| Übungen zu EMTP II | **Blatt Nr. 17** | Lösung |
|---|---|---|

**Aufgabe 52:**

(a) Int $= \int_a^b dx\ \left(\frac{1}{x^2} + \frac{1}{1+x}\right) = \left[\,-\frac{1}{x} + \ln(1+x)\,\right]\big|_{x=a}^{b}$

(b) Int $= \int_a^b dx\ \left(\frac{2x}{1+x^2} + \frac{2x}{1-x^2}\right) = \left[\,\ln(1+x^2) - \ln(1-x^2)\,\right]\big|_{x=a}^{b}$

(c) Int $= \int_a^b dx\ \left(\frac{1}{1+x} - \frac{2x}{1+x^2}\right) = \left[\,\ln(1+x) - \ln(1+x^2)\,\right]\big|_{x=a}^{b}$

**Aufgabe 53:**

(a) lese Integrand als $u'v$ mit $u' = \cos(x)\ ,\ v = x\ ;\ u = \sin(x)$

Int $= [\,x\ \sin(x)\,]\big|_{x=a}^{b} - \int_a^b dx\ \sin(x) = [\,x\ \sin(x) + \cos(x)\,]\big|_{x=a}^{b}$

(b) lese Integrand als $u'v$ mit $u' = e^{-x}\ ,\ v = x^2\ ;\ u = -e^{-x}$

Int $= [\,x^2(-e^{-x})\,]\big|_{x=a}^{b} - \int_a^b dx\, 2x(-e^{-x})$

nochmal, mit $v = 2x$

Int $= [\,x^2(-e^{-x}) + 2x(-e^{-x})\,]\big|_{x=a}^{b} - \int_a^b dx\, 2(-e^{-x}) = [\,-(x^2+2x+2)e^{-x}\,]\big|_{x=a}^{b}$

(c) lese Integrand als $u'v$ mit $u' = \sin(x)\ ,\ v = \sin(x)\ ;\ u = -\cos(x)$

Int $= [\,-sc\,]\big|_{x=a}^{b} + \int_a^b dx\, c^2 = [\,-sc\,]\big|_{x=a}^{b} + \int_a^b dx\,(1-s^2) = [\,-sc + x\,]\big|_{x=a}^{b} - \int_a^b dx\, s^2$

nun den letzten Term auf die linke Seite der Glg bringen (und durch 2 teilen)

Int $= \left[\,\frac{1}{2}(x - \sin(x)\cos(x))\,\right]\big|_{x=a}^{b}$

**Aufgabe 54:**

(a) Subst $x = \sin(y)\ ;\ dx = dy\ \cos(y)\ ;$ setze $A = \arcsin(a)\ ,\ B = \arcsin(b)$

Int $= \int_A^B dy \frac{\cos(y)}{\sqrt{1-sin^2(y)}} = \int_A^B dy = [\,y\,]\big|_{y=A}^{B} = [\,\arcsin(x)\,]\big|_{x=a}^{b}$

(b) Subst $u = \sin(x)\ ;\ du = dx\ \cos(x)\ ;$ setze $A = \sin(a)\ ,\ B = \sin(b)$

Int $= \int_A^B du\, u^2 = [\frac{1}{3}\,u^3]\big|_{y=A}^{B} = \left[\,\frac{1}{3}\ \sin^3(x)\,\right]\big|_{x=a}^{b}$

(c) Subst $t = \tan(x/2)\ ;\ dt = dx \frac{1}{2\cos^2(x/2)} = dx \frac{c^2+s^2}{2c^2} = dx \frac{1+t^2}{2}\ \Leftrightarrow\ dx = dt \frac{2}{1+t^2}$

und (s. Vorl) $\sin(x) = 2\sin(x/2)\cos(x/2) = \frac{2sc}{c^2+s^2} = \frac{2t}{1+t^2}$ ; setze $A = \tan(a/2)\ ,\ B = \tan(b/2)$

Int $= \int_a^b \frac{dx}{\sin(x)} = \int_A^B \frac{dt}{t} = [\,\ln|t|\,]\big|_{t=A}^{B} = [\,\ln|\tan(x/2)|\,]\big|_{x=a}^{b}$

**Aufgabe 55:**

vorweg: $E = V(a)$

(a) $T = 4\sqrt{\frac{m}{2}}\ \int_0^a \frac{dx}{\sqrt{\frac{m}{2}}\,\omega\sqrt{a^2-x^2}} \overset{x\to ax}{=} \frac{4}{\omega}\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{2\pi}{\omega}$

letzter Schritt: z.B. Subst $x = \sin(y)\ ,\ dx = dy\cos(y)\ ;$ Int $= \int_0^{\pi/2} dy \frac{c}{\sqrt{1-s^2}} = \int_0^{\pi/2} dy = \frac{\pi}{2}$

oder z.B. Int $= \int_0^1 dx\,\partial_x \arcsin(x) = \arcsin(1) - \arcsin(0) = \frac{\pi}{2} - 0$

(b) $T = 4\sqrt{\frac{m}{2}}\ \int_0^a \frac{dx}{\sqrt{\beta}\sqrt{1-x/a}} \overset{x\to ax}{=} 4\sqrt{\frac{m}{2}}\ \frac{a}{\sqrt{\beta}}\int_0^1 \frac{dx}{\sqrt{1-x}} = \sqrt{\frac{8ma^2}{\beta}}\ \int_0^1 dx\,\partial_x\ \left[-2\sqrt{1-x}\ \right] = \sqrt{\frac{32ma^2}{\beta}}$

(c) $T = 4\sqrt{\frac{m}{2}}\ \frac{a}{\sqrt{\beta}}\int_0^1 \frac{dx}{\sqrt{-\ln(x)}} = \sqrt{\frac{8\pi ma^2}{\beta}}$

letzter Schritt: z.B. Subst $x = e^{-y^2}\ ,\ dx = -2y\,dy\,e^{-y^2}\ ;$ Int $= 2\int_0^\infty dy\,e^{-y^2} = 2\frac{\sqrt{\pi}}{2}$

(d) $V = -\frac{\gamma m M_e}{|x|}\ ;\ T = 4\sqrt{\frac{m}{2}}\ \frac{a^{3/2}}{\sqrt{\gamma m M_e}}\int_0^1 dx\ \sqrt{\frac{x}{1-x}} = \sqrt{\frac{2\pi^2 a^3}{\gamma M_{\text{Erde}}}}$

letzter Schritt: Int$= -\int_0^1 dx\,\partial_x[\sqrt{x(1-x)} + \frac{1}{2}\arctan(\frac{1-2x}{2\sqrt{x(1-x)}})] = -\frac{1}{2}\arctan(-\infty) + \frac{1}{2}\arctan(\infty) = \frac{\pi}{2}$

oder z.B. per [Bronstein, 1.1.3.4.39]: $\int_0^1 dx\,x^\alpha(1-x)^\beta = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}$, bei $\alpha = \frac{1}{2}\ ,\ \beta = -\frac{1}{2}$ und mit $\Gamma(1/2) = \sqrt{\pi}\ ,\ \Gamma(3/2) = \sqrt{\pi/2}\ ,\ \Gamma(2) = 1$

oder z.B. per Subst $y = \sqrt{\frac{1-x}{x}}\ ,\ dy = \frac{1}{2}\frac{dx}{\sqrt{\frac{1-x}{x}}}(-\frac{1}{x^2}) = dx\sqrt{\frac{x}{1-x}}\ \frac{(1+y^2)^2}{-2}\ ;$ Int $= 2\int_0^\infty \frac{dy}{(1+y^2)^2} = \int_0^\infty dy\,\partial_y[\arctan(y) + \frac{y}{1+y^2}] = \arctan(\infty) = \frac{\pi}{2}$