5) perturbative expansion

(of $W[j]$ at $T \neq 0$, using Matsubara contour)

periodic Fourier (again)

$$\left.\begin{aligned}
&\phi(x) = \sum_K e^{-iKx}\tilde\phi(K) \quad , \quad \tilde\phi(K) = \int^\beta e^{iKx}\phi(x) \\
&\sum_K = T\sum_n \int\frac{d^3k}{(2\pi)^3} \quad , \quad K = (i\omega_n, \vec k) \quad , \quad \omega_n = \pi T\begin{cases} 2n & \text{Bose} \\ 2n+1 & \text{Fermi}\end{cases} \\
&x = (-i\tau, \vec x) \quad , \quad Kx = \omega_n\tau - \vec k\vec x \quad (\text{Mink.}) \quad , ((x^2 = -\tau^2 - \vec x^2)) \\
&\int^\beta e^{iKx} = \beta(2\pi)^3\,\delta_{n_K,0}\,\delta(\vec k) \hat{=} \beta V\,\delta_{n_K,0}\,\delta_{\vec k,0} = [K] \\
&\sum_K [K-Q] = 1 \quad , \quad \sum_K e^{iKx} = \delta(x)
\end{aligned}\right\} \quad (50)$$

(($\vec k$ in Box (L^3) is multiple of $\frac{2\pi}{L}$))

$W[\tilde j]$

quadratic part (in ϕ) of $\mathcal{L}$

- $\int^\beta(\mathcal{L}_0 + j\phi) = \sum_K\left[\frac{1}{2}\tilde\phi(-K)\frac{1}{\Delta(K)}\tilde\phi(K) + \tilde j(-K)\tilde\phi(K)\right]$ (($\Delta(K) = \Delta(-K)$))

$$\xrightarrow[\tilde\phi \to \tilde\phi - \tilde j\Delta]{\text{shift}} \sum_K\left[\frac{1}{2}\tilde\phi_-\frac{1}{\Delta}\tilde\phi - \frac{1}{2}\tilde j_-\Delta\tilde j\right]$$

- $W = \int D\phi\; e^{\int^\beta(\mathcal{L}[\phi] + j\phi)}$, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$ (quad.)

$$= e^{\int^\beta_{(x)}\mathcal{L}_{int}[\delta_{j(x)}]}\int D\phi\; e^{\int^\beta(\mathcal{L}_0[\phi] + j\phi)}$$

- analogous to $FT^2/23$, read $[K-Q] = \beta(2\pi)^3\delta_{n_{K-Q},0}\,\delta(\vec k - \vec q) =$

as

$$\left.\begin{aligned}
&\equiv \underbrace{\beta(2\pi)^3\,\delta_{\tilde j(Q)}}\,\tilde j(K) \\
&\equiv \big|_Q\,\tilde j(K) \quad ((\text{need simple notation}))
\end{aligned}\right\} \quad (51)$$

$\rightsquigarrow$ from $\delta_{j(x)}\,j(y) = \delta(x-y)$, after mult. with $\int^\beta_{(y)} e^{iKy}\int^\beta_{(x)} e^{-iQx}$

$$\Rightarrow \big|_Q = \int^\beta e^{-iQx}\,\delta_{j(x)} \quad , \quad \delta_{j(y)} = \sum_Q e^{iQy}\,\big|_Q \qquad (52)$$

$$\Rightarrow \boxed{W[\tilde j] = e^{\int^\beta_{(x)}\mathcal{L}_{int}\left[\sum_Q e^{iQx}\big|_Q\right]}\;\frac{1}{c}\;e^{-\sum_K\frac{1}{2}\tilde j(-K)\Delta(K)\tilde j(K)}} \qquad (53)$$

c such that $W[\tilde j = 0] = 1$; see next pg.

$$\big|_P\sum_K\tilde j(K)\,F(K) = F(P)$$

Greens

$$G_n(x_1 \ldots x_n) = \delta_{j(x_1)} \cdots \delta_{j(x_n)} W[j] \Big|_{j=0} \qquad \text{((note: no } \tfrac{1}{i}\text{'s like @ } T=0 \text{))}$$

$$\tilde{G}_n(K_1 \ldots K_n) = \int_{(x_1)}^{\beta} e^{iK_1x_1} \ldots \int_{(x_n)}^{\beta} e^{iK_nx_n} \underbrace{G_n(x_1 \ldots x_n)}$$

$$= \Big(\sum_{Q_1} e^{iQ_1x_1} \boldsymbol{\vert}_{Q_1}\Big) \cdots (\,n\,)\, W[\tilde{j}] \Big|_{\tilde{j}=0}$$

$$\Rightarrow \tilde{G}_n(K_1 \ldots K_n) = \boldsymbol{\vert}_{-K_1} \cdots \boldsymbol{\vert}_{-K_n} W[\tilde{j}] \Big|_{\tilde{j}=0} \qquad (54)$$

→ diagrammar (and combi factors etc.) works exactly as @ $T=0$, modulo periodicity. ((each closed loop → $\sum_K$; each vertex $[K_1 + \ldots + K_n]$))

- wow! All physics encoded in a simple way ($W[\tilde{j}]$ ..) now, ready for perturbative expansion (philo.: $\mathcal{L}_{int} \sim$ small parameter)

  cannot solve theory exactly → have to use some approximation scheme. others possible.. also lattice.

partition function

$$Z = \mathrm{Tr}\left(e^{-\beta \hat{H}}\right)$$

$$= \int D\phi \, e^{\int^\beta \mathcal{L}}$$

$$= \int D\phi \, e^{\int^\beta \mathcal{L}_0} \cdot \frac{\int D\phi \, e^{\int^\beta \mathcal{L}}}{\int D\phi \, e^{\int^\beta \mathcal{L}_0}} \equiv Z_0 \cdot Z_{int} \qquad (55)$$

((remember: absorbed norm. factor into measure $D\phi$.
has to be reinstalled in Z_0 only.))

$\frac{1}{Z}$ in $W[\tilde{j}]$

norm. factor in $W[\tilde{j}]$ follows from $W[j=0] \overset{!}{=} 1$

$$\rightsquigarrow W[j] = \frac{\int D\phi \, e^{\int^\beta (\mathcal{L}+j\phi)}}{\int D\phi \, e^{\int^\beta \mathcal{L}}} = \frac{1}{Z} \int D\phi \, e^{\int^\beta (\mathcal{L}+j\phi)}$$

$$= \frac{1}{Z} e^{\int^\beta \mathcal{L}_{int}[\delta_j]} \int D\phi \, e^{\int^\beta (\mathcal{L}_0+j\phi)} = \frac{1}{Z} e^{\int^\beta \mathcal{L}_{int}[\delta_j]} \underbrace{\left(\int D\phi \, e^{\sum_K \frac{1}{2} \tilde{\phi}_- \frac{1}{\Delta} \tilde{\phi}}\right)}_{= \int D\phi \, e^{\int^\beta \mathcal{L}_0} = Z_0} e^{-\sum_K \frac{1}{2} \tilde{j}_- \Delta \tilde{j}}$$

$$\Rightarrow W[\tilde{j}] = \frac{1}{Z_{int}} e^{\int^\beta \mathcal{L}_{int}[\ldots]} e^{-\sum \frac{1}{2} \tilde{j}_- \Delta \tilde{j}} \qquad (56)$$

... play a little ...

$\leftarrow$ quadratic part

like above, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$, $\int^\beta \mathcal{L}_0 = -\sum_k \frac{1}{2} \tilde{\phi}(-k) \frac{1}{G_0(k)} \tilde{\phi}(k)$

(introduced (bare) <u>propagator</u> , $G_0(k) = -\Delta(k)$ in previous notation)

$\rightsquigarrow Z_0 = \int D\tilde{\phi}\, e^{-\sum_k \frac{1}{2} \tilde{\phi}_- \frac{1}{G_0} \tilde{\phi}}$

$\rightsquigarrow Z_{int} = e^{Q}\, e^{+\sum_k \frac{1}{2} \tilde{j}_- G_0 \tilde{j}} \Big|_{\tilde{j}=0}$

where $Q \equiv e^{\int_{(x)}^\beta \mathcal{L}_{int} [\sum_q e^{iqx} l_q]}$ (57)

(stupid?) question : how does Z depend on <u>bare</u> propagator ?

$$\delta_{G_0(p)} \ln Z = \underbrace{\delta_{G_0(p)} \ln Z_0}_{A} + \underbrace{\delta_{G_0(p)} \ln Z_{int}}_{B}$$

$A = \frac{1}{Z_0} \delta_{G_0(p)} \int D\tilde{\phi}\, e^{-\sum \frac{1}{2} \tilde{\phi}_- \frac{1}{G_0} \tilde{\phi}}$

$\hookrightarrow \frac{1}{2} \frac{1}{G_0^2(p)} \frac{1}{\beta V} \tilde{\phi}(-p) \tilde{\phi}(p)$ $\longleftarrow$ comes from def. of $[K]$

$= \frac{1}{2G_0^2(p)} \frac{1}{\beta V} \frac{1}{Z_0} \int D\tilde{\phi}\, \tilde{\phi}(-p) \tilde{\phi}(p)\, e^{-\sum\left(\frac{1}{2}\tilde{\phi}_- \frac{1}{G_0} \tilde{\phi} - \tilde{j}\tilde{\phi}\right)} \Big|_{\tilde{j}=0}$

$= \quad \text{"} \quad l_p l_{-p} \underline{\int D\tilde{\phi}\, e^{\sum\left(\frac{1}{2}\tilde{\phi}_- \frac{-1}{G_0} \tilde{\phi} + \tilde{j}\tilde{\phi}\right)}} \Big|_{\tilde{j}=0}$

shift $= \left(\int D\tilde{\phi}\, e^{\sum \frac{1}{2} \tilde{\phi}_- \frac{-1}{G_0} \tilde{\phi}}\right) e^{\sum \frac{1}{2} \tilde{j}_- G_0 \tilde{j}}$

$= \frac{1}{2G_0^2(p)} \frac{1}{\beta V} l_p l_{-p}\, e^{\sum \frac{1}{2} \tilde{j}_- G_0 \tilde{j}} \Big|_{\tilde{j}=0}$

$\Big(l_p\, e^{\sum_k \frac{1}{2} \tilde{j}(-k) G_0(k) \tilde{j}(k)} = \left(\frac{1}{2} G_0(-p)\tilde{j}(-p) + \frac{1}{2} \tilde{j}(-p) G_0(p)\right) e^{\cdots}$

$= \tilde{j}(-p) G_0(p)\, e^{\cdots} \Big)$

$= \frac{1}{2G_0^2} \frac{1}{\beta V} l_p\, \tilde{j}(-p) G_0(p)\, e^{\cdots} \Big|_{\tilde{j}=0}$

$\hookrightarrow$ <u>has</u> to hit here , because of $\tilde{j}=0$

$= \frac{1}{2G_0^2} \frac{1}{\beta V} [0]\, G_0(p) = \frac{1}{2G_0(p)}$ (58)

$B = \frac{1}{Z_{int}} \delta_{G_0(p)}\, e^{Q}\, e^{\sum \frac{1}{2} \tilde{j}_- G_0 \tilde{j}} \Big|_{\tilde{j}=0}$

$\hookrightarrow \frac{1}{2} \frac{1}{\beta V} \underline{\tilde{j}(-p) \tilde{j}(p)}$

$= \tilde{j}(-p) \frac{1}{G_0(p)} l_{-p}$

$= \frac{1}{G_0(p)} \left(l_{-p} \underline{\tilde{j}(-p)} - [0] \right)$

$\hookrightarrow \frac{1}{G_0(p)} l_p$

$$\ldots\ \mathcal{B} = \frac{1}{2G_0^2(p)}\,\frac{1}{\beta V}\,\frac{1}{Z_{int}}\left(I_{-p}I_p - [0]\,G_0(p)\right)\underbrace{e^{Q}e^{\Sigma\frac{1}{2}\tilde{j}_-G_0\tilde{j}}}_{=Z_{int}\cdot W[\tilde{j}]}\Big|_{\tilde{j}=0}$$

$$= \frac{1}{2G_0^2(p)}\left(\frac{1}{\beta V}\underbrace{I_{-p}I_p\,W[\tilde{j}]\big|_{\tilde{j}=0}}_{=\tilde{G}_2(p,-p)} - G_0(p)\underbrace{W[\tilde{j}=0]}_{=1}\right) \qquad (59)$$

$$\Rightarrow \quad \boxed{2G_0^2(p)\,\delta_{G_0(p)}\ln Z = \frac{1}{\beta V}\,\tilde{G}_2(p,-p)} \qquad \text{((proves Kapusta (3.24)))} \quad (60)$$

((ln Z seems to be *the* quantity of interest ; also $F=-T\ln Z$, $p=\partial_V T\ln Z, ..$))

pert. exp. :

$$Z = Z_0\cdot Z_{int} = Z_0\cdot \underset{\uparrow\,\text{expand}}{e^{Q}}\,e^{\Sigma\frac{1}{2}jG_0j}\Big|_{j=0} = Z_0\cdot\left(1+\text{"int."}\right)$$

$$\rightsquigarrow \quad F = -T\ln Z = -T\ln Z_0 - T\ln Z_{int} = F_0 + \underbrace{F_{int}}_{=0+\text{"int"}}$$

⇒ conceptually exactly like $T=0$ perturbative expansion.
integrals → sum-integrals ; in general, much harder to evaluate!

6) $\lambda\phi^4$: first steps

toy model ; unphysical ; learn techniques ; gauge theories ⇝ §8 ff

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu\phi)\partial^\mu\phi - \frac{m^2}{2}\phi^2}_{\mathcal{L}_0}\ \underbrace{-\ \frac{\lambda}{4!}\phi^4}_{\mathcal{L}_{int}}$$

$$\int^\beta\mathcal{L}_0 = \underbrace{\int^\beta\sum_K e^{-iKx}\sum_Q e^{-iQx}}_{[K+Q]}\left[\frac{1}{2}(-iK_\mu)(-iQ^\mu) - \frac{m^2}{2}\right]\tilde{\phi}(K)\,\tilde{\phi}(Q)$$

$$= \sum_K \frac{1}{2}\tilde{\phi}(-K)\underbrace{\boxed{(K^2-m^2)}}_{=\frac{1}{\Delta(K)} = \frac{-1}{G_0(K)}}\tilde{\phi}(K) \qquad (61)$$

$$\left(\left(K^2 = -\omega_n^2 - \vec{k}^2\right)\right)$$

↳ "bare propagator"

Z (or $\ln Z$ or F or p ..) to leading order

- meet first frequency sum
- meet typical Bose integral
- understand physics : $\phi \to \frac{1}{2}\cdot$ blackbody

$$Z_0 = \mathcal{N}\int \mathcal{D}\tilde\phi \; e^{-\Sigma \frac{1}{2}\tilde\phi_- \frac{1}{G_0}\tilde\phi} \qquad , \; G_0(k) = \frac{-1}{k^2-m^2}$$

$\phi(x)$ real $\rightsquigarrow$ $\tilde\phi(-k) = \tilde\phi^*(k)$ ((prove via Fourier-def))

$$= \mathcal{N}\int \mathcal{D}\tilde\phi \; e^{-\Sigma\frac{1}{2}\frac{1}{G_0}|\tilde\phi|^2}$$ Gaussian integral! diagonal in Fourier space.

$$\left(\left(\int_{-\infty}^{\infty} dx_1 .. dx_n \; e^{-x_i D_{ij} x_j} = \pi^{n/2}(\det D)^{-\frac{1}{2}} \right)\right)$$

$$= \mathcal{N}\prod_n \prod_{\vec k} \left(\frac{\beta^2}{G_0(k)}\right)^{-\frac{1}{2}} \qquad (62)$$

$$\rightsquigarrow \ln Z_0 = \ln\mathcal{N} - \frac{1}{2}\underbrace{\sum_n \sum_{\vec k}}_{= V\int\frac{d^3k}{(2\pi)^3}} \ln\Big(\beta^2(\omega_n^2 + \underbrace{\vec k^2 + m^2}_{\equiv\,\omega^2})\Big)$$

((recall FT²/9 : $\vec k = \frac{2\pi}{L}(n_1,n_2,n_3)$, $\frac{2\pi}{1\mathrm{cm}} \approx .14\mathrm{K} \ll T$, $V = L^3$))

use $\sum_n \ln\left((2\pi n)^2 + (\beta\omega)^2\right) = \text{const} + \beta\omega + 2\ln(1-e^{-\beta\omega})$

"frequency sum" ; derivation : later. see Kapusta §2.3, (2.37)-(2.40)

$$= \text{const}' + V\int\frac{d^3k}{(2\pi)^3}\left[-\frac{1}{2}\beta\omega - \ln(1-e^{-\beta\omega})\right]$$

$$= \text{const}' + VT^3\int\frac{d^3x}{(2\pi)^3}\Big[\underbrace{-\frac{1}{2}\Gamma}_{} - \ln(1-e^{-\Gamma})\Big] \quad , \; \Gamma \equiv \sqrt{x^2+y^2}, \; y = \frac{m}{T}$$

↳ div! , but $F_0 = -T\ln Z_0$ $\rightsquigarrow$ includes zero-point E ✓

$$-\int\frac{d^3x}{(2\pi)^3}\ln(1-e^{-\Gamma}) = -\frac{1}{2\pi^2}\int_0^\infty dx\, x^2 \ln(1-e^{-\Gamma})$$

$$= \frac{1}{6\pi^2}\int_0^\infty dx\, x^3\,\partial_x \ln(1-e^{-\Gamma})$$

$$= \frac{1}{6\pi^2}\underbrace{\int_0^\infty dx \frac{x^4}{\sqrt{x^2+y^2}}\frac{1}{e^{\sqrt{x^2+y^2}}-1}}$$ ← Bose fct.

"typical Bose integral" $= \Gamma(5)\, h_5(y)$, see Kapusta App. A

$(y \ll 1)$ $\approx \frac{\pi^4}{15} - \left(\frac{\pi}{2}y\right)^2 + \dots$ (63)

↳ this result we already knew, compare formula (11)

$$\rightsquigarrow \frac{F_0}{V} = -\frac{T}{V}\ln Z_0 = \overset{\text{zero pt. } E}{\cancel{(\infty)}} - T^4 \frac{4}{\pi^2} h_5\left(\frac{m}{T}\right)$$

$$= -T^4\left(\frac{\pi^2}{90} - \frac{1}{24}\frac{m^2}{T^2} + \dots\right) \qquad (64)$$

$$\rightsquigarrow p_0 = -\partial_V F_0 = T^4\left(\frac{\pi^2}{90} - \dots\right) \qquad (65)$$

compare w/ (12). ⇝ have obtained $\frac{1}{2}$ · blackbody rad. here.
(clear: scalars have no spin d.o.f.,
photons have 2 transverse pol.)

2pt - Greens

$$\tilde{G}_2(K_1,K_2) = I_{-K_1} I_{-K_2} \underbrace{W[\tilde{j}]}_{} \Big|_{\tilde{j}=0}$$

$$W[\tilde{j}] = \frac{1}{Z_{int}} e^{Q} W_0[\tilde{j}] \approx \frac{1}{Z_{int}}\left(1 + Q + O(\lambda^2)\right) W_0[\tilde{j}]$$

$$\text{where } W_0[\tilde{j}] = e^{\sum_K \frac{1}{2}\tilde{j}(-K) G_0(K)\tilde{j}(K)} \qquad (66)$$

$$\text{and } Q = \int^\beta L_{int}$$

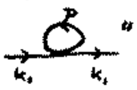
$$= -\frac{\lambda}{4!}\sum_{Q_1 \dots Q_4}[Q_1+Q_2+Q_3+Q_4]\, I_{Q_1} I_{Q_2} I_{Q_3} I_{Q_4} \qquad (67)$$

$$= \frac{1}{Z_{int}}\left(\underbrace{I_{-K_1} I_{-K_2} W_0[\tilde{j}]\Big|_{\tilde{j}=0}}_{} + I_{-K_1} I_{-K_2} Q W_0 \Big|_{\tilde{j}=0} + O(\lambda^2)\right)$$

$$I_{-K_1} I_{-K_2} W_0[\tilde{j}]\Big|_{\tilde{j}=0} = I_{-K_1} \tilde{j}(K_2) G_0(K_2) W_0\Big|_{\tilde{j}=0}$$

$$= [K_1+K_2]\, G_0(K_2) \underbrace{W_0[0]}_{=1}$$

$$= \frac{1}{Z_{int}}\left([K_1+K_2]\, G_0(K_1) + I_{-K_1} I_{-K_2} Q W_0\Big|_{\tilde{j}=0} + O(\lambda^2)\right) \qquad (68)$$

" $\xrightarrow{K_1}$ " $[K_1 + K_2]$ "  " …

higher orders in Z

$$Z_{int} = e^{Q} W_0[\tilde{j}]\Big|_{\tilde{j}=0}$$

$$= 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\Big(\underbrace{Q \dots Q}_{(n)} W_0[\tilde{j}]\Big)_{\tilde{j}=0} = 1 + \underbrace{Q W_0\big|_{\tilde{j}=0}}_{\text{"}\infty\text{"}} + \dots \qquad (69)$$

$$\left(\!\left(\ \lambda\phi^4 \rightsquigarrow Z_{int} = 1 + O(\lambda) \rightsquigarrow \frac{1}{Z_{int}} = 1 - O(\lambda) \rightsquigarrow \tilde{G}_2(K_1,K_2) = [K_1+K_2]\, G_0(K_1) + O(\lambda)\ \right)\!\right)$$

fine! everything consistent ✓