Monte Carlo Methods - Fall 2013/14

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Exercise Nr. 2

Discussion on October 28th, 14:00-15:00

4) Linear congruential random number generator *(4 points)*

One of the oldest pseudo-random number generator (PRNG) algorithms is the linear congruential generator (LCG), which yields a sequence of random numbers calculated with a linear equation:

$$X_{n+1} = (aX_n + b) \mod m$$

with X_0 the seed. An example is given in the pseudocode, generating an integer between 0 and $m-1$. (a, b have been carefully adjusted). Implement this RNG and check that the cycle length is at most m (it might be shorter for other choices of a, b.)

Explanation of the Pseudo-Code:

- $a \mod m$: modulus, returning the remainder r of integer division $a/m = b + r$
- $\texttt{real}(m)$: converts integer m into real number

procedure `naive-ran`
$m \leftarrow 134456$
$a \leftarrow 8121$
$b \leftarrow 28411$
input iran
$iran \leftarrow (a \cdot iran + b) \mod m$
$ran \leftarrow iran/\texttt{real}(m)$
output $iran, ran$

5) Gaussian random numbers *(3+2+3 points)*

We would like to obtain random numbers distributed with a probability

$$\pi(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

A straight-forward method to implement this distribution based on a uniform distribution is given in the pseudo-code, where the distribution above is squared ($x^2 \to x^2 + y^2$), and replaced by polar coordinates ($dxdy = rdrd\phi$).

procedure `gauss`
intput σ
$\phi \leftarrow \texttt{ran}(0, 2\pi)$
$\Upsilon \leftarrow -\texttt{log}\,\texttt{ran}(0,1)$
$r \leftarrow \sigma\sqrt{2\Upsilon}$
$x \leftarrow r\,\texttt{cos}\,\phi$
$y \leftarrow r\,\texttt{sin}\,\phi$
output $\{x, y\}$

a) Prove that both x and y are Gaussian distributed.

b) Think about a method to remove the use of trigonometric functions to speed up the algorithm.

c) Implement either the original or the improved code.

6) Random points in n-Sphere *(3+3+2 points)*

a) Prove that the volume of an n-dimensional unit sphere is given by the recurrence relation

$$V_d(1) = \frac{\pi}{d/2} V_{d-2}(1) \qquad \text{with} \quad V_1(1) = 2, \quad V_2(1) = \pi.$$

b) Implement the Hit or Miss Monte Carlo to estimate the volume of the n-sphere.

c) Show that the acceptance rate (hit events/total events) decreases drastically as n is increased. Plot the volume and the acceptance rate as a function of $n = 1 \ldots 50$.

Stanislaw Marcin Ulam
(April 13, 1909 - May 13, 1984)

Ulam received a doctoral degree (1933) at the Polytechnic Institute in Lvov (now Lviv). At the invitation of John von Neumann, he worked at the Institute for Advanced Study, Princeton, New Jersey, U.S., in 1936. He lectured at Harvard University in 1939-40 and taught at the University of Wisconsin at Madison from 1941 to 1943. In 1943 he became a U.S. citizen and was recruited to work at Los Alamos on the development of the atomic bomb. He remained at Los Alamos until 1965 and taught at various universities thereafter.

Ulam had a number of specialties, including set theory, mathematical logic, functions of real variables, thermonuclear reactions, topology, and the Monte Carlo theory.

Working with physicist Edward Teller, Ulam solved one major problem encountered in work on the fusion bomb by suggesting that compression was essential to explosion and that shock waves from a fission bomb could produce the compression needed. He further suggested that careful design could focus mechanical shock waves in such a way that they would promote rapid burning of the fusion fuel. Teller suggested that radiation implosion, rather than mechanical shock, be used to compress the thermonuclear fuel. This two-stage radiation implosion design, which became known as the Teller-Ulam configuration, led to the creation of modern thermonuclear weapons. Ulam's work at Los Alamos had begun with his development (in collaboration with von Neumann) of the Monte Carlo method, a technique for finding approximate solutions to problems by means of artificial sampling. Through the use of electronic computers, this method became widespread, finding applications in weapons design, mathematical economy, and operations research. Ulam also improved the flexibility and general utility of computers and wrote a number of papers and books on aspects of mathematics. The latter include A Collection of Mathematical Problems (1960), Stanislaw Ulam: Sets, Numbers, and Universes (1974), and Adventures of a Mathematician (1976).

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