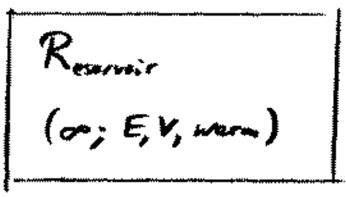
2) (review of) Stat Mech

(.. who remembers Stat Mech by heart? → quick redo!)

the central eq: $S = -\sum_n p_n \ln(p_n) = -\mathrm{Tr}\,(\rho \ln(\rho))$ (1)

for every ensemble (here: canonical)

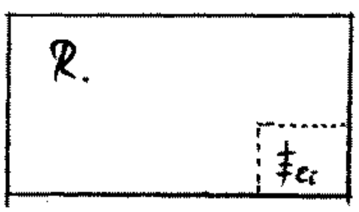


\# of states $= \eta$

equal prob. $\left(\frac{1}{\eta}\right)$ for every state

$\rightsquigarrow S = -\sum_{n=1}^{\eta} \frac{1}{\eta} \ln \frac{1}{\eta} = \ln \eta$

now, look @ small subsystem (1 particle?) (canonical: $\updownarrow E$ only)



$\eta = \eta_1 + \eta_2 + \eta_3 + \dots$

$\llcorner$ # of R-states, if subsystem is in state 1

$$S = \ln \eta = \underbrace{\ln \eta_i}_{= S_R(E-\varepsilon_i) = S_R(E) - \varepsilon_i \partial_E S_R(E)} - \ln \underbrace{\boxed{\frac{\eta_i}{\eta}}}_{= p_i \text{ (for subsystem to be in state } i)}$$

((E ≫ ε_i always true))

$\rightsquigarrow \ln p_i = -[S - S_R(E)] - \beta\varepsilon_i$, where $\beta \equiv \partial_E S_R$ (2)

$\rightsquigarrow p_i = \underbrace{e^{-[\;]}}_{= 1/Z} e^{-\beta\varepsilon_i} \quad \rightarrow \quad \rho = \frac{1}{Z} e^{-\beta H}$ (3a)

(p_i's: eigenvalues of ρ)

normalize: $1 \overset{!}{=} \sum_i p_i = \frac{1}{Z} \sum_i e^{-\beta\varepsilon_i} \rightarrow Z = \mathrm{Tr}\, e^{-\beta H}$ (3b) expectation value (= ensemble average)

$$\langle \text{any} \rangle = \sum_p p_n (\text{any})_n$$

any = operator A : $(\text{any})_n = \int dx\, \varphi_n^* A \varphi_n$, $H\varphi_n = E_n \varphi_n$

$$\langle A \rangle = \sum_n p_n \int dx\, \varphi_n^* A \varphi_n$$

$$= \sum_n \frac{1}{Z} e^{-\beta E_n} \int dx\, \varphi_n^* A \varphi_n$$

$$= \sum_n \int dx\, \varphi_n^* \frac{1}{Z} e^{-\beta H} A \varphi_n$$

$$= \sum_n \int dx\, \varphi_n^* \rho A \varphi_n \quad \left(= \sum_n \langle n | \rho A | n \rangle \right)$$

$$\langle A \rangle = \mathrm{Tr}\,(\rho A) \qquad (4)$$

ex $A = \mathcal{T}\, \bar\psi(x_1)\psi(x_2)$

note that for $T \to 0$ ($\beta \to \infty$), $Z = \sum_n e^{-\beta E_n} \to e^{-\beta E_0}$,

$p_0 = 1$, others $= 0$

$\to$ propagator $\langle 0 | \mathcal{T} \bar\psi(x_1)\psi(x_2) | 0 \rangle$ is $T \to 0$ result of eq. (4)

ex $A = H$

$$E = \langle H \rangle = \mathrm{Tr}\left(\frac{1}{Z} e^{-\beta H} H\right) = \frac{1}{Z}\, \partial_{-\beta}\, \mathrm{Tr}\, e^{-\beta H}$$

$$E = -\partial_\beta \ln Z \qquad (5) \qquad (\!(\beta\partial_\beta = -T\partial_T)\!)$$

thermodyn. relations

def. $F \equiv -T \ln Z \qquad (6)$

$\rightsquigarrow \quad (S, p, -\mu) = -(\partial_T, \partial_V, \partial_N)\, F \qquad (7)$

- check S: $p_i = \frac{1}{Z} e^{-\beta \varepsilon_i}$, $\ln p_i = -\ln Z - \beta \varepsilon_i$

$$S = -\sum_i p_i \ln p_i = -\sum_i \frac{1}{Z} e^{-\beta\varepsilon_i}(-\ln Z - \beta\varepsilon_i) = \ln Z + \frac{\beta}{Z} \underbrace{\sum_i \varepsilon_i e^{-\beta\varepsilon_i}}_{= -\partial_\beta \sum_i e^{-\beta\varepsilon_i} = -\partial_\beta Z}$$

$$= \ln Z - \beta\partial_\beta \ln Z = \ln Z + T\partial_T \ln Z$$

$$= \partial_T\, T \ln Z = -\partial_T F \quad ✓$$

• derive p: $V = A\cdot L$ $\quad \frac{(1)}{Z}$

force = E-difference , $A\cdot p_\lambda = -\partial_L \varepsilon_\lambda$ pressure (of mode λ), not probab.

average → $A\cdot p = A\cdot\langle p_\lambda\rangle = \sum_\lambda \frac{e^{-\beta\varepsilon_\lambda}}{Z}(-\partial_L\varepsilon_\lambda) \overset{\partial_L = A\partial_V}{=} -A\sum_\lambda \frac{e^{-\beta\varepsilon_\lambda}}{Z}\partial_V\varepsilon_\lambda$

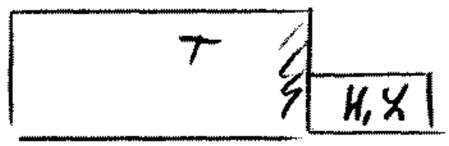
note that $\partial_V F = -T\partial_V \ln Z = -T\frac{1}{Z}\partial_V Z = (-T)\frac{1}{Z}\sum_n e^{-\beta\varepsilon_n}(-\beta)\partial_V\varepsilon_n = -p$ ✓✓

• μ is def if $Z = Z(N)$
conserved quantity (in R-system)

→ defines μ - 'chemical potential'

usually N = part. # (grand canonical ; canonical: $\mu = 0$)

specify systems (coupled to R)



ex.1) two-state system $\frac{\overline{\;\;}\,\varepsilon}{\underline{\;\;}\,0}$ (no interaction)

$H = -b\sigma^z \quad , \quad Z = e^{\beta b} + e^{-\beta b} \qquad \left(\sigma^z = \begin{pmatrix}1 & \\ & -1\end{pmatrix}\right)$

ex.2) harm. oscillator

 , $\varepsilon_n = n\cdot\hbar\omega = n\cdot\hbar c k$

$Z = \sum_n e^{-\beta n\hbar\omega} = \frac{1}{1-e^{-\beta\hbar\omega}}$ (geom. series!)

$\langle n\rangle = \frac{1}{\hbar\omega}\langle\varepsilon\rangle \overset{(5)}{=} \partial_{-\beta\hbar\omega}\ln Z = \dots = \frac{1}{e^{\beta\hbar\omega}-1} \equiv n_{Bose}(\hbar\omega)$

$$n_{Bose}(x) = \frac{1}{e^{\beta x}-1} \qquad (8)$$

ex.3a) again: one bosonic d.o.f.

any # ; $|n\rangle$, $\langle n|n'\rangle = \delta_{nn'}$, $\sum_{n=0}^{\infty}|n\rangle\langle n| = 1$

$b^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \quad , \quad b|n\rangle = \sqrt{n}\,|n-1\rangle$

$\hat{N} = b^+ b$: $b^+ b |n\rangle = b^+ \sqrt{n} |n-1\rangle = n|n\rangle$

$\rightsquigarrow$ $bb^+ - b^+ b = [b, b^+] = 1$

$|n\rangle = \frac{1}{\sqrt{n!}} (b^+)^n |0\rangle$

$H = \frac{\omega}{2}(b^+ b + b b^+) = \omega(\hat{N} + \cancel{\tfrac{1}{2}})$ drop (normal ordering : $:bb^+: = \pm b^+ b$ Fermi/Bose; $\rightsquigarrow$ omission of a const.)

$Z = \sum_{n=0}^{\infty} \langle n | e^{-\beta\omega\hat{N}} | n \rangle = \frac{1}{1-e^{-\beta\omega}}$ (✓)

$\langle \hat{N} \rangle = \sum \langle n | \frac{1}{Z} e^{-\beta\omega\hat{N}} \hat{N} | n \rangle = \partial_{-\beta\omega} \ln Z = \frac{1}{e^{\beta\omega}-1}$ (✓)

ex. 3b) one fermionic d.o.f. ← [homework problem]

Pauli exclusion principle $\rightsquigarrow$ only states $|0\rangle$, $|1\rangle$

$f^+|0\rangle = |1\rangle$, $f|1\rangle = |0\rangle$, rest $= 0$

$\rightsquigarrow$ $ff = f^+ f^+ = 0$

$\hat{N} = f^+ f$: $f^+ f |0\rangle = 0$, $f^+ f |1\rangle = |1\rangle$

$\rightsquigarrow$ $ff^+ + f^+ f = \{f, f^+\} = 1$

$H = \frac{\omega}{2}(f^+ f - f f^+) = \omega(\hat{N} - \cancel{\tfrac{1}{2}})$ drop

$Z = \sum_{n=0}^{1} \langle n | e^{-\beta\omega\hat{N}} | n \rangle = 1 + e^{-\beta\omega}$

$\langle \hat{N} \rangle = \sum \langle n | \frac{1}{Z} e^{-\beta\omega\hat{N}} \hat{N} | n \rangle = \partial_{-\beta\omega} \ln Z = \frac{1}{e^{\beta\omega}+1} = n_{Fermi}(\omega)$

$n_{Fermi}(x) = \frac{1}{e^{\beta x}+1}$ (9)

ex. 4) QED$|_{\text{fields only } (\gamma, \cancel{e})}$: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

= blackbody radiation  → E only $\rightsquigarrow$ canonical

PBC : $j = 1,2$; $\vec{k} = \frac{2\pi}{L}(n_1, n_2, n_3)$

$\varepsilon_{\vec{k},j} = n \cdot \hbar\omega_{\vec{k}_j} = n \cdot \hbar c k$

$Z = \sum_{\vec{k},j} Z^{(1\,\text{oscillator})} = \sum_{\vec{k},j} \frac{1}{1 - e^{-\beta\hbar\omega_{\vec{k}j}}}$

$\langle n \rangle_{\vec{k}_j} = \frac{1}{e^{\beta\hbar c k} - 1} = n(\hbar c k)$

$$E = \sum_{\vec{k},i} \hbar c k \langle n \rangle_{\vec{k},i} = 2 \frac{V}{(2\pi)^3} \int d^3k \, \frac{\hbar c k}{e^{\beta \hbar c k} - 1}$$

$$= \frac{V}{(\hbar c)^3} \frac{\pi^2}{15} T^4 \qquad (10)$$

$\left[\sum \to \int : \frac{2\pi}{L} \ll T \right.$

$\left. L = 1\,\mathrm{cm} \leadsto \frac{2\pi}{L} = 0{,}14\,K \right]$

useful int's $\int_0^\infty dx \frac{x}{e^x - 1} = \frac{\pi^2}{6}, \quad \int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15} \qquad (11)$

(note that for $\hbar \to 0$, $E \to \infty \Rightarrow$ classical ED is wrong)

$F = E - TS = E + T\partial_T F$

$F \overset{?}{=} E - TS$

$\hookrightarrow \left(-T \ln Z \overset{?}{=} E - T(\ln Z + T\partial_T \ln Z) \Leftrightarrow E \overset{?}{=} T^2 \partial_T \ln Z = -\partial_\beta \ln Z \checkmark \right)$

$(1 - T\partial_T) F = E \qquad (E \sim T^4 \leadsto F \sim T^4 \leadsto T\partial_T = 4)$

$$F = -\frac{V\pi^2}{45} T^4 \quad , \quad p \left(= -\partial_V F \right) = \frac{\pi^2}{45} T^4 \qquad (12)$$

$\left(p \overset{!}{=} \frac{(1\,\mathrm{kg})\, g}{\mathrm{cm}^2} = \text{air pressure} \Rightarrow T \approx 10^5\, K \right)$

[homework problem]

ex. 5) Standard Model

$$\mathcal{L}_{SM} = \bar{\psi} i\gamma^\mu D_\mu \psi + \underbrace{(D_\lambda \phi)^\dagger D_\lambda \phi + \mu^2 \phi^2 - \lambda \phi^4}_{\mathcal{L}_{Higgs}}$$

$$\underbrace{-\frac{1}{2} \sum_{W,B,\text{gluons}} Tr (G_{\mu\nu} G^{\mu\nu})}_{T_{W,B,\text{gluons}}} - \underbrace{\left[\bar{\psi}_R C \vec{\phi}^\dagger \vec{\psi}_L + h.c. \right]}_{\mathcal{L}_{Yukawa}} \qquad (13)$$

$Z, A \leftarrow$ light!

90 components ; $3 \cdot (3 + 4 \cdot 3) \cdot 2 = 6 \cdot 15$

families			colours	spins
$\nu\nu\nu$ | uct	ν_L	u_L		
$e\mu\tau$ | dsb	e_L	d_L		
	e_R	u_R		
		d_R		

18 parameter. (11 masses, 3 couplings, 4 Cabibbo angles)