$= \frac{1}{\Gamma(n)}\underbrace{\boxed{\sum_{j=0}^\infty (j+1)^{1-n}}}_{=\zeta(n-1)}\ \underbrace{\boxed{\int_0^\infty dx\, x^{n-2}e^{-x}}}_{=\Gamma(n-1)}$$

(Riemann) Zeta function, $\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z}$

$$= \frac{\zeta(n-1)}{n-1}$$

(($h_3(0) = \frac{1}{2}\zeta(2) = \frac{\pi^2}{12}$; $h_5(0) = \frac{1}{4}\zeta(4) = \frac{\pi^4}{360}$))

<u>b)</u> calc. integral for $n=1$

$$h_1(y) = \int_0^\infty dx\, \frac{1}{\Gamma}\,\frac{1}{e^{\Gamma}-1} \qquad , \Gamma \equiv \sqrt{x^2+y^2}$$

use $\frac{1}{e^x-1} = \frac{1}{x} - \frac{1}{2} + \sum_{n=1}^\infty \frac{2x}{(2\pi n)^2+x^2}$

$$= \underbrace{\int_0^\infty dx\, \frac{1}{\Gamma}\left(-\frac{1}{2}\right) x^{-\varepsilon}}_{(*)} + \underbrace{\int_0^\infty dx\, \frac{1}{\Gamma}\left(\frac{1}{\Gamma} + \sum_{n=1}^\infty \frac{2\Gamma}{(2\pi n)^2+\Gamma^2}\right) x^{-\varepsilon}}_{(**)}$$

⟵ convergence factor (h_1 is finite!)

$$(*) = -\frac{1}{2}y^{-\varepsilon}\underbrace{\boxed{\int_0^\infty dx\, \frac{x^{-\varepsilon}}{\sqrt{x^2+1}}}}_{=\frac{1}{2}B\left(\frac{1-\varepsilon}{2},\frac{\varepsilon}{2}\right)} = -\frac{1}{2\varepsilon} + \frac{1}{2}\ln\frac{y}{2} + O(\varepsilon)$$

$$(**) = \underbrace{\boxed{\int_0^\infty dx\, \frac{x^{-\varepsilon}}{x^2+1}}}_{\substack{\text{finite at } \varepsilon\to 0 \\ =\frac{1}{2}B\left(\frac{1-\varepsilon}{2},\frac{1+\varepsilon}{2}\right)}}\left(y^{-1-\varepsilon} + 2\sum_{n=1}^\infty \sqrt{(2\pi n)^2+y^2}^{\,-1-\varepsilon}\right)$$

extract div. part; set $\varepsilon=0$ in rest

$$= \frac{2}{(2\pi)^{1+\varepsilon}}\underbrace{\boxed{\sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}}}}_{=\zeta(1+\varepsilon)} + \underbrace{2\sum_{n=1}^\infty\left(\frac{1}{\sqrt{(2\pi n)^2+y^2}} - \frac{1}{2\pi n}\right)}$$

(($(1+x)^{-\frac{1}{2}} = 1 + \sum_{j=1}^\infty c_j x^j$, $c_j = (-)^j\frac{(2j-1)!!}{2^j\, j!}$))

$$= \frac{1}{\pi}\sum_{j=1}^\infty c_j\left(\frac{y}{2\pi}\right)^{2j}\underbrace{\boxed{\sum_{n=1}^\infty \frac{1}{n^{2j-1}}}}_{=\zeta(2j-1)}$$

$$(**) = \frac{\pi}{2y} + \frac{1}{2\varepsilon} + \frac{\gamma}{2} - \frac{1}{2}\ln(2\pi) + \frac{1}{2}\sum_{j=1}^\infty c_j\,\zeta(2j-1)\left(\frac{y}{2\pi}\right)^{2j} + O(\varepsilon)$$

$$\Rightarrow h_1(z) = \frac{1}{4z} + \frac{1}{2}\left(\ln\frac{z}{2} + \gamma\right) + \sum_{j=1}^\infty \frac{c_j}{2}\,\zeta(2j-1)\, z^{2j} \qquad , z \equiv \frac{y}{2\pi} \qquad (75)$$