9.5. Dark matter abundance in cosmology

In this section we continue the considerations of Sec. 9.4. Denoting by \( n(t, q) \) the phase space density of right-handed neutrinos in either polarization state,

\[
n(t, q) = \sum_{s=1,2} \frac{dN^s(t, x, q)}{d^3x d^3q},
\]

(9.126)

Eq. (9.96) is of the form

\[
\frac{\partial n(t, q)}{\partial t} = R(T, q),
\]

(9.127)

where \( R \) is given in Eq. (9.125),

\[
R(T, q) = \frac{2n_F(q^0)}{(2\pi)^3 2q^0} \sum_{\alpha=1}^{3} |F_{\alpha}|^2 Tr \left\{ Q a_L \left[ \rho_{\alpha\alpha}(-Q) + \rho_{\alpha\alpha}(Q) \right] a_R \right\}.
\]

(9.128)

This applies in flat spacetime, at a given temperature \( T \). Note that in thermal equilibrium, in the free limit, we would expect \( n(t, q) = 2n_F(q^0) / (2\pi)^3 \), where the factor 2 comes the polarization sum, and the factor \((2\pi)^3\) from that according to Eq. (9.126), \( \sum_{s=1,2} N^s = \int d^3x d^3q n(t, q) \). For cosmological applications, the first task now is to generalize Eq. (9.127) to an expanding Universe, where the temperature furthermore is a function of time.

In order to carry out the generalization, we first have to give a meaning to our variables, the time \( t \) and the momenta \( q \). In the following we mean by these the physical time and momenta, i.e. the ones defined in a local Minkowskian frame. However, as is well known, local Minkowskian frames at different times are inequivalent in an expanding background; in particular, the physical momenta redshift. Carrying out the derivation of the rate equation in this situation is a topic of general relativity rather than thermal field theory, so we only quote the result here: the upshot is that the time derivative gets replaced with \( \partial / \partial t \to \partial / \partial t - H q_i \partial / \partial q_i \)


\[
\left( \frac{\partial}{\partial t} - H q_i \frac{\partial}{\partial q_i} \right) n(t, q) = R(T, q),
\]

(9.129)

where \( H \) is the Hubble parameter, \( H(t) \equiv \dot{a}(t) / a(t) \), and \( q_i \) are the spatial components of \( q \). We will see presently that this form is in any case consistent with the expected redshift \( q(t) = q(t_0)a(t_0) / a(t) \), where \( a(t) \) is the scale factor.

Eq. (9.129) can be written in a simpler form through a change of variables. We note, first of all, that \( R(T, q) \) and consequently also \( n(t, q) \) are only functions of \( q \equiv |q| \). Changing correspondingly the notation to \( n(t, q) \), \( R(T, q) \), Eq. (9.129) becomes

\[
\left( \frac{\partial}{\partial t} - H q \frac{\partial}{\partial q} \right) n(t, q) = R(T, q).
\]

(9.130)

Introducing then an ansatz \( n(t, q) = n(t, q(t_0) a(t_0) / a(t)) \), and noting that

\[
\frac{d}{dt} \left[ q(t_0) a(t_0) / a(t) \right] = -q(t_0) a(t_0) \dot{a}(t) / a^2(t) = -Hq ,
\]

(9.131)

Eq. (9.130) can be written as

\[
\frac{d}{dt} n(t, q(t_0) a(t_0) / a(t)) = R \left( T, q(t_0) a(t_0) / a(t) \right).
\]

(9.132)
This integrates immediately to
\[ n(t_0, q(t_0)) = \int_0^{t_0} dt \, R(T(t), q(t_0) \frac{a(t_0)}{a(t)}) , \tag{9.133} \]
where we assumed the initial condition \( n(0, q) = 0 \), i.e., that there are no right-handed neutrinos in the beginning.

At this point we need to recall the basic cosmological relations between the time \( t \) and the temperature \( T \). Assuming a homogeneous and isotropic metric,
\[ ds^2 = dt^2 - a^2(t) \, dx_k^2 , \tag{9.134} \]
and the energy-momentum tensor of an ideal fluid,
\[ T_{\mu}^{\nu} = \text{diag}(e, -p, -p, -p) , \tag{9.135} \]
where \( e \) denotes the energy density and \( p \) the pressure, the Einstein equations, \( G_{\mu}^{\nu} = 8\pi G T_{\mu}^{\nu} \), reduce to
\[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3} e , \tag{9.136} \]
\[ d(ea^3) = -p \, da . \tag{9.137} \]

We will assume a flat Universe in the following, \( k = 0 \), and denote
\[ \frac{1}{m_{Pl}^2} \equiv G , \tag{9.138} \]
where \( m_{Pl} = 1.2 \times 10^{19} \text{ GeV} \) is the Planck mass.

We now combine the Einstein equations with basic thermodynamic relations. Assuming a system with at most very small chemical potentials, the energy and entropy densities are related by
\[ e = Ts - p , \tag{9.139} \]
where \( s = dp/dT \) is the entropy density. The derivative of Eq. (9.139) with respect to \( T \) yields
\[ \frac{de}{dT} = T \, \frac{ds}{dT} = Te , \tag{9.140} \]
where \( c \) is the heat capacity. Moving all the terms in Eq. (9.137) to the left-hand side, we get
\[ a^3 \, de + 3(e + p)a^2 \, da = 0 \tag{9.141} \]
\[ \Rightarrow \quad de = -3T s \, \frac{da}{a} \tag{9.142} \]
\[ \Rightarrow \quad \frac{dT}{dt} \, \frac{de}{dT} = -3T s \, \frac{\dot{a}}{a} . \tag{9.143} \]

Inserting Eqs. (9.136), (9.138) and (9.140) results finally in
\[ \frac{dT}{dt} = -\frac{\sqrt{24\pi} \, s(T) \, \sqrt{c(T)}}{m_{Pl} \, \frac{c(T)}{T}} . \tag{9.144} \]

In addition, Eq. (9.137) can also be written in terms of the well-known entropy conservation law:
\[ 0 = \frac{d(ea^3)}{dt} + p \frac{d(a^3)}{dt} \]
\[ = \frac{d((e + p)a^3)}{dt} - a^3 \, dp \]
\[ = \frac{d(Tsa^3)}{dt} - a^3 \, sdT \]
\[ = T \frac{d(sa^3)}{dT} , \tag{9.145} \]
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where we inserted Eq. (9.139) and the definition \( s = \frac{dp}{dT} \). Eq. (9.145) can in turn be expressed as

\[
\frac{a(t)}{a(t_0)} = \left[ \frac{s(T_0)}{s(T)} \right]^\frac{1}{3},
\]

where \( t \) and \( T \) are related through Eq. (9.144).

It is furthermore conventional to introduce the effective numbers of massless bosonic degrees of freedom \( g_{\text{eff}}(T) \) and \( h_{\text{eff}}(T) \) via the relations

\[
e(T) \equiv \frac{\pi^2 T^4}{30} g_{\text{eff}}(T), \quad s(T) \equiv \frac{2\pi^2 T^3}{45} h_{\text{eff}}(T),
\]

where the prefactors follow by applying Eq. (9.139) and the line below it to the free result \( p(T) = \pi^2 T^4/90 \) from Eq. (2.80). Given the equation-of-state of the plasma [i.e. the relation between the pressure and the temperature, \( p = p(T) \)], \( g_{\text{eff}}(T) \) and \( h_{\text{eff}}(T) \) can accordingly to Eq. (9.139) be found from

\[
g_{\text{eff}}(T) = \frac{30}{\pi^2 T^2} \frac{dp}{dT}, \quad h_{\text{eff}}(T) = \frac{45}{2\pi^2 T^3} \frac{dp}{dT}.
\]

Furthermore, we note that \( s(T)/T^4(T) = \rho'(T)/T\rho(T) = \rho'(T)/\rho(T) = \frac{\partial p}{\partial \rho} e = c_s^2(T) \), where we identified the standard expression for the sound speed squared.

Combining Eqs. (9.144), (9.146) and the definitions above, we can now change variables in Eq. (9.133), arriving at \( (q \equiv q(t_0))\)

\[
n(t_0, q) = \int_{T_0}^\infty \frac{dT}{T^3} \sqrt{\frac{5}{4\pi^3 c_s^2(T) \sqrt{g_{\text{eff}}(T)}}} R \left( T, q \frac{T}{T_0} \left[ \frac{h_{\text{eff}}(T)}{h_{\text{eff}}(T_0)} \right]^\frac{1}{3} \right),
\]

The integral over Eq. (9.149) with the measure \( d^3 q \) gives the number density of right-handed neutrinos, denoted by \( n(t_0) \), which can subsequently be conveniently normalized with respect to the total entropy density, Eq. (9.147), which produces the so-called yield parameter, \( Y \):

\[
Y(t_0) \equiv \frac{n(t_0)}{s(t_0)} = \frac{45 \times 4\pi}{2\pi^2 T_0^3 h_{\text{eff}}(T_0)} \sqrt{\frac{5}{4\pi^3}} \int_0^\infty dq q^2 \int_{T_0}^\infty \frac{dT}{T^3} \sqrt{\frac{m_{\text{Pl}}^3}{c_s^2(T) \sqrt{g_{\text{eff}}(T)}}} R \left( T, q \frac{T}{T_0} \left[ \frac{h_{\text{eff}}(T)}{h_{\text{eff}}(T_0)} \right]^\frac{1}{3} \right)
\]

\[
= \frac{45 \sqrt{5}}{\pi^{5/2}} \int_0^\infty \frac{dT}{T^3} \int_0^\infty dz z^2 \frac{m_{\text{Pl}}}{c_s^2(T) \sqrt{g_{\text{eff}}(T)}} R(T, T z),
\]

where in the last step we substituted \( q = z T_0 [h_{\text{eff}}(T_0)/h_{\text{eff}}(T)]^{1/3} \). Note that Eq. (9.150) obtains a constant value at low temperatures if \( T \to 0 \). Therefore the yield parameter is a good (i.e. \( T_0 \)-independent) characterization of the dark matter relic density.

In order to write the result more explicitly, we need to specify the function \( R(T, q) \). The relevant information is contained in Eqs. (8.88), (8.89), (9.128). The Dirac algebra in Eq. (9.128) can be carried out:

\[
\text{Tr} \left\{ Q a_L \left[ P_1 \right] a_R \right\} = 2Q \cdot P_1.
\]

Furthermore, this can be written in various ways depending on the channel:

\[
\delta^{(4)}(P_1 + P_2 - Q) 2Q \cdot P_1 = \delta^{(4)}(P_1 + P_2 - Q) [P_1^2 + Q^2 - (Q - P_1)^2],
\]

\[
\delta^{(4)}(P_2 - P_1 - Q) 2Q \cdot P_1 = \delta^{(4)}(P_2 - P_1 - Q) [(Q + P_1)^2 - P_1^2 - Q^2],
\]

\[
\delta^{(4)}(P_1 - P_2 - Q) 2Q \cdot P_1 = \delta^{(4)}(P_1 - P_2 - Q) [P_2^2 + Q^2 - (Q - P_1)^2].
\]
These factors are all constants, independent of the momenta \( p_1, p_2 \). Thereby we arrive at

\[
R(T, q) = \frac{1}{(2\pi)^3 2p^0} \sum_{\alpha=1}^3 |F_\alpha|^2 \sum_c \left( m_{\phi_c}^2 - m_{\ell_c}^2 - M^2 \right) \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} \times \left\{ \right. \\
- (2\pi)^4 \delta^{(4)}(P_1 + P_2 - Q) n_{F1} n_{B2} + \left. \right\} ,
\]

where \( E_1 \equiv \sqrt{m_{\phi_c}^2 + p^2} \), \( E_2 \equiv \sqrt{m_{\ell_c}^2 + (p + q)^2} \). In passing, it is interesting to note that Eq. (9.153) resembles very much a collision term of a Boltzmann equation. We briefly recall the structure of the latter in Sec. 9.6.

Let us analyse Eq. (9.153) more precisely. Suppose that \( q^0 > 0 \). The first question is, when do the different channels get realized? Since all particles are massive, we can go to the rest frame of one of them; it is then clear that the first channel gets realized for \( M > m_{\ell_c} + m_{\phi_c} \); the second for \( m_{\phi_c} > M + m_{\ell_c} \); the third for \( m_{\ell_c} > M + m_{\phi_c} \); and the last one never. So, assuming that the scalar mass (the Higgs mass) is larger than those of the produced particles, \( m_{\phi_c} \gg M, m_{\ell_c} \), we can focus on the second channel, and the integral to be considered is

\[
I = \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(4)}(P_2 - P_1 - Q) n_B(u \cdot P_2)[1 - n_F(u \cdot P_1)] .
\]

Here \( u \) is the four-velocity of the thermal bath, and we have written the integral in a frame-independent way.

Remarkably, the integral in Eq. (9.154) can be written in a very simple form, in the high-temperature limit where the masses \( M^2 = Q^2 \) and \( m_{\ell_c}^2 = P_1^2 \) of the produced particles can be neglected\(^{47}\). Denoting

\[
p \equiv |p_1| , \quad q \equiv |q| ,
\]

we get

\[
I = \int \frac{d^3p_1}{(2\pi)^3 2p} \int \frac{d^3p_2}{(2\pi)^3 2E_2} (2\pi)^3 \delta^{(3)}(p_1 + q - p_2) (2\pi) \delta(p + q - E_2) n_B(p)[1 - n_F(p)]
\]

\[
= \frac{1}{(4\pi)^2} \int \frac{d^3p_1}{p(p + q)} \delta\left(p + q - \sqrt{m_{\phi_c}^2 + (p_1 + q)^2}\right) n_B(p)[1 - n_F(p)]
\]

\[
= \frac{1}{8\pi} \int_0^{+1} d\varrho \delta\left(p + q - \sqrt{m_{\phi_c}^2 + p^2 + q^2 + 2pq}\right) n_B(p + q)[1 - n_F(p)] ,
\]

where in the last step radial coordinates were introduced, with \( q \) as the \( z \)-axis. The delta function gets realized when

\[
p^2 + q^2 + 2pq = m_{\phi_c}^2 + p^2 + q^2 + 2pqz ,
\]

\(^{47}\) M. Shaposhnikov, unpublished notes.
i.e. \( z = 1 - m_{\phi_c}^2/2pq \). This belongs to the interval \((-1, 1)\) if \( p > m_{\phi_c}^2/4q \), so that

\[
I = \frac{1}{8\pi} \int_{m_{\phi_c}^2/4q}^{\infty} \frac{dp}{p + q} \left| \frac{d}{dz} \sqrt{m_{\phi_c}^2 + p^2 + q^2 + 2pqz} \right|^{n_b(p + q)[1 - n_F(p)]}.
\]

(9.158)

The derivative is trivial,

\[
\frac{d}{dz} \sqrt{\cdots} = \frac{pq}{p + q},
\]

(9.159)

whereby we finally arrive at

\[
I = \frac{1}{8\pi q} \int_{m_{\phi_c}^2/4q}^{\infty} dp n_b(p + q)[1 - n_F(p)]
= T \frac{8\pi q}{\int_{0}^{\infty} dx n_B(Tx + q + m_{\phi_c}^2)/4q)[1 - n_F(Tx + m_{\phi_c}^2/4q)]}.
\]

(9.160)

In the last step we substituted \( p + m_{\phi_c}^2/4q + Tx \).

We can easily work out an upper bound for this integral if we make the further approximation that \( T \ll m_{\phi_c} \). Indeed, \( 1 - n_F \leq 1 \), and

\[
q + \frac{m_{\phi_c}^2}{4q} = m_{\phi_c} + \frac{2}{m_{\phi_c}} \left( q - \frac{m_{\phi_c}}{2} \right)^2 + \cdots \geq m_{\phi_c}.
\]

(9.161)

Thereby the distribution function \( n_B \) can to a good approximation be replaced by the Boltzmann distribution, whereby

\[
I \lesssim \frac{T}{8\pi q} \int_{0}^{\infty} dx e^{-\beta \left( q + \frac{m_{\phi_c}^2}{4q} \right)} = \frac{T}{8\pi q} e^{-\beta \left( q + \frac{m_{\phi_c}^2}{4q} \right)}.
\]

(9.162)

Inserting Eq. (9.162) into Eq. (9.153), we get

\[
R(T, q) \lesssim \frac{1}{(2\pi)^{3/2} q^3} \sum_{a=1}^{3} |F_a|^2 \sum_{c} m_{\phi_c} \phi_c \frac{T}{8\pi q} e^{-\beta \left( q + \frac{m_{\phi_c}^2}{4q} \right)}.
\]

(9.163)

Combining now with Eq. (9.150) produces

\[
Y(t_0) \lesssim \frac{45\sqrt{5}}{128\pi^2} \frac{1}{128\pi^2} \sum_{a=1}^{3} |F_a|^2 \sum_{c} m_{\phi_c} \int_{T_0}^{\infty} \frac{dt}{T^2} \int_{T_0}^{\infty} \frac{dz}{T^2} \frac{m_{p_1}}{c_1^2(T)h_{\text{eff}}(T)\sqrt{g_{\text{eff}}(T)}} \left[ \frac{m_{\phi_c}}{T} \right] \approx \frac{45\sqrt{5}}{64\pi^{13/2}} \sum_{a=1}^{3} |F_a|^2 \sum_{c} m_{\phi_c} \int_{T_0}^{\infty} \frac{dt}{T^2} \frac{m_{p_1}}{c_1^2(T)h_{\text{eff}}(T)\sqrt{g_{\text{eff}}(T)}} K_1 \left( \frac{m_{\phi_c}}{T} \right),
\]

(9.164)

where \( K_1 \) is a Bessel function. For \( m_{\phi_c}/T \gg 1 \), we can finally approximate

\[
K_1 \left( \frac{m_{\phi_c}}{T} \right) \approx \sqrt{\frac{\pi T}{2m_{\phi_c}}} e^{-\beta m_{\phi_c}},
\]

(9.165)

and if we approximate \( c_1^2 \), \( h_{\text{eff}} \) and \( g_{\text{eff}} \) to be slowly varying, the remaining integral can be carried out:

\[
\int_{T_0}^{\infty} \frac{dt}{T^2} f(T) \sqrt{\frac{\pi T}{2m_{\phi_c}}} e^{-\beta m_{\phi_c}} \approx \frac{1}{m_{\phi_c}^3} \frac{1}{2} \sqrt{\frac{\pi}{2}} \int_{0}^{\infty} dx x^{-7/2} f(m_{\phi_c} x) e^{-x} \approx \frac{1}{m_{\phi_c}^3} \frac{1}{2} \sqrt{\frac{\pi}{2}} \int_{0}^{\infty} dy y^{3/2} f \left( \frac{m_{\phi_c}}{y} \right) e^{-y} \approx \frac{1}{m_{\phi_c}^3} \sqrt{\frac{\pi}{2}} \left( \frac{5}{2} \right) f \left( \frac{m_{\phi_c}}{2} \right).
\]

(9.166)
where we replaced \( y \) in the argument of \( f \) by the value where the integral has obtained about 50% of its total magnitude. Thereby most of the dark matter abundance is indeed generated at temperatures \( T \lesssim m_{\phi_c} \), and

\[
Y(t_0) \lesssim \frac{135\sqrt{5}}{256\sqrt{2}\pi^{11/2}} \sum_{\alpha=1}^{3} |F_{\alpha}|^2 \sum_{c} \frac{m_{\text{Pl}}}{m_{\phi_c}} \frac{1}{c_{\alpha}^2 \left( \frac{m_{\phi_c}}{2} \right) h_{\text{eff}} \left( \frac{m_{\phi_c}}{2} \right) \sqrt{g_{\text{eff}} \left( \frac{m_{\phi_c}}{2} \right)}} , \tag{9.167}
\]

where we inserted \( \Gamma(5/2) = 3\sqrt{\pi}/4 \).

Eq. (9.167) displays clearly the variables on which the dark matter abundance related to a specific mechanism depends on: the coupling constants \( |F_{\alpha}|^2 \), the mass of the decaying particle \( m_{\phi_c} \), as well as the thermal history of the Universe (the functions \( c_{\alpha}^2, h_{\text{eff}}, g_{\text{eff}} \)).
9.6. Appendix: relativistic Boltzmann equation

We recall here briefly the structure of the collision terms of a relativistic Boltzmann equation, and compare the result with the quantum field theoretic formula in Eq. (9.153).

To understand the logic of the Boltzmann equation, a possible starting point is Fermi’s Golden Rule for a decay rate:

$$\Gamma_{1\rightarrow n}(Q) = \frac{c}{2E_q} \int d\Phi_{1\rightarrow n} |\mathcal{M}_{1\rightarrow n}|^2,$$  

(9.168)

where the phase space integration measure is defined as

$$\int d\Phi_{1+m\rightarrow n} = \int \left\{ \prod_{a=1}^m \frac{d^3q_a}{(2\pi)^32E_{q_a}} \right\} \left\{ \prod_{i=1}^n \frac{d^3p_i}{(2\pi)^32E_{p_i}} \right\} (2\pi)^4\delta^{(4)}(\sum_{i=1}^n P_i - Q - \sum_{a=1}^m Q_a).$$  

(9.169)

Moreover, $c$ is a statistical factor and $\mathcal{M}$ is the scattering amplitude.

Let now $f(x, q)$ be a particle distribution function; we assume its normalization to be so chosen that the total number density of particles at $x$ is given by

$$n(x) = \int \frac{d^3q}{(2\pi)^3} f(x, q).$$  

(9.170)

In particular, in thermal equilibrium, $f(x, q) \equiv n_F(E_q)$ (or $f(x, q) \equiv n_B(E_q)$ for bosons) is independent of the position $x$, and determined uniquely by the temperature (and by possible chemical potentials). At the same time, in vacuum, assuming a single plane wave regularized by a finite volume $V$, we would have

$$f(x, q_0) = (2\pi)^3\frac{1}{V}\delta^{(3)}(q - q_0);$$  

(9.171)

the logic is that then $n(x)$, as defined by Eq. (9.170), evaluates to $1/V$, while $f(x, q_0) = 1$, where we made use of $\delta^{(3)}(0) = V/(2\pi)^3$.

To convert Eq. (9.168) into a Boltzmann equation, we identify the decay rate $\Gamma = -\partial f/f$ for the plane wave case, and deform then the structure to be Lorentz-covariant:

$$E_q \Gamma \Rightarrow -E_q \frac{\partial f_N}{\partial t} \frac{1}{f_N} \Rightarrow -q^\alpha \frac{\partial f_N}{\partial x^\alpha} \frac{1}{f_N}.$$  

(9.172)

We also modify the right-hand side of Eq. (9.168) by allowing for $1 + m$ particles in the initial state, and adding Bose enhancement and Fermi blocking factors. Thereby

$$q^\alpha \frac{\partial f_N}{\partial x^\alpha} = -\frac{c}{2} \sum_{m,n} \int d\Phi_{N+m\rightarrow n}$$

$$\times \left\{ |\mathcal{M}|^2_{N+m\rightarrow n} f_N f_a \cdots f_m (1 \pm f_i) \cdots (1 \pm f_n) 
\quad - |\mathcal{M}|^2_{n\rightarrow N+m} f_i \cdots f_n (1 \pm f_i) (1 \pm f_a) \cdots (1 \pm f_m) \right\}.  

(9.173)

Here + applies for bosons and − for fermions.

Let us compare this with Eq. (9.153). We may observe that Eq. (9.153) corresponds to the gain terms of Eq. (9.173) (i.e. the last row), since the right-handed neutrinos are (by assumption) non-thermal and escape: $f_N \equiv 0$. At the same time, to obtain a complete match, we should work out the scattering matrix elements, $|\mathcal{M}|^2$, and the statistical factors, $c$. The “strength” of the quantum field theoretic computation leading to Eq. (9.153) is that these automatically obtain their correct values; its weakness is the allowing for (perhaps partial) equilibration is not possible with Eq. (9.153), but can be achieved through the non-linear dependence of Eq. (9.173) on $f_N$. 

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