n matrix el. $\langle ..|u|..\rangle$ $\leftrightarrow$ n different π-integrations

$$\rightsquigarrow \langle\{\phi'\}|U|\{\phi\}\rangle \sim \int_{\vec r} \mathcal{D}\phi_{n-1} \cdots \int_{\vec r}\mathcal{D}\phi_1 \int_{\vec p}\mathcal{D}\pi_n \cdots \int_{\vec r}\mathcal{D}\pi_1$$
$$* \, e^{i\int d^3r \sum_{j=1}^{n}\{\pi_j\underbrace{(\phi_j-\phi_{j-1})}_{=\delta t\frac{\phi_j-\phi_{j-1}}{\delta t}\to\delta t\,\dot\phi_j} - \delta t[\frac12\pi_j^2+\underbrace{V(\phi_{j-1})}_{=V(\phi_j)+\cdots}]\}}$$

((here, $\phi_n\equiv\phi'$, $\phi_0\equiv\phi$))

$$\rightsquigarrow \int_{\vec r}\mathcal{D}\phi' \langle\{\phi'\}|U|\{\phi\}\rangle \sim \int\mathcal{D}\phi\int\mathcal{D}\pi \; e^{i\int_t^{t'}dt\int d^3r\,\{\pi\dot\phi-[\frac12\pi^2+V(\phi)]\}} \tag{40}$$

space-time-funct. int's

t is time-label of ϕ
t' 〃 ϕ'

$H_{class}\to\mathcal{L}$

$\{\pi\dot\phi - \underbrace{[\frac12\pi^2+V(\phi)]}_{=\mathcal{X}(\pi,\phi)}\} = \mathcal{L}$? No! (correct procedure: $\dot q=\partial_p H$, eliminate p in $L=p\dot q-H$)

$\{\} = -\frac12(\pi-\dot\phi)^2 + \underbrace{\frac12\dot\phi^2 - V(\phi)}_{=\mathcal{L}(\dot\phi,\phi)}$ (($\dot\phi=\partial_\pi\mathcal{X}=\pi \rightsquigarrow \mathcal{L}=\pi\dot\phi-\mathcal{X} \overset{\pi=\dot\phi}{=} \frac12\dot\phi^2-V(\phi)$))

$\Rightarrow \int\mathcal{D}\pi$ in $\langle ..|U|..\rangle$ is Gaussian $\rightsquigarrow$ gives const

$$\Rightarrow \boxed{\int_{\vec r}\mathcal{D}\phi' \langle\{\phi'\}|e^{-i\hat H(t'-t)}|\{\phi\}\rangle \sim \int\mathcal{D}\phi\; e^{i\int_t^{t'}dt\int d^3x\,\mathcal{L}(\dot\phi,\phi)}} \tag{41}$$

one $\int\mathcal{D}\phi$ only, Bose-periodicity

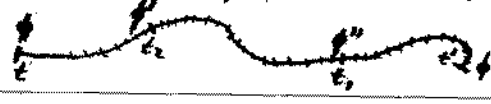
recall (pg. FT^2/19) $M_1 = \int_{\vec r}\mathcal{D}\phi''\int_{\vec r}\mathcal{D}\phi'\,\phi''(\vec r_1)\phi'(\vec r_2)$ largest time smallest time
$$* \langle\{\phi\}|e^{-i\hat H(t'-t_1)}|\{\phi''\}\rangle\langle\{\phi''\}|e^{-i\hat H(t_1-t_2)}|\{\phi'\}\rangle\langle\{\phi'\}|e^{-i\hat H(t_2-t)}|\{\phi\}\rangle$$
$$= \int\mathcal{D}\phi\;\phi(x_1)\phi(x_2)\; e^{i\int dt\int d^3x\,\mathcal{L}(\dot\phi,\phi)} \quad \text{with} \quad \phi(t,\vec r)=\phi(t',\vec r)$$

contour from t to t'

measure $\mathcal{D}$ absorbs all prop. factors from above.

also, $M_2 = M_1^{\,x_1\leftrightarrow x_2} = M_1$

(write left $\langle\{\phi\}|$ as $\int_{\vec r}\mathcal{D}\phi'''\,\delta(\phi'''(\vec r)-\phi(\vec r))$ $\rightsquigarrow$ this gives the 3rd $\int\mathcal{D}\phi$ needed.
ϕ''' 'lives' at time t', ϕ lives at time t $\Rightarrow$ Bose per. follows from δ-fct!)



recall (p. $FT^2/19$)

$$W_{(j^2\text{-part})} = \Big(\lim_{\beta\to\infty}\Big)\int_{P} D\phi \,\Big(-\frac{1}{2}\Big)\int_{C} d^4x_1 \int_{C} d^4x_2\; j(x_1)\, j(x_2)\,\underbrace{\big\{ \Theta(t_1-t_2)M_1 + \Theta(t_2-t_1)M_2 \big\}}_{\hat{=}\, M_1} \qquad (42)$$

generalize to n j's : every j finds ϕ in M_1 $\Rightarrow \Big(\int_C d^4x\, j(x)\,\phi(x)\Big)^n \frac{i^n}{n!}$

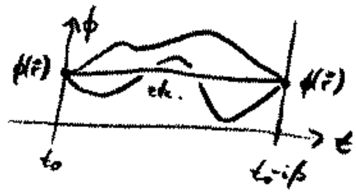
$\rightsquigarrow$ can write as exponential again

$$\boxed{W[j] = \Big(\lim_{\beta\to\infty}\Big) \int D\phi \; e^{\,i\int_C d^4x\, (\mathcal{L} + j\phi)} \;,\quad \phi(t_0 - i\beta, \vec{x}) = \phi(t_0, \vec{x}) \quad (\text{Bose})} \qquad (43)$$

(Minus for Fermi fields)

<u>comment:</u> the 'last' functional integration is constrained:

fix $\vec{x}$, $\phi(\vec{x})$ $\rightarrow$



$\rightsquigarrow$ paths in field-over-time diagram
(that's why it's called <u>path</u> integral)

<u>philosophy</u> periodicity and contour important for $T \neq 0$.

cold physics ($T=0$) : $\phi(t_0 - i\infty, \vec{x}) = \phi(t_0, \vec{x})$ unimportant
in C, think of real part only

but: <u>clean</u> field theory is $T \to 0$ limit !

<u>outlook</u> fields <u>periodic</u> in (imag.) time
$\longrightarrow$ <u>discrete</u> Fourier transform$|_t$ (<u>discrete</u> frequencies!)

use of $W[j]$ - representation

→ pert. expansion of an interacting theory ! (T=0 here) (T≠0 → pg. 25)

fields in PI are complex #'s ⇝ no ordering problems !!!

$$W[j] = \int \mathcal{D}\phi \; e^{i\int d^4x\, \mathcal{L}_{int}[\phi]}\, e^{i\int d^4x\, (\mathcal{L}_0[\phi] + j\phi)}$$

$$\boxed{W[j] = e^{i\int d^4x\, \mathcal{L}_{int}[\frac{1}{i}\delta_{j(x)}]} \int \mathcal{D}\phi\; e^{i\int d^4x\,(\mathcal{L}_0 + j\phi)}} \qquad (44)$$

will look even nicer in Fourier space:

- $\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx}\tilde{\phi}(k)$, $\tilde{\phi}(k) = \int d^4x\, e^{ikx}\phi(x)$ (45a)
- $\delta_{j(x)} j(y) = \delta(x-y) \quad \| \cdot \int d^4y\, e^{iky} \quad \rightsquigarrow \quad \delta_{j(x)}\tilde{j}(k) = e^{ikx} \quad \| \cdot \int d^4x\, e^{-iqx}$

$$\rightsquigarrow \quad \int d^4x\, e^{-iqx}\delta_{j(x)}\tilde{j}(k) = (2\pi)^4\delta(k-q) = (2\pi)^4\,\delta_{\tilde{j}(q)}\tilde{j}(k)$$

$$\Rightarrow \quad \delta_{\tilde{j}(k)} = \frac{1}{(2\pi)^4}\int d^4x\, e^{-ikx}\delta_{j(x)} \quad , \quad \delta_{j(x)} = \int d^4k\, e^{ikx}\delta_{\tilde{j}(k)} \qquad (45b)$$

((note difference in (2π)'s and sign $e^{\pm}$ wrt. Fourier))

- $\int \mathcal{D}\phi(x) \to \int \mathcal{D}\tilde{\phi}(k)$ (covers space of field config's equally well)
- $\mathcal{L}_0$ is quadratic part in fields.

$$i\int d^4x\,(\mathcal{L}_0 + j\phi) = i\int \frac{d^4k}{(2\pi)^4}\left[\tfrac{1}{2}\tilde{\phi}(-k)\frac{1}{\Delta(k)}\tilde{\phi}(k) + \tilde{j}(-k)\tilde{\phi}(k)\right]$$

defines 'Kernel' $\frac{1}{\Delta(k)}$. $\Delta(-k) = \Delta(k)$ due to $\int d^4k$

$$\xrightarrow[\tilde{\phi}\to\tilde{\phi}-\tilde{j}\Delta]{\text{shift}} i\int \frac{d^4k}{(2\pi)^4}\left[\tfrac{1}{2}\tilde{\phi}(-k)\frac{1}{\Delta(k)}\tilde{\phi}(k) - \tfrac{1}{2}\tilde{j}(-k)\Delta(k)\tilde{j}(k)\right] \qquad (45c)$$

⇒ $\int \mathcal{D}\tilde{\phi}$ is Gaussian ! ⇝ can be done ⇝ j-indep. const.

$$\boxed{W[\tilde{j}] = \frac{1}{c}\, e^{i\int d^4x\, \mathcal{L}_{int}[\frac{1}{i}\int d^4k\, e^{ikx}\delta_{\tilde{j}(k)}]}\; e^{-i\int \frac{d^4q}{(2\pi)^4}\frac{1}{2}\tilde{j}(-q)\Delta(q)\tilde{j}(q)}} \qquad (46)$$

(c such that $W[\tilde{j}=0] = 1$)

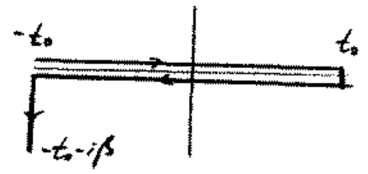
(→ Greens just from further δ_j's on this monstrum...)

$$G_n = \left(\tfrac{1}{i}\delta_{j(x_1)}\right)\cdots\left(\tfrac{1}{i}\delta_{j(x_n)}\right) W[j]\Big|_{j=0}$$

$$\rightsquigarrow \tilde{G}_n = \left(\int d^4x_1\, e^{ik_1x_1}\tfrac{1}{i}\int d^4q_1\, e^{iq_1x_1}\delta_{\tilde{j}(q_1)}\right)\cdots\left(\;n\;\right) W[\tilde{j}]\Big|_{\tilde{j}=0}$$

$$= \left((2\pi)^4\tfrac{1}{i}\delta_{\tilde{j}(-k_1)}\right)\cdots\left(\;n\;\right) W[\tilde{j}]\Big|_{\tilde{j}=0} \qquad (47)$$

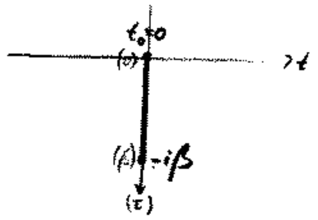
contours $(T\neq 0)$



(e.g. $\sigma=\frac{\beta}{2}$)

"real time"

- propagators → matrices
- ϕ-parts factorize → $\mathcal{N}$
  [A. Das, (2.28) pg. 50]
- propagators have physical interpretation: $(T=0)+(T\neq 0)$-piece
- useful for calculating dynamic processes



"imaginary time" / Matsubara

- useful for calculating static properties
- much simpler than real time
- analogous to $T=0$ case

def imag time τ: $\boxed{t=-i\tau}$, $0\le\tau<\beta$, τ real

$$\hat\phi^H = e^{H\tau}\hat\phi(0,\vec x)\, e^{-H\tau} \quad ; \quad i\int_C d^4x = i\int_C dt\int d^3r = \int_0^\beta d\tau \int d^3r \equiv \int^\beta \qquad (48)$$

$$W[j] = \int D\phi\; e^{\int^\beta(\mathcal{L}+j\phi)}$$

$$G_n = \langle \mathcal{T}\,\hat\phi^H(1)\dots\hat\phi^H(n)\rangle = \delta_{j(1)}\dots\delta_{j(n)}\, W[j]\big|_{j=0}$$

"Matsubara frequencies"

fields periodic ⇝ Fourier discrete in frequencies

$$f(\tau+\beta)=f(\tau) \qquad f(\tau) = T\sum_{n=-\infty}^{\infty} e^{i\omega_n\tau}\tilde f(\omega_n) \quad , \omega_n = \frac{2\pi n}{\beta} = 2\pi n T \qquad (49)$$

$$\Big(\Big(f(\tau+\beta) = -f(\tau) \qquad\qquad , \omega_n = \frac{\pi}{\beta}(2n+1) \Big)\Big) \leftarrow \text{Fermi}$$

$$\rightsquigarrow \tilde f(\omega_n) = \int_0^\beta d\tau\, e^{-i\omega_n\tau} f(\tau)$$

$$\rightsquigarrow \boxed{\begin{array}{ll} f(x) = \sum\!\!\!\!\!\!\int_K e^{-iKx}\tilde f(K) & \sum\!\!\!\!\!\!\int_K = T\sum_n \int\frac{d^3k}{(2\pi)^3} \\ \tilde f(K) = \int^\beta e^{iKx} f(x) & \int^\beta = \int_0^\beta d\tau\int d^3x \end{array}} \qquad (50)$$

$$K=(i\omega_n,\vec k)\ ,\quad x=(-i\tau,\vec x)\ ,\quad Kx = \omega_n\tau - \vec k\vec x \quad \text{(Mink. metrics)}$$

$$\Big(\Big(\text{Delta-fct: } \int^\beta e^{iKx} = \int_0^\beta d\tau\int d^3x\; e^{i\omega_n\tau}e^{-i\vec k\vec x} = \beta\,\delta_{n_k,0}\,(2\pi)^3\delta(\vec k) \equiv [K] \Big)\Big)$$