4) Greens @ T, Path Integral

it should come as no surprise (see ex on $FT^2/6$), that

$$G_n = Tr\left(\frac{e^{-\beta\hat{H}}}{Z}\, \mathcal{T}\, \hat{\phi}^H(x_1)\dots\hat{\phi}^H(x_n)\right)$$

$\rightsquigarrow G_0 = 1$, G_2 ! ; $\beta\to\infty$ $(T\to 0)$ $\Rightarrow \langle 0|..|0\rangle$, $Z=1$ $(E_0=0)$

automatically covered! choice of E scale

the (∞ many) Greens fcts have the generating functional

$$W[j] \stackrel{!}{=} \frac{1}{Z} Tr\left(e^{-\beta\hat{H}}\, \mathcal{T}\, e^{i\int d^4x\, j(x)\hat{\phi}^H(x)}\right), \quad Z = Tr(..)\big|_{j=0} \quad ((W[0]=1))$$

$((\langle A\rangle = Tr(\rho A)))$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!}\int d^4x_1\dots d^4x_n\, \underbrace{Tr\left(\frac{e^{-\beta\hat{H}}}{Z}\,\mathcal{T}\,\hat{\phi}^H(x_1)\dots\hat{\phi}^H(x_n)\right)}_{\stackrel{!}{=} G_n} j(x_1)\dots j(x_n)$$

$$\Rightarrow G_n = \left(\frac{1}{i}\delta_{j(x_1)}\right)\dots\left(\frac{1}{i}\delta_{j(x_n)}\right) W[j]\,\Big|_{j=0}$$

functional derivative: $\quad \partial_{f_j} f_k = \delta_{jk}$

$$\text{generalise} \longrightarrow \quad \delta_{f(x)} f(y) \equiv \delta(x-y) \tag{28}$$

next, express Tr in $W[j]$ in suitable basis

$\Rightarrow$ PI (path integral; functional integral)

first comments:
- PI is essential in QFT
  gauge theories $\rightsquigarrow$ Faddeev-Popov
- PI is beautiful & elegant
- PI gives direct approach to diagrams of a pert. treatment of a many-particle-theory
- Lit can be tough to understand

Abers + Lee, Physics Reports 9C (1973) 1
Itzykson + Zuber, Quantum Field Theory (1985)
Cheng + Li, Gauge theory of elem. part. phys (1984)
Feynman + Hibbs, QMech and Path Integrals (1965)

• here: approach you might not have seen before; treats all Greens at once, includes $T \neq 0$ case.

(($\mathcal{L} \xrightarrow[\text{PI}]{\text{recipe} \to \hat\phi^H} G_n$; does all the work for us))

(again)

$$\boxed{\begin{aligned} G_n(x_1 \dots x_n) &= \mathrm{Tr}\left(\frac{e^{-\beta \hat H}}{Z}\,\mathcal{T}\hat\phi^H(x_1)\dots\hat\phi^H(x_n)\right) \xrightarrow[T\to 0]{\beta\to\infty} \langle 0_H|\mathcal{T}\hat\phi^H(x_1)\dots\hat\phi^H(x_n)|0_H\rangle \\ &= \left(\frac{1}{i}\delta_{j(x_1)}\right)\cdots\left(\frac{1}{i}\delta_{j(x_n)}\right) W[j]\Big|_{j=0} \\ W[j] &= \frac{1}{Z}\,\mathrm{Tr}\left(e^{-\beta\hat H}\,\mathcal{T} e^{i\int d^4x\, j(x)\hat\phi^H(x)}\right) \xrightarrow[T\to 0]{\beta\to\infty} \langle 0_H|\mathcal{T} e^{i\int d^4x\, j(x)\hat\phi^H(x)}|0_H\rangle \\ &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^4x_1\dots d^4x_n\; G_n(x_1\dots x_n)\, j(x_1)\dots j(x_n)\end{aligned}} \quad (29)$$

$\hat\phi^H$: Bose-field operators with full Heisenberg-time-dependence

here, treat only Bose case (ex: "ϕ^4": $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}g^2\phi^4$)

$j(x)$: 'source field'. arbitrary. complex.

philosophy: interested in physics of a finite x-sector ($x = t, \vec r$)

⇝ $j \to 0$ rapidly outside

⇝ can forget about adiabatic turning on/off of interaction : $H \neq H(t)$

ground state $|0_H\rangle$ of H non-degenerate, $H|0_H\rangle = E_0|0_H\rangle \overset{\text{choice of E-scale}}{=} 0$

more philo: one does not know much about $|0_H\rangle$.

that's why the trace ($\hat{=}$ freedom of choice of basis) helps!

next: choice of basis (in the space of field operators $\hat\phi^H$)

Basis for Tr

motivation: should have connection to $\hat{\phi}^H$.

observe: $\hat{\phi}^H(x)\big|_{t=0} = \hat{\phi}(\vec{r}) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\varepsilon_k}} \left(\hat{b}_{\vec{k}}\, e^{i\vec{k}\vec{r}} + \hat{b}^\dagger_{\vec{k}}\, e^{-i\vec{k}\vec{r}}\right)$

Schrödinger operators — ϕ^4: $\varepsilon_{\vec{k}} = \sqrt{m^2+\vec{k}^2}$

$\hat{\phi}$'s are hermitean, and obey comm. rel's (at equal time = 0)

$$[\hat{\phi}(\vec{r}), \hat{\pi}(\vec{r}\,')] = i\delta(\vec{r}-\vec{r}\,'),\ \underline{[\hat{\phi}(\vec{r}), \hat{\phi}(\vec{r}\,')]=0},\ [\hat{\pi}(\vec{r}), \hat{\pi}(\vec{r}\,')] = 0$$

$\hookrightarrow$ $\exists$ system of simultaneous eigenfunctions

((recall $[\hat{A}, \hat{B}] = 0 \Rightarrow \exists\ |a,b\rangle$ with $\hat{A}|a,b\rangle = a|a,b\rangle$, $\hat{B}|a,b\rangle = b|a,b\rangle$))

$\rightsquigarrow$ state in basis $\Leftrightarrow$ set of real eigenvalues $\phi(\vec{r})$
"field config." $\{\phi\}$

$$\Rightarrow \underset{e^{i\hat{H}t}}{\hat{\phi}(\vec{r})}\ \underset{e^{-i\hat{H}t}e^{i\hat{H}t}}{|\{\phi\}\rangle} = \underset{e^{i\hat{H}t}}{\phi(\vec{r})}\ \underset{e^{-i\hat{H}t}e^{i\hat{H}t}}{|\{\phi\}\rangle}$$

$\hat{\phi}^H(\vec{r})\,|\{\phi\},t\rangle = \phi(\vec{r})\,|\{\phi\},t\rangle$, where $|\{\phi\},t\rangle \equiv e^{i\hat{H}t}|\{\phi\}\rangle$

$\hookrightarrow$ pick as basis, at some $t = t_0$.

$$\Rightarrow \boxed{\begin{array}{l} \text{basis} \equiv |\{\phi\}, t_0\rangle \\ \text{summation in Tr} = \mathcal{N} \prod_{\vec{r}} \int d\phi(\vec{r}) =: \int_{\vec{r}} \mathcal{D}\phi \end{array}} \qquad (30)$$

step 3 comments:

c1) $\int d\phi$: integrate over <u>continuum</u> of (real) ϕ-values ??

look at spectrum of $\hat{\phi}(\vec{r})\,|\{\phi\}\rangle = \phi(\vec{r})\,|\{\phi\}\rangle$!

$$\hat{\phi}(\vec{r}) = \int \frac{d\vec{k}}{\sqrt{(\,)2\varepsilon_k}} \left(\boxed{\hat{b}\, e^{i\vec{k}\vec{r}}} + \hat{b}^\dagger e^{-i\vec{k}\vec{r}}\right)$$

$\hookrightarrow \sqrt{\frac{\varepsilon_k}{2}}\, x_k + \sqrt{\frac{1}{2\varepsilon_k}}\, \partial_{x_k}$ coord. space rep.

fix $\vec{r}$.
<u>one</u> of the $\hat{\phi}$-op's

$$= \int \frac{d\vec{k}}{\sqrt{(2\pi)^3}}\, x_k$$

$\rightsquigarrow$ $\hat{\phi}(\vec{r})\, \varphi(\{x_k\}) = a\, \varphi(\{x_k\}) \Rightarrow \varphi = \ldots \cdot \delta(\hat{\phi}(\vec{r}) - a)$

$\Rightarrow$ eigenvalues a of $\hat{\phi}(\vec{r})$ are <u>continuous</u> on real axis !

c2) time evolution goes like $e^{-i\hat{H}t}$. above, 'wrong' sign !?

$\rightsquigarrow$ interpretation of argument t in $|\{\phi\},t\rangle$:

$X\langle x|h\rangle \equiv x\langle x|h\rangle$

coord. op. of 1D QM — general Hilbert-vector

$\rightsquigarrow (x-a)\langle x|a\rangle = 0 \quad \Rightarrow \quad \langle x|a\rangle = \delta(x-a)$

δ-fct broadens.

$|a,t_1,t_2\rangle = e^{-i\hat{H}(t_1-t_2)}\underbrace{|a\rangle}_{\delta\text{-fct}} \qquad ((\rightsquigarrow |a,t_2,t_2\rangle = |a\rangle \checkmark))$

time when $|a\rangle$ was δ

look @ packet at time t_1

$\Rightarrow |a,0,t\rangle = \underline{e^{i\hat{H}t}}|a\rangle$ means observe (at $t=0$) a state that was localized at t.

$\rightsquigarrow |\{\phi\},t\rangle$ — time of localization.

c3) symbols.

$\int\mathcal{D} \rightsquigarrow$ one integral at every point $\vec{r}$

$\mathcal{N}$: measure factor. absorbed in $\mathcal{D}$. ((other authors: $\mathcal{N}\int\mathcal{D}$))

often: $\cancel{\mathcal{N}}$, because $\underline{\ln W}$ wanted.

index $\int_{\vec{r}}$: which space gets integrals

$((\text{later: } \prod_t \int_{\vec{r}} \mathcal{D}\phi = \int_x \mathcal{D}\phi \equiv \int\mathcal{D}\phi \ ))$

imaginary time β

states: $W[j] = \left(\lim_{\beta\to\infty}\right)^{T=0} \int_{\vec{r}} \mathcal{D}\phi \ \langle\{\phi\},t_0| e^{-\beta\hat{H}} \mathcal{T} e^{i\int d^4x\, \hat{\phi}^H(x) j(x)} |\{\phi\},t_0\rangle$

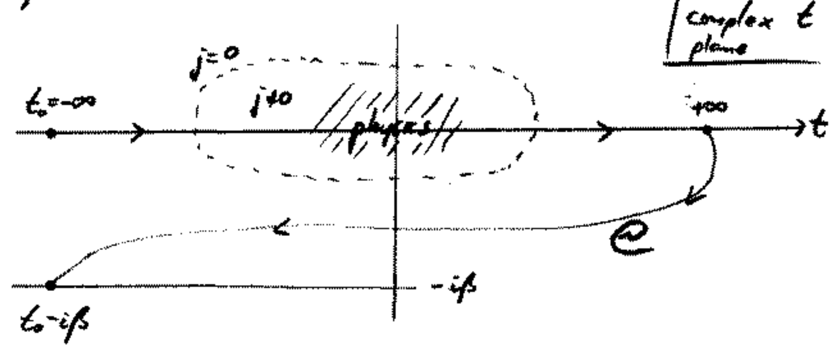
$\hookrightarrow \langle\{\phi\}| e^{-i\hat{H}t_0} e^{-\beta\hat{H}} = \langle\{\phi\}| e^{-i\hat{H}(t_0-i\beta)} = \langle\{\phi\},t_0-i\beta|$

times in $\hat{\phi}^H, j$: $-\infty\ldots\infty$.

$j\to 0$ outside 'physics' $\rightsquigarrow$ can read $\infty \mathrel{\hat{=}}$ pretty large 'phys-like'

choose $t_0 = -\infty$ ('phys-like')

now, relate the 4 time-labels :



(31)

⤳ contour $\mathcal{C}$ in complex t-plane.

⤳ generalize time ordering $\mathcal{T}$ to path ordering along $\mathcal{C}$.

⤳ (since $j=0$ for added contour) $\int d^4x = \int_{-\infty}^{\infty} dt \int d^3x \to \int_{\mathcal{C}} dt \int d^3x \equiv \int_{\mathcal{C}} d^4x$ (32)

states: $$\boxed{W[j] = \left(\lim_{\beta\to\infty}\right) \int_{\vec r} \mathcal{D}\phi \, \langle\{\phi\}, -\infty - i\beta |\, \mathcal{T}\, e^{i\int_{\mathcal{C}} d^4x\, \hat\phi^H(x)\, j(x)} \,|\{\phi\}, -\infty\rangle} \quad (33)$$

outlook: $\langle\ldots\rangle \to e^{i\int_{\mathcal{C}} d^4x\, (\mathcal{L}+j\phi)}$

discretize contour $\mathcal{C}$

want to take care of $\langle\ldots\rangle$ in $W[j]$. expand $e^{\cdots}$

consider $$W_{\text{quadratic in } j} = (\,)\int_{\vec r}\mathcal{D}\phi \left(-\tfrac12\right)\int_{\mathcal{C}} d^4x_1 \int_{\mathcal{C}} d^4x_2\, j(x_1)\, j(x_2) \left\{\Theta(t_1-t_2) M_1 + \Theta(t_2-t_1) M_2\right\} \quad (34)$$

$$M_1 \equiv \langle\{\phi\}|\, e^{-i\hat H(t'-t_1)}\, \hat\phi(\vec r_1)\, e^{-i\hat H(t_1-t_2)}\, \hat\phi(\vec r_2)\, e^{-i\hat H(t_2-t)} \,|\{\phi\}\rangle$$

$t' \equiv -\infty - i\beta$ biggest time (on $\mathcal{C}$)

$t \equiv -\infty$ smallest time (on $\mathcal{C}$)

$$M_2 = M_1^{\,x_1 \leftrightarrow x_2}$$

op's $\hat\phi \to$ eigenvalues ϕ if complete systems $1 = \int \mathcal{D}\phi\, |\{\phi\}\rangle\langle\{\phi\}|$ inserted

$$M_1 = \int_{\vec r}\mathcal{D}\phi'' \int_{\vec r}\mathcal{D}\phi'\ \phi''(\vec r_1)\, \phi'(\vec r_2) \quad (35)$$
$$\times \langle\{\phi\}|e^{-i\hat H(t'-t_1)}|\{\phi''\}\rangle \langle\{\phi''\}|e^{-i\hat H(t_1-t_2)}|\{\phi'\}\rangle \langle\{\phi'\}|e^{-i\hat H(t_2-t)}|\{\phi\}\rangle$$

if 7 $\hat\phi$'s ⤳ 7 $\mathcal{D}\phi$ integrals, 8 matrix elements

⤳ study $$\boxed{\langle\{\phi'\}|\,U\,|\{\phi\}\rangle, \quad U \equiv e^{-i\hat H(t'-t)}} \quad (36)$$

answer on $FT^2/21$

discretize: $t'-t = n\,\delta t$

$\hat{H} \neq \hat{H}(t) \Rightarrow \mathcal{U} = u^n, \quad u = e^{-i\hat{H}\delta t}$

$$\langle\{\phi'\}|\mathcal{U}|\{\phi\}\rangle = \int_{\vec{r}} \mathcal{D}\phi_{n-1} \dots \int_{\vec{r}} \mathcal{D}\phi_1 \, \langle\{\phi'\}|u|\{\phi_{n-1}\}\rangle$$
$$\cdot \langle\{\phi_{n-1}\}|u|\{\phi_{n-2}\}\rangle \dots \langle\{\phi_1\}|u|\{\phi\}\rangle \qquad (37)$$

$$\langle\{\phi_1\}|u|\{\phi\}\rangle \underset{\delta t \to 0}{\overset{n\to\infty}{=\!=}} \langle\{\phi_1\}|1 - i\hat{H}\delta t|\{\phi\}\rangle = \langle\{\phi_1\}|\{\phi\}\rangle - i\delta t\langle\{\phi_1\}|\hat{H}|\{\phi\}\rangle$$

$\hat{H} \to H_{class}$

in a bosonic field theory, $\hat{H} = \int d^3r\, \hat{\mathcal{H}} = \int d^3r \left[\frac{1}{2}\hat{\pi}^2 + V(\hat{\phi})\right]$

$\left(\pi = \partial_{\dot\phi}\mathcal{L} = \dot\phi \quad (\phi^4)\right)$

in $\hat{H}|\{\phi\}\rangle$, $V(\hat{\phi}) \to \underset{\text{eigenvalue}}{V(\phi)}$

same for $\hat{\pi}$: 3^{rd} equal time commutator $[\pi,\pi] = 0$

$\Rightarrow \exists$ simultaneous system of eigenfcts

$\hat{\pi}(\vec{r})|\{\pi\}\rangle = \pi(\vec{r})\,|\{\pi\}\rangle$

continuous, real, $-\infty \dots \infty$

$$\langle\{\phi_1\}|u|\{\phi\}\rangle = \underline{\int_{\vec{r}} \mathcal{D}\pi} \, \langle\{\phi_1\}|\{\pi\}\rangle\underline{\langle\{\pi\}|\{\phi\}\rangle}\; e^{-i\delta t\overbrace{\int d^3r\left[\frac{1}{2}\pi^2 + V(\phi)\right]}^{H_{class}\,!}} \qquad (38)$$

(inserted 1) $\qquad 1 - i\delta t \xrightarrow{\text{back}} e^{-i\delta t\dots}$

$\{\phi\}$ - representation

(need scalar product $\langle\{\phi\}|\{\pi\}\rangle$)

not used so far: $[\hat{\phi}(\vec{r}), \hat{\pi}(\vec{r}')] = i\delta(\vec{r}-\vec{r}') \overset{\text{reads}}{=} i\,\delta_{\phi(\vec{r})}\,\phi(\vec{r}')$

$\rightsquigarrow \hat{\phi}(\vec{r}) \to \phi(\vec{r}), \quad \hat{\pi}(\vec{r}) \to -i\delta_{\phi(\vec{r})}$

commutators ✓

$-i\delta_{\phi(\vec{r})}\langle\{\phi\}|\{\pi\}\rangle = \pi(\vec{r})\langle\{\phi\}|\{\pi\}\rangle \qquad$ diff. eq!

$\rightsquigarrow$ sol'n: $\langle\{\phi\}|\{\pi\}\rangle \sim e^{+i\int d^3r\, \pi(\vec{r})\phi(\vec{r})}$

$$\Rightarrow \langle\{\phi_1\}|u|\{\phi\}\rangle \sim \int_{\vec{r}} \mathcal{D}\pi \; e^{i\int d^3r\,\{\pi\phi_1 - \pi\phi - \delta t[\frac{1}{2}\pi^2 + V(\phi)]\}} \qquad (39)$$