3) (review of) Quantum Field Theory

from now on, $\hbar = c = k_B = 1$

in this chapter, $T = 0$

outlook:

$$\left.\begin{array}{l} \mathcal{L} \xrightarrow[\text{definition}]{\text{'recipe'; } \hat{\phi}^H} G_n \xrightarrow[\text{experience}]{\text{response fct}} \text{physics} \\ \qquad\qquad\qquad\quad \downarrow \\ \qquad\qquad\qquad\;\; \text{pert.} \dashrightarrow \text{phys. as } \Sigma \end{array}\right\} \quad (14)$$

auxiliary fcts

(for a quick intro, take) simplest Field Theory (?)

• free bosonic field in 0+1 D

$$\left.\begin{array}{l} \int d^3r \to 1 \;, \quad \sqrt{m^2 + \vec{k}^2} \to m \\ \mathcal{L} = L = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m^2\phi^2 \end{array}\right\} \quad (15)$$

(eqn of motion via $\partial_\mu \partial_{\partial_\mu \varphi} \mathcal{L} = \partial_\varphi \mathcal{L} \;\Rightarrow\; \ddot{\phi} = -m^2\phi$)

$\rightsquigarrow$ harm. osz! $\phi \hat{=} q$, $m \hat{=} \omega$, $\Pi \hat{=} p$

(conj. momenta via $\Pi_\mu = \partial_{\dot{\varphi}_\mu} \mathcal{L} \;\Rightarrow\; \Pi = \dot{\phi}$)

$$H_{class.} = \Pi\dot{\phi} - L = \frac{1}{2}\Pi^2 + \frac{1}{2}m^2\phi^2$$

$$\Pi \to -i\partial_\phi \;, \quad H_{class} \to H = -\frac{1}{2}\partial_\phi^2 + \frac{1}{2}m^2\phi^2$$

"QM at one point, in 'field direction'"

equal time commutators $[\phi, \Pi] = i$, rest $= 0$

($[\phi_s(t,\vec{r}), \Pi_s(t',\vec{r}')] = i\delta(\vec{r}-\vec{r}')\delta(t-t')\delta_{ss'}$, $\{\ldots,\ldots\}$ for Fermi)

algebraical method

$$b = \sqrt{\tfrac{m}{2}}\,\phi + \frac{1}{\sqrt{2m}}\partial_\phi \;, \quad b^+ = \ldots - \ldots$$

$$\rightsquigarrow \phi = \frac{1}{\sqrt{2m}}(b + b^+) \quad \leftarrow \text{coordinate operator}$$

$$\rightsquigarrow H = \frac{m}{2}(bb^+ + b^+b) \;, \quad [b, b^+] = 1$$

Schrödinger functional
$H\psi(\phi) = E\,\psi(\phi)$, ψ_n, E_n
$\psi[\phi(\vec{r})]$

change of representation ("2nd quant.")

$|n\rangle$ cons ((complete orthonormal system)) , $|n\rangle = \frac{1}{\sqrt{n!}}(\hat{b}^+)^n|0\rangle$, $b|0\rangle = 0$

$$\langle n'|\hat{b}|n\rangle = \langle n'|\frac{1}{\sqrt{n!}}\hat{b}(\hat{b}^+)^n|0\rangle \quad , \quad [\hat{b},\hat{b}^+] = 1$$
$$= \langle n'|\left(\frac{1}{\sqrt{n!}}(\hat{b}^+)^{n-1} + \frac{1}{\sqrt{n!}}\hat{b}^+\hat{b}(\hat{b}^+)^{n-1}\right)|0\rangle$$
$$\vdots$$
$$= \langle n'|\left(\frac{n}{\sqrt{n!}}(\hat{b}^+)^{n-1} + \frac{1}{\sqrt{n!}}(\hat{b}^+)^n\hat{b}\right)|0\rangle$$
$$= \sqrt{n}\langle n'|n-1\rangle$$

$\rightsquigarrow \hat{b}|n\rangle = \sqrt{n}|n-1\rangle$ ((and $\hat{b}^+|n\rangle = \sqrt{n+1}|n+1\rangle \rightsquigarrow \hat{b}^+\hat{b}|n\rangle = n|n\rangle$

$$\hat{H} = \frac{m}{2}(\hat{b}\hat{b}^+ + \hat{b}^+\hat{b}) \quad , \quad [\hat{b},\hat{b}^+] = 1$$
$$= m\left(\tfrac{1}{2} + \hat{b}^+\hat{b}\right) \quad , \quad :\hat{H}: \equiv m\hat{b}^+b \tag{16}$$

normal ordering ((creation op's to left $:\hat{b}\hat{b}^+: \equiv \hat{b}^+\hat{b}$))

$\hat{=}$ omission of ∞ constants (choice of E scale)

$$\phi \to \hat{\phi} = \frac{1}{\sqrt{2m}}(\hat{b} + \hat{b}^+)$$

Heisenberg rep.

$$\hat{\phi}^H = e^{i\hat{H}t}\hat{\phi}e^{-i\hat{H}t} = e^{im(\frac{1}{2}+\hat{b}^+\hat{b})t}\frac{1}{\sqrt{2m}}(\hat{b}+\hat{b}^+)e^{-im(\frac{1}{2}+\hat{b}^+\hat{b})t}$$

"$\cdot|n\rangle$" $\;\rightarrow \left\{e^{im(\frac{1}{2}+n-1)t}\frac{1}{\sqrt{2m}}\hat{b} + e^{im(\frac{1}{2}+n+1)t}\frac{1}{\sqrt{2m}}\hat{b}^+\right\}e^{-im(\frac{1}{2}+n)t}$

$$= \frac{1}{\sqrt{2m}}\left(\hat{b}e^{-imt} + \hat{b}^+e^{imt}\right) \quad \text{"field operator"} \tag{17}$$

$\rightarrow$ homework problems (1), (2)

$\Rightarrow$ 'recipe' for canonical quantization

$\boxed{1}$ $\mathcal{L} \xleftrightarrow{\partial_\mu \partial_{\partial_\mu \varphi} \mathcal{L} = \partial_\varphi \mathcal{L}}$ field eqns.

$\boxed{2}$ $\Pi_i = \partial_{\dot\varphi_i} \mathcal{L}$, $\mathcal{H} = \Pi_i \dot\varphi_i - \mathcal{L}$

$\boxed{3}$ Bose $[\varphi_s(t,\vec r), \Pi_{s'}(t',\vec r')] = i\delta(t-t')\delta(\vec r-\vec r')\delta_{ss'}$ (18)

Fermi $\{$ '' , '' $\}$ = ''

$\boxed{4}$ expand φ, Π in terms of solutions of field eqns.

coefficients $\sim$ creation / annihilation op's

$\varphi \to$ field operator

---

some theories one should know

- harmonic oscillator: $\mathcal{L} = \frac{1}{2}\dot\phi^2 - \frac{1}{2}m^2\phi^2$ (19)

  (0+1 D free field theory)

- ϕ^4 theory : $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$, $\phi(x)$ (20)

  (the toy-model . ϕ scalar, real $\sim$ neutral particles)

- QED : $\mathcal{L} = \bar\psi(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ (21)

  (electrons + photons)

  $\not{D} = \gamma^\mu(\partial_\mu - ieA_\mu)$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

- QCD : $\mathcal{L} = \bar\psi(i\not{D} - m)\psi - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}$ (22)

  (quarks + gluons)

  $A_\mu = A^a_\mu T^a$, $F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \equiv F^a_{\mu\nu}T^a$

  generators of $SU(N)$, $a = 1 .. N^2-1$ (reality : $N=3$ colours $\rightsquigarrow$ 8 gluons)

- Higgs model: $\mathcal{L} = (D_\mu\phi)^* D^\mu\phi + \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ (23)

  (neutral / charged scalar + (SSB!) massive gauge bosons)

- Standard Model (see page $FT^2/4$, chapter 2)

why be happy with $\hat\phi^H$?

$\longrightarrow$ def. Greens fcts !

time ordering: bigger times to the left

$$G_n(x_1,\dots,x_n) = \langle 0|\,\mathcal{T}\,\hat\phi^H(x_1)\dots\hat\phi^H(x_n)\,|0\rangle \qquad (24)$$

philosophy: $\mathcal{L} \xrightarrow{\;\to\hat\phi^H\to\;} \boxed{G_n} \longrightarrow$ physics

$\boxed{G_n}$ $\to$ part. $\longrightarrow \Sigma$ physics

ex $G_2(t) = \langle 0|\,\mathcal{T}\,\hat\phi(t)\,\hat\phi(0)\,|0\rangle$

((transl. inv.: $t_1,t_2 \to G_2(t_1-t_2)$))

$$= \frac{1}{2m}\,\theta(t)\,\langle 0|\left(b e^{-imt} + \cancel{b^+} e^{imt}\right)\left(\cancel{b}+b^+\right)|0\rangle$$

$$+\frac{1}{2m}\,\theta(-t)\,\langle 0|\left(b+\cancel{b^+}\right)\left(\cancel{b}e^{-imt}+b^+e^{imt}\right)|0\rangle$$

$$= \frac{1}{2m}\left(\theta(t)e^{-imt}+\theta(-t)e^{imt}\right)\underbrace{\langle 0|\,bb^+\,|0\rangle}_{=1}$$

$$= \frac{1}{2m}\,e^{-im|t|} \qquad (25a)$$

$$\left(\partial_t^2+m^2\right) i G_2(t) = \delta(t) \qquad \text{(that's the reason for the name)}$$

Fourier transform ?

convergence factor. see below!

$$\tilde G_2(\omega) = \int dt\, e^{i\omega t}\,\frac{1}{2m}\,e^{-im|t|}\left(e^{-\varepsilon|t|}\right)$$

$$= \frac{1}{2m}\left(\frac{e^{(\,)t}}{(i\omega+im+\varepsilon)}\Bigg|_{t=-\infty}^{0} + \frac{e^{(\,)t}}{(i\omega-im-\varepsilon)}\Bigg|_{t=0}^{\infty}\right)$$

$$= \frac{1}{2m}\left(\frac{1}{i\omega+im+\varepsilon} - \frac{1}{i\omega-im-\varepsilon}\right) = \frac{i+\varepsilon/m}{\omega^2-(m-i\varepsilon)^2}$$

$$= \frac{i}{\omega^2-m^2+i\varepsilon} + O(\varepsilon) \qquad (25)$$

Feynman propagator (in 0+1 D)

response function (example for "$G_n \to$ physics")

$$H_0 = \underbrace{H}_{\text{some mean, unsolvable, many-body system}} + \underbrace{H_{ext}}_{\text{source; perturbation}} = H - J(t)\phi \qquad (\mathcal{L} \to \mathcal{L} + J\phi)$$

interaction picture wrt H_{ext}

ask for $A(t) = \langle 0 | \hat\phi^H | 0 \rangle_0$ $\qquad |0; t=-\infty\rangle_0 = |0\rangle$ $\quad (H_{ext}(t=-\infty) = 0)$

$\partial_t |q,t\rangle = -iH(t)|q,t\rangle$

$$= \langle 0 | (1 + \dots) \hat\phi^H(t) \Big(1 - i\int_{-\infty}^{t} dt' \underbrace{\hat H_{ext}(t')}_{= -J(t')\hat\phi^H(t')}\Big) |0\rangle$$

$$= i\int_{-\infty}^{t} dt' \langle 0 | [\hat\phi^H(t), \hat\phi^H(t')] | 0 \rangle \underbrace{J(t')}_{\equiv e^{\varepsilon t'} j(t')} + \underbrace{O(J^2)}_{\text{small!}}$$

$e^{\varepsilon t} a(t) \equiv$

from t'-integration limit

$$\Rightarrow a(t) = \int dt' \underbrace{\theta(t-t')\, e^{-\varepsilon(t-t')}\, i\langle 0 | [\ ,\ ] | 0 \rangle}_{\equiv \chi(t-t') \quad \text{response fct}} j(t') \qquad (26)$$

here (harm. oscillator) one can actually compute χ:

$$\chi(t) = \theta(t)\, e^{-\varepsilon t} \frac{\sin(mt)}{m}$$

$$\tilde\chi(\omega) = -\frac{1}{(\omega + i\varepsilon)^2 - m^2}$$

causality

$$i\tilde G_2(\omega) = \tilde\chi(\omega) \quad \text{for } \mathrm{Re}\,\omega > 0 \qquad (G_n \to \text{physics!}) \qquad (27)$$

outlook Temperature? $\to$ next chapter.

generate all Greens, without canonical quant.

$\to$ path integral! $(T, T=0)$ at once; next chapter