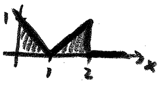
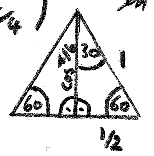
ENTP II – Klausur 16.1.2009 Lösungsvorschläge

① (a) $= \partial_z\left[(z-1)e^z\right]$ (b) $= \int_{-\pi}^{\pi} dx\, \overset{u}{\sin(x)}\,\overset{g}{\cos(x)} = 0$ (c) =  $= 1$

② $= \left[\frac{1}{2i}\left(e^{ikx}-e^{-ikx}\right)\right]^2 = -\frac14\left(-2+e^{i2kx}+e^{-i2kx}\right) = \frac12-\frac12\cos(2kx)$

③ $= \int_{+\pi/6}^{\pi/4} dx\, \underbrace{\frac{\sin x}{\cos x}}_{=-\partial_x \ln(\cos(x))} = \ln\left(\frac{\cos\pi/6}{\cos\pi/4}\right) = \ln\left(\frac{\sqrt3/2}{1/\sqrt2}\right) = \frac12\ln\left(\frac32\right)$,  $\Rightarrow 1^2 = 2\cos^2\frac\pi4 \Rightarrow \cos\frac\pi4 = 1/\sqrt2$

 $\Rightarrow 1^2 = \frac14+\cos^2\frac\pi6 \Rightarrow \cos\frac\pi6 = \frac{\sqrt3}{2}$

④ (a) $= \left(x\partial_x+y\partial_y+z\partial_z\right)\sqrt{x^2+y^2+z^2} = |\vec r|$ (b) $= \sum_{i=0}^\infty \left(\frac12\vec r\cdot\vec\nabla\right)^i|\vec r| = \sum\left(\frac12\right)^i r = \frac{1}{1-\frac12}r = 2r$

(c) $\left(r\vec e_r\times\left(\vec e_r\partial_r+\vec e_\vartheta\frac1r\partial_\vartheta+\vec e_\varphi\frac{1}{r\sin\vartheta}\partial_\varphi\right)\right)\cdot r\vec e_r = \left(\vec e_\varphi\partial_\vartheta - \vec e_\vartheta\frac{1}{\sin\vartheta}\partial_\varphi\right)\cdot r\vec e_r = r\vec e_\varphi\vec e_\vartheta - r\vec e_\vartheta\frac{1}{\sin\vartheta}\sin\vartheta\,\vec e_\varphi = 0$

⑤ $T(x,t) = e^{tD\partial_x^2}\,T(x,0) = T_0\left(1+\sum_{n=1}^\infty\frac{(tD\partial_x^2)^n}{n!}\right)\left(1+\cos(2kx)\right)$

$= T(x,0)+T_0\sum_1 \frac{(tD(-4k^2))^n}{n!}\cos(2kx) = T_0\left(1+e^{-4tDk^2}\cos(2kx)\right)$

bei $t\to\infty$: überall T_0

⑥ (a) $L\,G(t,a) \overset{!}{=} \delta(t-a)$; hom: $\dot G_{hom}+\frac{f_2(t)}{f_1(t)}=0 \Rightarrow G_{hom} = e^{-\int_{t_0}^t dx\, \frac{f_2(x)}{f_1(x)}}$

Vdk: $G = G_{hom}\cdot u(t)$, $LG = \cancel{f_1\dot G_h u} + f_1 G_h\dot u + \cancel{f_2 G_h u} \overset{!}{=}\delta(t-a)$,

$\dot u = \frac{\delta(t-a)}{f_1(t)G_h(t)} = \frac{\delta(t-a)}{f_1(a)G_h(a)}$, $u = \frac{\Theta(t-a)}{f_1(a)G_h(a)}$, also $G(t,a) = \frac{\Theta(t-a)}{f_1(a)}\, e^{-\int_a^t dx\,\frac{f_2(x)}{f_1(x)}}$

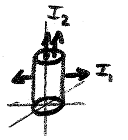
(b) $G(t,a) = \frac{\Theta(t-a)}{a}\,e^{c\int_a^t dx\,\frac1x} = \frac{\Theta(t-a)}{a}\,e^{c(\ln t-\ln a)} = \frac{t^c}{a^{c+1}}\,\Theta(t-a)$

⑦ (a) $q_{sp}(x) = \int_{-\infty}^\infty da\, f(a)\,G(x-a) \overset{a\to x-a}{=} \int_{-\infty}^\infty da\, f(x-a)\,G(x) = \int_0^\infty da\, a\, e^{-\beta a}f(x-a)$

(b) $q_{sp}(x) = \int_0^\infty da\, a(x-a)e^{-\beta a} = -\left(x\partial_\beta+\partial_\beta^2\right)\int_0^\infty da\, e^{-\beta a} = -\left(x\partial_\beta+\partial_\beta^2\right)\frac1\beta = \frac{x}{\beta^2}-\frac{2}{\beta^3}$

⑧ rhs hat Zyl.-Symmetrie, ist z-unabh. ⇒ Ansatz: $\vec v(\vec r) = v(\varrho)\,\vec e_z$

$\to \vec\nabla\times\vec v = \begin{pmatrix}\partial_y v\\-\partial_x v\\0\end{pmatrix} = \begin{pmatrix}y\\-x\\0\end{pmatrix}\frac1\varrho v'(\varrho) \overset{!}{=} \alpha\,\delta(\varrho-R)\begin{pmatrix}-y\\x\\0\end{pmatrix}\frac1\varrho$, $v' = -\alpha\,\delta(\varrho-R)$, $v(\varrho) = C-\alpha\,\Theta(\varrho-R)$

⑨ $I_1 = \int_{\text{Zyl. Mantel}(R)} df\, \underbrace{\vec e_\varrho\cdot\vec j}_{=j_0(-z^3\varrho)}\Big|_{\varrho=R} = \int_0^H dz\, 2\pi R\left(-j_0z^3R\right) = -\frac\pi2\, j_0R^2H^4$ 

$I_2 = \int_{\text{Zyl. Deckel}(H)} df\,\underbrace{\vec e_3\cdot\vec j}_{=j_0(r^4-z^4)}\Big|_{z=H} = \int_0^{2\pi}d\varphi\int_0^R d\varrho\,\varrho\, j_0\left((\varrho^2+H^2)^2-H^4\right) = \ldots = 2\pi j_0\frac{R^4}{6}\left(R^2+3H^2\right)$

⑩ Gem y ⇒ $u(x)\equiv y'(x)$, $2-x = \frac{u'}{u^2} = -\partial_x\frac1u = \partial_x\left(c_1+2x-\frac12x^2\right) \Rightarrow u = \frac{1}{\frac12x^2-2x-c_1}$,

$u(0)=\frac12 \Rightarrow c_1=-2$, $u = \frac{2}{(2-x)^2}$; $y(x) = \frac{2}{2-x}+c_2$, $y(0)=1\Rightarrow c_2=0$

⑪ $\tilde f(\vec k) = \int d^3r\, e^{-i\vec k\vec r}f(r) = 2\pi\int_0^\infty r^2 f(r)\int_{-1}^1 du\, e^{-ikru} = \frac{2\pi}{ik}\int_0^\infty dr\, r f(r)\left(e^{ikr}-e^{-ikr}\right)$

$= \frac{2\pi}{ik}\int_{R_1}^{R_2}dr\, e^{ikr-\gamma r} + \text{dito}_{k\to-k} = \frac{2\pi}{ik}\,\frac{e^{(ik-\gamma)R_2}-e^{(ik-\gamma)R_1}}{ik-\gamma} + \text{dito}_{k\to-k}$

Ergebnis reell, da $k\to-k \;\hat=\; i\to -i$, und $a+a^* = 2\,\mathrm{Re}[a]$.

Ergebnisse: ab 22. Jan, online + Aushang E6; Einsicht: Prüfungsamt D3-155