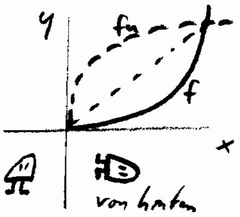
Umkehrfkt  $f_u(x) :=$ an Diagonale gespiegeltes $f(x)$




$$\left.\begin{array}{l} y = f(x) \\ \Rightarrow x = f_u(y) \end{array}\right\} \quad \begin{array}{l} y = f(f_u(y)) \\ x = f_u(f(x)) \end{array}$$

$$f_u'(x) = \frac{1}{f'(f_u(x))}\ ,\quad \text{denn}$$

$\partial_x$ auf $x = f(f_u(x))$ gibt $1 = f'(f_u(x))\cdot f_u'(x)$ ■

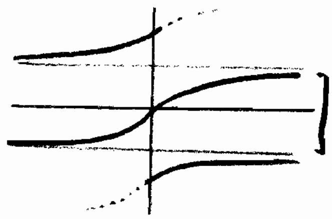
(( Bsp: $f = x^2$, $f_u = \sqrt{x}$, $f_u' = \frac{1}{2\sqrt{x}}$ ))

Anw.-Bsp  $f(x) = \tan(x)$ , $f_u(x) =: \arctan(x)$

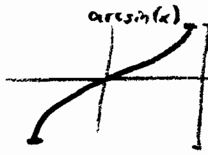
$$\partial_x \tan(x) = \left(\frac{s}{c}\right)' = \frac{c^2+s^2}{c^2} = 1 + \tan^2(x)$$

$$\partial_x \arctan(x) = \frac{1}{1+\tan^2(\arctan(x))} = \frac{1}{1+x^2}$$

"Stammfunktion" zur Lorentz-Kurve




$$\partial_x \arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$




## 5.2. e-Funktion

14 Wünsche an jede neue Fkt — am Bsp dieser neuen.

s.z.B. Gliederung in [Abramowitz/Stegun, Handbook of math. fcts]

- ●1 Name: e-Funktion
- ●2 Bezeichnung: exp(x) (vorläufig)
- ●3 Bedarf: (a) Wachstum $\boxed{\dot N = \alpha N,\ N(0) = N_0}$  $[\alpha] = 1/\text{Zeit}$

  Ans $N(t) = N_0 f(\alpha t)$ ; $N_0 f'\cdot \alpha = \alpha N_0 f$ , $N_0 f(0) = N_0$

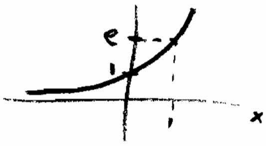
  $\Rightarrow f$ – ER: $f'(x) = f(x)$, $f(0) = 1$

  (b) $m\dot v = -m\lambda v$ , $v(0) = v_0$

  Ans $v(t) = v_0 f(-\lambda t)$ ; $-\lambda v_0 f' = -\lambda v_0 f$ , $f(0) = 1$

●4 Def: exp(x) := Lsg von $\boxed{f'(x)=f(x),\ f(1)=1}$

●5 Verlauf:




$f(1) =: e \approx 2.72$

●6 Funktionale Beziehung

$$\frac{\exp(x+y)}{\exp(y)} =: g(x)\ ,\quad g(0)=1\ ,\quad g'(x) = \frac{\exp'(x+y)}{\exp(y)} = g(x)$$

$\Rightarrow$ g erfüllt auch $\boxed{\cdots}$

$\Rightarrow$ $\exp(x+y) = \exp(x)\exp(y)$

$\Rightarrow$ $\exp(x) = e^x$

((denn: $\exp(\frac{m}{n}) = \exp(\frac{1}{n}+\frac{1}{n}+\dots+\frac{1}{n}) = [\exp(\frac{1}{n})]^m = ([\exp(\frac{1}{n})]^n)^{\frac{m}{n}}$
$= [\exp(1)]^{\frac{m}{n}}$ ))

●7 Ableitung: $e^x$

●8 Stammfunktion: $e^x$

●9 Dgln $\partial_x e^x = e^x$ , $\partial_x^2 e^{\pm x} = e^{\pm x}$

(( $\ddot{x} = +\omega^2 x$ mit Ansatz $Ae^{\omega t} + Be^{-\omega t}$ ))

(( $\dot{N} = \alpha N - \beta$, $N(0) = N_0$ mit (a) Ansatz (b) $N = u + a$ (c) $N = e^{\alpha t} w$

(d) „direkt" $N(t) = \frac{\beta}{\alpha} + \left(\text{Dgl-ohne-}\beta\ \text{Lsg}\right)$

$= \frac{\beta}{\alpha} + \left( \quad \right) e^{\alpha t}$

$= \frac{\beta}{\alpha} + \left( \quad -\frac{\beta}{\alpha} \right) e^{\alpha t}$

$= \frac{\beta}{\alpha} + \left( N_0 - \frac{\beta}{\alpha} \right) e^{\alpha t}$ ))

●10 Reihe: $f = c_0 + c_1 x$ in $\boxed{f' = f,\ f(0) = 1}$ einsetzen,

gibt $c_1 = c_0 + c_1 x$ , $c_0 = 1$

↳ oh! Brauche auch $+c_2 x^2$ in $f$!

$c_1 + 2c_2 x = c_0 + c_1 x + c_2 x^2$

↳ oh! Brauche ... <u>alle</u> Potenzen!

$f(x) = \sum_{n=0}^{\infty} c_n x^n$ in $\boxed{f' = f,\ f(0) = 1}$ ersetzen.

$$\sum_{n=0}^{\infty} c_n\, n\, x^{n-1} = \sum_{m=0}^{\infty} c_m x^m, \quad m = n-1$$

$$\sum_{n=1}^{\infty} c_n\, n\, x^{n-1} = \sum_{n=1}^{\infty} c_{n-1} x^{n-1}$$

$\Rightarrow \quad c_n = \frac{1}{n} c_{n-1}$ für $n = 1, 2, \ldots$ und $c_0 = 1$

$c_0 = 1,\ c_1 = 1,\ c_2 = \frac{1}{2},\ c_3 = \frac{1}{3}\cdot\frac{1}{2},\ \ldots \quad c_n = \frac{1}{n!}$ mit $n! := 1\cdot 2 \cdot \ldots \cdot n \ (n \geq 1)$
$0! := 1$

also $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ ; kann auch als exp-Def nehmen.

die Summe hat $\forall x$ einen endlichen Wert
("die Reihe <u>konvergiert</u> $\forall x$"), denn

gegeb. $x$, $a$ = natürl. Zahl $\gtrsim x$,

$$\sum \frac{1}{n!} x^n = 1 + \ldots + \frac{a^{10a}}{(10a)!}\left[1 + \frac{a}{10a+1} + \frac{a^2}{(10a+1)(10a+2)} + \ldots\right]$$

und $[\cdot] < 1 + \frac{1}{10} + \frac{1}{100} + \ldots = 1.111\ldots$

- •11 Werte: z.B. per Reihe
- •12 Asymptotik, $x \to \infty$:

$\frac{1}{\varepsilon}$

$$\frac{1}{x^m} e^x \to \infty, \quad x^m e^{-x} \to 0 \quad (e^{-x} \text{ erschlägt jede Potenz})$$

---

$e^x$ <u>als</u> Asymptotik $\boxed{f' = f,\ f(0) = 1}$ per Computer,

[illegible]

$$f(x+\varepsilon) = f(x) + \varepsilon f(x) = (1+\varepsilon) f(x) \qquad (\varepsilon \to 0)$$

$$f(x + N\varepsilon) = (1+\varepsilon)^N f(x)$$

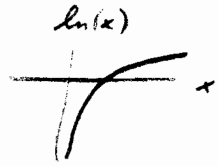
$x=0$: $f(\underline{N\varepsilon}) = (1+\varepsilon)^N$

↳ nenne dies wieder $x$; trotz $\varepsilon \to 0$: $N$ beliebig groß, $N = \frac{x}{\varepsilon}$

$$f(x) = e^x = \lim_{N\to\infty} \left(1 + \frac{x}{N}\right)^N$$

ist <u>auch</u> $e^x$-Def; (eher exotisch)

**•13** Umkehrfkt:

,



$$\ln(e^x) = x\,, \quad e^{\ln(x)} = x$$

$$\partial_x \ln(x) = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

$$\ln(xy) = \ln\left(e^{\ln x} e^{\ln y}\right) = \ln\left(e^{\ln x + \ln y}\right) = \ln(x) + \ln(y)$$

$$\ln\left(\tfrac{1}{x}\right) \overset{x = e^u}{=} \ln\left(e^{-u}\right) = -u = -\ln(x)$$

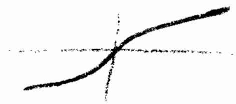
$$10^x = \left[e^{\ln(10)}\right]^x = e^{x \underbrace{\ln(10)}_{\approx 2.3026}}$$

$$\ln(a + b\varepsilon) = \ln\left(a\left[1 + \tfrac{b}{a}\varepsilon\right]\right) = \ln(a) + \ln\left(1 + \tfrac{b}{a}\varepsilon\right)$$
$$= \ln(a) + \tfrac{b}{a}\varepsilon + O(\varepsilon^2)$$ → s. S. 43

**•14** verwandte Fkn

$$e^x = \underbrace{\tfrac{1}{2}\left(e^x + e^{-x}\right)}_{\equiv \cosh(x)} + \underbrace{\tfrac{1}{2}\left(e^x - e^{-x}\right)}_{\equiv \sinh(x)}$$

→ Umkehrfkt: arsinh(x) "Area Sinus Hyperbolicus"

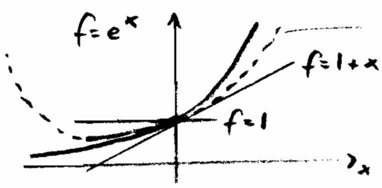


$$\cosh' = \sinh\,, \quad \sinh' = \cosh$$

$$\cosh^2 = \tfrac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) = 1 + \sinh^2$$

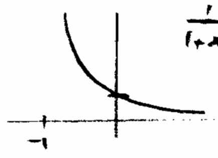
## 5.3. Potenzreihen

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$ nicht nur für $e^x$ ?

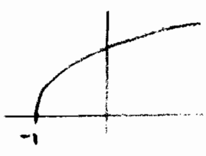



→ $f = 1 + x + \frac{x^2}{2} = \frac{(1+x)^2 + 1}{2}$

Strategie funktioniert doch auch bei:




$$\frac{1}{1+x} = 1 - x + x^2 + \ldots$$




$$\sqrt{1+x} = 1 + \tfrac{1}{2}x - x^2 + \ldots$$