Elements of fluid dynamics for the modeling of heavy-ion collisions

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Elements of fluid dynamics for the modeling of heavy-ion collisions

Three lectures:

- Overview on the dynamics of relativistic perfect fluids

- Application of perfect relativistic fluid dynamics to the description of high-energy nucleus–nucleus collisions

- Beyond perfect relativistic fluid dynamics

Elements of fluid dynamics for the modeling of heavy-ion collisions

Three lectures:

- Overview on the dynamics of relativistic perfect fluids

phenomenological approach

  - reminder(?) on non-relativistic fluid dynamics ("hydrodynamics")
  - fundamental equations of perfect relativistic fluid dynamics
  - an example of solution

- Application of perfect relativistic fluid dynamics to the description of high-energy nucleus-nucleus collisions

- Beyond perfect relativistic fluid dynamics

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- Overview on the dynamics of relativistic perfect fluids
- Application of perfect relativistic fluid dynamics to the description of high-energy nucleus–nucleus collisions

  phenomenology & experiment

  - fluid dynamical modeling of the fireball created in the collisions
  - yields a surprisingly good rendering of some measurements
  - … at the cost of introducing elements that are not within perfect hydrodynamics
- Beyond perfect relativistic fluid dynamics

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theoretical approaches

  - many ideas: dissipative fluid dynamics / including hydrodynamical fluctuations / or "non-hydro modes"... ☛ out-of-equilibrium physics
  - ... coming from the underlying "microscopic" physics (transport) or invoking strong-coupling scenarios

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Lecture I:

- Overview on the dynamics of relativistic perfect fluids

  phenomenological approach

  - reminder(?) on non-relativistic fluid dynamics ("hydrodynamics")
  - fundamental equations of perfect relativistic fluid dynamics
  - an example of solution (possibly in lecture II)

- Application of perfect relativistic fluid dynamics to the description of high-energy nucleus-nucleus collisions

- Beyond perfect relativistic fluid dynamics

Elements of fluid dynamics...

General references:

- A few textbooks aimed at physicists:
  - Landau & Lifshitz, vol.6: "Fluid Mechanics"

    (L&L-style..., a short chapter on relativistic hydro)
  - Guyon, Hulin, Petit, Mitescu: "Physical Hydrodynamics"

    (more phenomenological, only non-relativistic fluid dynamics)
  - Rezzola & Zanotti, "Relativistic Hydrodynamics"

    (by astrophysicists with an interest in numerical fluid dynamics)
  - General Relativity / Cosmology textbooks often contain a chapter on relativistic fluid dynamics

    Weinberg, "Gravitation and Cosmology"

    Misner, Thorne, Wheeler, "Gravitation"

Elements of fluid dynamics...

- Review articles
  - Andersson & Comer, "Relativistic fluid dynamics: Physics for many scales" arXiv:gr-qc/0605010
  - Romatschke, "New developments in relativistic viscous hydrodynamics" arXiv:0902.3663
  - Ollitrault, "Relativistic hydrodynamics for heavy ion collisions" arXiv:0708.2443
  - ... and many others
- Online lecture notes
  - a few chapters in Blandford & Thorne, "Applications of Classical Physics" (will soon become a book; use your favorite search engine...)
  - N.B. @ http://www.physik.uni-bielefeld.de/~borghini/Teaching/Hydrodynamics (sorry for the lack of modesty)

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Lecture I:

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phenomenological approach

  - reminder(?) on non-relativistic fluid dynamics ("hydrodynamics")
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Non-relativistic fluid dynamics

Classical definition:

A “fluid” is a continuous medium that keeps on deforming as long as it is subject to tangential forces (“shear stresses” ≠ normal stresses):

gas, liquid, plasma…

☛ ≠ deformable solid (elastic / plastic), which will reach an equilibrium

Non-relativistic fluid dynamics

Classical definition:

A "fluid" is a **continuous medium** that keeps on deforming as long as it is subject to tangential forces ("shear stresses" ≠ normal stresses):

gas, liquid, plasma...

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Continuous medium?

- for the mathematician, this makes life easier
  ☛ "differentiable medium" is even better
- for the physicist, this is a model, thus open to discussion if need be
  ☛ atoms exist, don't they?
  Well, so do fields, so who knows...

Non-relativistic fluid dynamics

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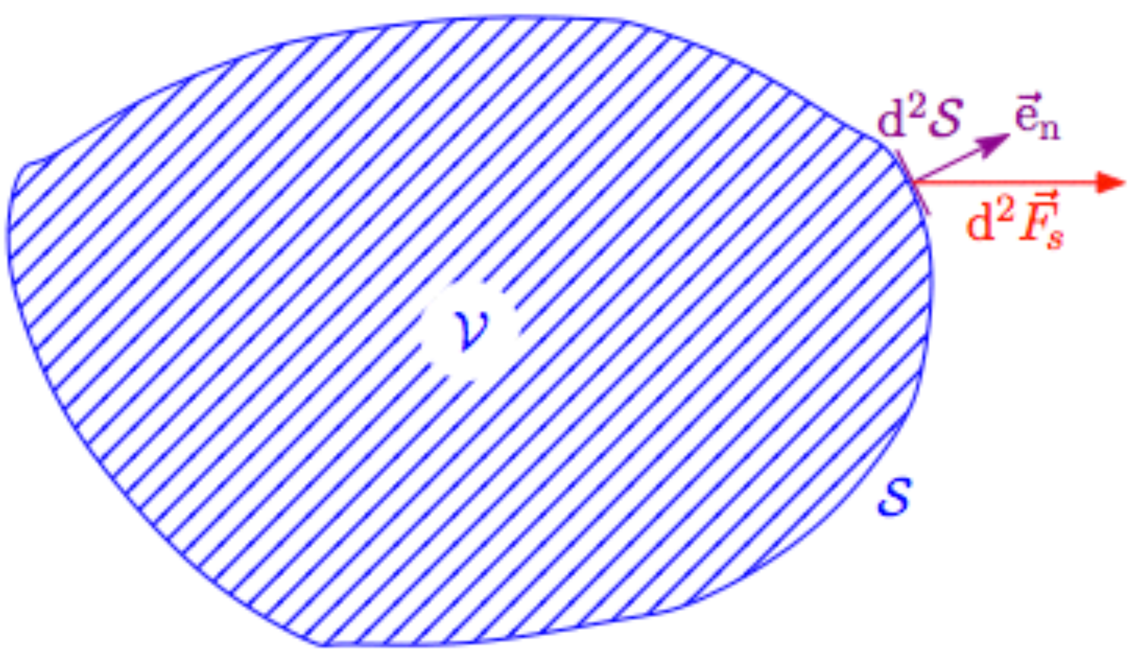
Idea: decompose the contact force — exerted by the neighboring fluid element(s) or by a wall / obstacle — per unit surface into a normal and a tangential part:



Non-relativistic fluid dynamics

Classical definition:

A "fluid" is a continuous medium that keeps on deforming as long as it is subject to tangential forces ("shear stresses" ≠ normal stresses):

☞ In a fluid at rest, the contact forces are normal!

(hydrostatic) pressure forces

One introduce a pressure field $\mathcal{P}(t,\vec{r})$ at each point of the fluid

☛ will quickly be continuous, differentiable... at least by parts

Non-relativistic fluid dynamics

The "degrees of freedom" = dynamical variables

A "fluid" is a continuous medium characterized by its

- mass density $\rho(t,\vec{r})$
- pressure $\mathcal{P}(t,\vec{r})$
- local flow velocity $\vec{\mathrm{v}}(t,\vec{r})$
- energy density $e(t,\vec{r})$
- ... further dynamical fields?

at each time and position.

☛ all of them are assumed to behave smoothly, at least by parts

☞ These "mesoscopic" quantities deserve a better definition, in terms of more microscopic ones... Will come later!

Non-relativistic fluid dynamics

The “degrees of freedom” = dynamical variables

A “fluid” is a continuous medium characterized by its

- mo
- pr
- loc
- en
- ...

at ea

parts

Throughout these lectures, I consider a single, simple fluid:

- single “component”: one mass density / particle number and one velocity field only

$\Longleftrightarrow$

no mixture!

- simple: electrically neutral, non-magnetic...

☞ These “mesoscopic” quantities deserve a better definition, in terms of more microscopic ones... Will come later!

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?

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Model of a continuous medium

… the hidden physical assumptions

System (of particles / fields) can be divided in thought in cells that fulfill two contradictory conditions:

- they must be large enough that quantities defined as averages over their content (e.g.: mass density, average particle velocity…) have small fluctuations

  ☞ cell size $\gg$ mean free path $\ell_{\rm mfp}$

- they must be small enough to remain statistically homogeneous, i.e. the system "mesoscopic" properties do not vary too much over the cell

  ☞ cell size $\ll$ scale L of macroscopic gradients

Model of a continuous medium

an important dimensionless number

Two length scales:

- mean free path* $\ell_{\rm mfp}$
- size L over which macroscopic fields vary ($\sim 1/|\partial_{\vec{r}}|$)

☞ The description as a continuous medium is meaningful iff

Knudsen number $\mathrm{Kn} \equiv \dfrac{\ell_{\rm mfp}}{L} \ll 1$

necessary consistency check (easily written... non-trivial!)

*strictly speaking, is well defined only for "dilute" systems

Non-relativistic fluid dynamics

The "degrees of freedom" = dynamical variables

A "fluid" is a continuous medium characterized by its

- mass density $\rho(t,\vec{r})$
- pressure $\mathcal{P}(t,\vec{r})$
- local flow velocity $\vec{\mathrm{v}}(t,\vec{r})$
- energy density $e\,(t,\vec{r})$
- … further dynamical fields?

at each time and position.

☞ We now need equations for these various fields!

Complicated-looking(?) equations, yet with a clear physical meaning!

Non-relativistic fluid dynamics

The dynamical equations of motion

Invoke the most basic laws of physics: conservation equations!

- mass / particle number conservation
- momentum conservation (or more generally, Newton's 2nd law)
- energy conservation

each of which are expressed locally.

Non-relativistic fluid dynamics

The dynamical equations of motion

Invoke the most basic laws of physics: conservation equations!

- mass / particle number conservation (equivalent since $\rho = nm$)
- momentum conservation (or more generally, Newton's 2nd law)
- energy conservation

each of which are expressed locally.

"Continuity equation":

$$\frac{\partial \rho(t,\vec{r})}{\partial t} + \vec{\nabla}\cdot\left[\rho(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right] = 0$$

$$\frac{\partial n(t,\vec{r})}{\partial t} + \vec{\nabla}\cdot\left[n(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right] = 0$$

(Remember charge conservation in electrodynamics!)

Non-relativistic fluid dynamics

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"Continuity equation":

$$\partial_t \rho(t,\vec{r}) + \partial_i \big[\rho(t,\vec{r})\, \mathrm{v}^i(t,\vec{r})\big] = 0$$

$$\partial_t n(t,\vec{r}) + \partial_i \big[n(t,\vec{r})\, \mathrm{v}^i(t,\vec{r})\big] = 0$$

with a sum over repeated indices (here i=1,2,3)

Rem.: in non-Cartesian coordinates, replace partial by covariant derivatives

Non-relativistic fluid dynamics

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Euler equation:

$$\rho(t,\vec{r})\underbrace{\left[\partial_t\vec{\mathrm{v}}(t,\vec{r}) + \left[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\mathrm{v}}(t,\vec{r})\right]}_{\text{local fluid acceleration}} = -\underbrace{\vec{\nabla}\mathcal{P}(t,\vec{r})}_{\text{pressure forces}} + \underbrace{\vec{f}_{\!\mathcal{V}}(t,\vec{r})}_{\text{volume forces (gravity...)}}$$

Non-relativistic fluid dynamics

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component-wise:

$$\rho(t,\vec{r})\left[\partial_t\mathrm{v}^i(t,\vec{r}) + \mathrm{v}^j(t,\vec{r})\partial_j\mathrm{v}^i(t,\vec{r})\right] = -\partial^i\mathcal{P}(t,\vec{r}) + (f_{\!\mathcal{V}})^i(t,\vec{r})$$

Non-relativistic fluid dynamics

The dynamical equations of motion

Invoke the most basic laws of physics: conservation equations!

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Euler equation:

$$\rho(t,\vec{r})\left[\partial_t\vec{\rm v}(t,\vec{r}) + \left[\vec{\rm v}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\rm v}(t,\vec{r})\right] = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \vec{f}_{\!\mathcal{V}}(t,\vec{r})$$

or Navier–Stokes equation:

$$\rho(t,\vec{r})\left[\partial_t\vec{\rm v}(t,\vec{r}) + \left[\vec{\rm v}(t,\vec{r})\cdot\vec{\nabla}\right]\vec{\rm v}(t,\vec{r})\right] = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \vec{f}_{\rm visc.}(t,\vec{r}) + \vec{f}_{\!\mathcal{V}}(t,\vec{r})$$

with
$$\vec{f}_{\rm visc.}(t,\vec{r}) \equiv \eta\vec{\nabla}^2\vec{\rm v}(t,\vec{r}) + \left(\zeta + \frac{\eta}{3}\right)\vec{\nabla}\left[\vec{\nabla}\cdot\vec{\rm v}(t,\vec{r})\right]$$

Non-relativistic fluid dynamics

The dynamical equations of motion

Invoke the most basic laws of physics: conservation equations!

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with $$\vec{f}_{\rm visc.}(t,\vec{r}) = \eta\vec{\nabla}^2\vec{\rm v}(t,\vec{r}) + \left(\zeta + \frac{\eta}{3}\right)\vec{\nabla}\big[\vec{\nabla}\cdot\vec{\rm v}(t,\vec{r})\big]$$

WHAT THE...?

Non-relativistic fluid dynamics

The dynamical equations of motion

- Momentum conservation (or more generally, Newton's 2nd law) is expressed locally by...
  - the Euler equation

$$\rho(t,\vec{r})\Big[\partial_t\vec{\mathrm{v}}(t,\vec{r}) + \big[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathrm{v}}(t,\vec{r})\Big] = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \vec{f}_{\!\mathcal{V}}(t,\vec{r})$$

☞ if the fluid is "perfect" (or "ideal")

  - the Navier–Stokes equation

$$\rho(t,\vec{r})\Big[\partial_t\vec{\mathrm{v}}(t,\vec{r}) + \big[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathrm{v}}(t,\vec{r})\Big] = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \vec{f}_{\mathrm{visc.}}(t,\vec{r}) + \vec{f}_{\!\mathcal{V}}(t,\vec{r})$$

☞ if the fluid is "Newtonian"

  - the Burnett / super-Burnett equation... ... if the fluid is...

not kidding!

(Simple) fluid models

may be seen as properties of specific flows (more later)

- Perfect / ideal fluids:

  … are such that there are no dissipative effects in them:

  ☛ neither shear stresses (friction) nor heat conduction

  ☞ "dry water"

  (Feynman, Lectures on Physics vol.II, chap.40)

(Simple) fluid models

may be seen as properties of specific flows (more later)

- Perfect / ideal fluids:

  ... are such that there are no dissipative effects in them:

  ☛ neither shear stresses (friction) nor heat conduction

  $$\Longleftrightarrow$$

  At each point in the fluid, the properties as seen by an observer at rest w.r.t. the fluid—i.e. comoving with it—are (locally) isotropic.

In particular, the momentum flux-density tensor $T^{ij}(t,\vec{r})$ is isotropic:

☛ amount of i-th component of momentum transported in direction j

Conversely, "anisotropic fluids" are not perfect. (So what?)

(Simple) fluid models

may be seen as properties of specific flows (more later)

- Perfect / ideal fluids:

... have a locally isotropic momentum flux-density tensor:

(amount of i-th component of momentum transported in direction j)

$$T^{ij}(t,\vec{r}) = \underbrace{\mathcal{P}(t,\vec{r})}_{\text{thermal transport}}\delta^{ij} + \underbrace{\rho(t,\vec{r})\,\mathsf{v}^i(t,\vec{r})\,\mathsf{v}^j(t,\vec{r})}_{\text{convective transport}}$$

☞ only normal surface forces (no tangential stresses): pressure

Rems.:

- $T^{ij}(t,\vec{r})$ is symmetric.
- In non-Cartesian coordinates, replace δ^{ij} by the inverse metric tensor $g^{ij}(t,\vec{r})$.

(Simple) fluid models

may be seen as properties of specific flows (more later)

- Dissipative fluids:

... have a possibly anisotropic momentum flux-density tensor:

☛ because they admit dissipative currents

$$T^{ij}(t,\vec{r}) = \underbrace{\mathcal{P}(t,\vec{r})\delta^{ij}}_{\text{thermal transport}} + \underbrace{\rho(t,\vec{r})\mathsf{v}^i(t,\vec{r})\mathsf{v}^j(t,\vec{r})}_{\text{convective transport}} + \underbrace{\pi^{ij}(t,\vec{r})}_{\text{diffusive transport}}$$

What is the form of the viscous stress tensor π^{ij}?

☞ models (many!)

(Simple) fluid models

may be seen as properties of specific flows (more later)

- Newtonian fluids:

... have an anisotropic momentum flux-density tensor T^{ij} including a viscous stress tensor π^{ij} depending linearly on the 1st-order spatial derivatives of the flow velocity:

$$T^{ij}(t,\vec{r}) = \mathcal{P}(t,\vec{r})\delta^{ij} + \rho(t,\vec{r})\mathrm{v}^i(t,\vec{r})\mathrm{v}^j(t,\vec{r}) + \pi^{ij}(t,\vec{r})$$

with $$\pi^{ij}(t,\vec{r}) \equiv -\eta(t,\vec{r})\underbrace{\left[\partial^j\mathrm{v}^i(t,\vec{r}) + \partial^i\mathrm{v}^j(t,\vec{r}) - \tfrac{2}{3}\delta^{ij}\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\right]}_{\text{symmetric, traceless}} - \zeta(t,\vec{r})\underbrace{\delta^{ij}\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})}_{\propto\,\delta^{ij}\,!}$$

where $\underbrace{\text{shear viscosity } \eta \text{ \& bulk viscosity } \zeta}_{\text{transport coefficients}}$ are independent of $\vec{\mathrm{v}}$.

(Simple) fluid models

may be seen as properties of specific flows (more later)

- Newtonian fluids:

  ... have a shear viscosity η and a bulk viscosity ζ.

  ☞ modify the surface forces:

  - Normal force: with an "effective pressure"

$$\mathcal{P}(t,\vec{r}) - \left[\zeta(t,\vec{r}) - \frac{2}{3}\eta(t,\vec{r})\right]\underbrace{\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})}_{\text{local expansion rate of the fluid}}$$

  - Tangential force: friction

    "wet water" (Feynman, Lectures on Physics vol.II, chap.41)

    ... on a surface in the xy-plane, with the velocity along x, and a velocity gradient along the z direction:

$$F_x \propto \eta\, \partial_z \mathrm{v}_x$$

(Simple) fluid models

may be seen as properties of specific flows (more later)

- Newtonian fluids:

  ... have an anisotropic momentum flux-density tensor T^{ij} including a viscous stress tensor π^{ij} depending linearly on the 1st-order spatial derivatives of the flow velocity $\vec{\rm v}$:

  ☞ define "first-order dissipative hydrodynamics"

- Dissipative fluids:

  ... with a momentum flux-density tensor with a viscous stress tensor depending linearly on the 2nd-order spatial derivatives of $\vec{\rm v}$

  ☞ define "second-order dissipative hydrodynamics"

- and so on... You can also devise other models (see e.g. Lecture III)

(Simple) fluid models

Ideal, first-, second-, third-order hydro...

- increasing complication (= precision of the description?)
- including higher and higher gradients

When are "low order" description appropriate?

☛ the gradients must be small!

$$\Longleftrightarrow$$

size L over which macroscopic fields vary ($\sim 1/|\partial_{\vec{r}}|$) is large

...as was already argued above!

(+ looks like an expansion: is there any convergence?)

- and so on... You can also devise other models (see e.g. Lecture III)

Non-relativistic fluid dynamics

The dynamical equations of motion

- Momentum conservation (or more generally, Newton's 2nd law) is expressed locally by...
  - the Euler equation ☞ perfect / ideal hydro

$$\rho(t,\vec{r})\Big[\partial_t\vec{\mathrm{v}}(t,\vec{r}) + \big[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathrm{v}}(t,\vec{r})\Big] = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \vec{f}_{\!\mathcal{V}}(t,\vec{r})$$

for perfect / ideal fluids

  - the Navier–Stokes equation ☞ 1st order dissipative hydro

$$\rho(t,\vec{r})\Big[\partial_t\vec{\mathrm{v}}(t,\vec{r}) + \big[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathrm{v}}(t,\vec{r})\Big] = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \vec{f}_{\mathrm{visc.}}(t,\vec{r}) + \vec{f}_{\!\mathcal{V}}(t,\vec{r})$$

for Newtonian fluids

  - the Burnett / super-Burnett equation... ☞ 2nd / 3rd order hydro

Non-relativistic fluid dynamics

The dynamical equations of motion

- Momentum conservation (or more generally, Newton's 2nd law):
  - the Euler equation (perfect / ideal hydro)

$$\rho(t,\vec{r})\Big[\partial_t\vec{\mathrm{v}}(t,\vec{r}) + \big[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathrm{v}}(t,\vec{r})\Big] = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \vec{f}_{\!\mathcal{V}}(t,\vec{r})$$

  - the Navier–Stokes equation (1st order dissipative hydro)

$$\rho(t,\vec{r})\Big[\partial_t\vec{\mathrm{v}}(t,\vec{r}) + \big[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathrm{v}}(t,\vec{r})\Big] = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \vec{f}_{\mathrm{visc.}}(t,\vec{r}) + \vec{f}_{\!\mathcal{V}}(t,\vec{r})$$

$$\vec{f}_{\mathrm{visc.}}(t,\vec{r}) \equiv \eta\vec{\nabla}^2\vec{\mathrm{v}}(t,\vec{r}) + \left(\zeta + \frac{\eta}{3}\right)\vec{\nabla}\big[\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\big]$$

  - the Burnett / super-Burnett equations... (2nd / 3rd order hydro)

are reformulations of

$$\underbrace{\partial_t\big[\rho(t,\vec{r})\mathrm{v}^i(t,\vec{r})\big]}_{\text{momentum density}} + \underbrace{\partial_j T^{ij}(t,\vec{r})}_{\text{momentum flux density}} = (f_{\!\mathcal{V}})^i(t,\vec{r})$$

with constant transport coefficients in the dissipative case

Model of a continuous medium

a second important dimensionless number

Write down the Navier–Stokes equation in the simpler case $\vec{\nabla}\cdot\vec{\mathrm{v}} = 0$ & in the absence of volume forces:

$$\rho(t,\vec{r})\Big[\partial_t\vec{\mathrm{v}}(t,\vec{r}) + \big[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathrm{v}}(t,\vec{r})\Big] = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \eta\vec{\nabla}^2\vec{\mathrm{v}}(t,\vec{r})$$

"incompressible Navier–Stokes equation"

Solve it... become rich!

(Millenium Clay problem...)

Model of a continuous medium

a second important dimensionless number

Write down the Navier–Stokes equation in the simpler case $\vec{\nabla}\cdot\vec{\mathrm{v}} = 0$ & in the absence of volume forces:

$$\rho(t,\vec{r})\Big[\partial_t\vec{\mathrm{v}}(t,\vec{r}) + \big[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathrm{v}}(t,\vec{r})\Big] = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \eta\vec{\nabla}^2\vec{\mathrm{v}}(t,\vec{r})$$

Divide lengths, flow velocity & pressure by typical values (depend on the fluid and flow under consideration!)

☛ yields dimensionless equation

$$\partial_{t^*}\vec{\mathrm{v}}^*(t^*,\vec{r}^*) + \big[\vec{\mathrm{v}}^*(t^*,\vec{r}^*)\cdot\vec{\nabla}^*\big]\vec{\mathrm{v}}^*(t^*,\vec{r}^*) = -\frac{\vec{\nabla}^*\mathcal{P}^*(t^*,\vec{r}^*)}{\rho} + \frac{\eta}{\rho\,\mathrm{v}_c\,L_c}\vec{\nabla}^{*2}\vec{\mathrm{v}}^*(t^*,\vec{r}^*)$$

Model of a continuous medium

a second important dimensionless number

Write down the Navier–Stokes equation in the simpler case $\vec{\nabla}\cdot\vec{\mathrm{v}} = 0$ & in the absence of volume forces:

$$\rho(t,\vec{r})\Big[\partial_t\vec{\mathrm{v}}(t,\vec{r}) + \big[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathrm{v}}(t,\vec{r})\Big] = -\vec{\nabla}\mathcal{P}(t,\vec{r}) + \eta\vec{\nabla}^2\vec{\mathrm{v}}(t,\vec{r})$$

Divide lengths, flow velocity & pressure by typical values (depend on the fluid and flow under consideration!)

☛ yields dimensionless equation

$$\partial_t\vec{\mathrm{v}} + (\vec{\mathrm{v}}\cdot\vec{\nabla})\vec{\mathrm{v}} = -\frac{\vec{\nabla}\mathcal{P}}{\rho} + \frac{\eta}{\rho\,\mathrm{v}_c\,L_c}\vec{\nabla}^2\vec{\mathrm{v}}$$

involving the **Reynolds number** $\mathrm{Re} \equiv \dfrac{\rho\,\mathrm{v}_c\,L_c}{\eta}$

Model of a continuous medium

a second important dimensionless number

The **Reynolds number** $\mathrm{Re} \equiv \frac{\rho \, \mathsf{v}_c \, L_c}{\eta}$

measures the importance of viscous effects in the flow:

- for Re ≪ 1: viscous flow
- for Re ≫ 1: inviscid ("ideal") flow...

☛ You may probably describe the fluid as perfect (not everywhere...)

Non-relativistic fluid dynamics

The dynamical equations of motion

Invoke the most basic laws of physics: conservation equations!

- mass / particle number conservation (equivalent since $\rho = nm$)
- momentum conservation (or more generally, Newton's 2nd law)
- energy conservation

each of which are expressed locally.

for a perfect fluid (I omit the variables and drop the volume forces...):

$$\partial_t \underbrace{\left(e + \tfrac{1}{2}\rho\vec{v}^2\right)}_{\text{energy density}} + \vec{\nabla}\cdot\Big[\underbrace{\left(e + \mathcal{P} + \tfrac{1}{2}\rho\vec{v}^2\right)\vec{v}}_{\text{energy flux density}}\Big] = 0$$

component-wise:

$$\partial_t\left(e + \tfrac{1}{2}\rho\vec{v}^2\right) + \partial_i\Big[\left(e + \mathcal{P} + \tfrac{1}{2}\rho\vec{v}^2\right)v^i\Big] = 0$$

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- momentum conservation (or more generally, Newton's 2nd law)
- energy conservation

each of which are expressed locally.

for a Newtonian fluid (I omit the variables and drop the volume forces...):

energy flux density:

$$\partial_t\Big(e+\tfrac{1}{2}\rho\vec{\mathrm{v}}^2\Big)+\vec{\nabla}\cdot\Bigg\{\Big(e+\mathcal{P}+\tfrac{1}{2}\rho\vec{\mathrm{v}}^2\Big)\,\vec{\mathrm{v}}-\kappa\vec{\nabla}T-\bigg[\zeta-\frac{2\eta}{3}\bigg](\vec{\nabla}\cdot\vec{\mathrm{v}})\,\vec{\mathrm{v}}-\eta\bigg[(\vec{\mathrm{v}}\cdot\vec{\nabla})\vec{\mathrm{v}}+\vec{\nabla}\bigg(\frac{\vec{\mathrm{v}}^2}{2}\bigg)\bigg]\Bigg\}=0$$

Non-relativistic fluid dynamics

The dynamical equations of motion

Invoke the most basic laws of physics: conservation equations!

- mass / particle number conservation (equivalent since $\rho = nm$)
- momentum conservation (or more generally, Newton's 2nd law)
- energy conservation

each of which are expressed locally.

for a Newtonian fluid (I omit the variables and drop the volume forces...):

$$\partial_t\left(e+\tfrac{1}{2}\rho\vec{\mathrm{v}}^2\right)+\vec{\nabla}\cdot\Big\{\left(e+\mathcal{P}+\tfrac{1}{2}\rho\vec{\mathrm{v}}^2\right)\vec{\mathrm{v}}-\boxed{\kappa\vec{\nabla}T}$$

→ heat transport

κ heat conductivity

diffusive energy transport by the viscous forces

$$-\left[\zeta-\frac{2\eta}{3}\right](\vec{\nabla}\cdot\vec{\mathrm{v}})\,\vec{\mathrm{v}}$$

$$-\eta\left[(\vec{\mathrm{v}}\cdot\vec{\nabla})\vec{\mathrm{v}}+\vec{\nabla}\left(\frac{\vec{\mathrm{v}}^2}{2}\right)\right]\Big\}=0$$

Non-relativistic fluid dynamics

dynamical variables and equations

A fluid is characterized at each time and position by its

- mass density $\rho(t,\vec{r})$ / particle number $n(t,\vec{r})$ (equivalent!)
- pressure $\mathcal{P}(t,\vec{r})$
- local flow velocity $\vec{\mathrm{v}}(t,\vec{r})$
- energy density $e(t,\vec{r})$

and possibly transport coefficients (η, ζ, κ...): material properties!

☞ **6** dynamical fields

These are governed by local expressions of conservation equations

- mass / particle number conservation
- momentum conservation
- energy conservation

☞ **5** equations only

Non-relativistic fluid dynamics

dynamical variables and equations

6 dynamical fields — $\rho(t,\vec{r})$ or $n(t,\vec{r})$, $\vec{\mathrm{v}}(t,\vec{r})$, $\mathcal{P}(t,\vec{r})$, and $e(t,\vec{r})$ — yet only 5 coupled (conservation) equations relating them to each other.

☞ one more equation needed!

- One possible easy way out:

  only investigate fluid motions with a given kinematic constraint

  ☛ steady flows ($\partial_t=0$), incompressible flows ($\vec{\nabla}\cdot\vec{\mathrm{v}}=0$), irrotational / potential flows ($\vec{\nabla}\times\vec{\mathrm{v}}=\vec{0}$: no vorticity)...

  Nature is not always that nice!

- More general, yet not innocent:

  There exists a relation between internal energy density, pressure, and particle number, "the" equation of state (EoS).

  [= a combination of the thermal & mechanical EoS $e(n,T)$ & $\mathcal{P}(n,T)$]

Non-relativistic fluid dynamics

dynamical variables and equations

An equation of state is indeed a relation between e, $\mathcal{P}$ & n or ρ.

Yet its use presupposes that the fluid is at thermodynamic equilibrium (or more precisely, at local thermodynamic equilibrium at each point & instant).

☞ strong assumption!

Actually already hidden in the transport coefficients (η, ζ, κ...):

- material properties—like the equation(s) of state;
- that quantify dissipative currents describing the linear response of the system to (small) departures from thermodynamic equilibrium.

☛ are well-defined close to thermodynamic equilibrium.

But now, we may use the whole bunch of thermodynamic relations.

Thermodynamics

Differential of internal energy U:

$$\mathrm{d}U = -\mathcal{P}\,\mathrm{d}\mathcal{V} + T\,\mathrm{d}S + \mu\,\mathrm{d}N$$

$\mathcal{P}$ pressure, $\mathcal{V}$ volume, T temperature, S entropy, μ chemical potential; N is the number of particles.

☛ this number is not conserved in a relativistic system: should be replaced by a conserved quantum number (e.g., baryon number).

☛ In a relativistic system, U also includes the mass energy of the constituents.

Internal energy: $$U = -\mathcal{P}\mathcal{V} + TS + \mu N \qquad (1)$$

Gibbs–Duhem relation: $$\mathcal{V}\,\mathrm{d}\mathcal{P} = S\,\mathrm{d}T + N\,\mathrm{d}\mu$$

Thermodynamics

In fluid dynamics, the useful quantities are rather the densities:

- internal energy density $e \equiv U/\mathcal{V}$,
- entropy density $s \equiv S/\mathcal{V}$,
- (baryon) number density $n \equiv N/\mathcal{V}$.

☞ Eq.(1) gives $$e = -\mathcal{P} + Ts + \mu n$$

Gibbs-Duhem becomes $$\mathrm{d}\mathcal{P} = s\,\mathrm{d}T + n\,\mathrm{d}\mu$$

leading to: $$\mathrm{d}e = T\,\mathrm{d}s + \mu\,\mathrm{d}n$$

Thermodynamics and fluid dynamics

Using the local Gibbs–Duhem relation $\mathrm{d}\mathcal{P} = s\,\mathrm{d}T + n\,\mathrm{d}\mu$, one can study the behavior of entropy in moving fluids:

- in perfect fluids, the continuity equation, Euler equation, & energy conservation equation automatically lead to entropy conservation

$$\partial_t s(t,\vec{r}) + \vec{\nabla}\cdot\left[s(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right] = 0$$

- in turn, in Newtonian fluids, the continuity, Navier–Stokes, & energy conservation equations lead, if the transport coefficients η, ζ, κ are taken to be positive, to the production of entropy

$$\partial_t s(t,\vec{r}) + \vec{\nabla}\cdot\left[s(t,\vec{r})\,\vec{\mathrm{v}}(t,\vec{r})\right] \geq 0$$

which makes sense, since these coefficients characterize dissipative currents.

Non-relativistic fluid dynamics

... describes the motion of continuous media in local thermodynamic equilibrium at each t & $\vec{r}$, using a set of 6 equations:

- 5 dynamical relations, expressing the local conservations of mass (or particle number), momentum, & energy

  ☛ take different forms in perfect / 1st-order / 2nd-order hydro

- and an equation of state.

These equations govern the coupled evolutions of

- mass density ρ (or equivalently particle number density n),
- internal energy density e,
- pressure $\mathcal{P}$,
- flow velocity $\vec{\rm v}$.

Besides its equation of state, the fluid is characterized by transport coefficients (η, ζ, κ...).

Elements of fluid dynamics for the modeling of heavy-ion collisions

Lecture I:

- Overview on the dynamics of relativistic perfect fluids

  phenomenological approach

  - reminder(?) on non-relativistic fluid dynamics ("hydrodynamics")
  - fundamental equations of perfect relativistic fluid dynamics
  - an example of solution

- Application of perfect relativistic fluid dynamics to the description of high-energy nucleus-nucleus collisions

- Beyond perfect relativistic fluid dynamics

~~Non-~~Relativistic fluid dynamics

... describes the motion of continuous media in local thermodynamic equilibrium at each t & $\vec{r}$, using a set of ~~6 equations:~~ 5 + N$_f$ equations

- 4 + N$_f$ dynamical relations ~~5 dynamical relations~~, expressing the local conservations of ~~mass (or particle number)~~ quantum numbers), momentum, & energy

  ☛ take different forms in perfect / 1st-order / 2nd-order hydro

- and an equation of state.

These equations govern the coupled evolutions of

- ~~mass density ρ (or equivalently~~ ~~particle~~ quantum number ~~density n)~~ densities n_a,
- internal energy density e,
- pressure $\mathcal{P}$,
- flow velocity $\vec{v}$. "repackaged" within a 4-velocity

Besides its equation of state, the fluid is characterized by transport coefficients (η, ζ, κ...).

Relativistic fluid dynamics

dynamical variables and equations

A relativistic fluid is a continuous medium characterized by currents:

- a quantum number 4-current $\mathrm{N}_a(\mathrm{x})$, with components $N_a^\mu(\mathrm{x})$, for each conserved quantum number a, such that (in Minkowski coordinates)
  - $N_a^0(\mathrm{x})$ is the local density of quantum number a, and
  - the $N_a^i(\mathrm{x})$ are the components of the local flux density of a;
- an energy-momentum tensor $\mathrm{T}(\mathrm{x})$, with components $T^{\mu\nu}(\mathrm{x})$, such that
  - $T^{00}(\mathrm{x})$ is the local energy density;
  - $T^{0j}(\mathrm{x})$ is the density of the j^{th} component of momentum;
  - the $T^{i0}(\mathrm{x})$ are the components of the energy flux density;
  - the $T^{ij}(\mathrm{x})$ are the components of the momentum flux-density.

Greek resp. Latin indices run from 0 resp. 1 to 3.

Relativistic fluid dynamics

dynamical variables and equations

A relativistic fluid is a continuous medium characterized by currents:

- a quantum number 4-current $\mathrm{N}_a(\mathrm{x})$, with components $N_a^\mu(\mathrm{x})$, for each conserved quantum number a, whose conservation equation reads

$$\partial_\mu N_a^\mu(\mathrm{x}) = 0$$

- an energy-momentum tensor $\mathrm{T}(\mathrm{x})$, with components $T^{\mu\nu}(\mathrm{x})$, whose conservation equation reads

$$\partial_\mu T^{\mu\nu}(\mathrm{x}) = 0$$

☛ take different forms in perfect / 1st-order / 2nd-order hydro

Rem.: in non-Minkowski coordinates, replace partial by covariant derivatives: $\partial_\mu \to \mathrm{d}_\mu$

Relativistic perfect fluid dynamics

In perfect / ideal fluids:

... one can at each point x find a reference frame LR(x), such that an observer at rest in LR(x) sees the instantaneous local properties of the fluid as isotropic, i.e. (in Minkowski coordinates)

- the spatial components of every quantum number 4-current $N_a(x)$ vanish in LR(x):

$$N_a^\mu(\mathrm{x})\big|_{\mathrm{LR(x)}} = \begin{pmatrix} n_a(\mathrm{x}) \\ \vec{0} \end{pmatrix}$$

- in the energy-momentum tensor T(x), all T^{i0}(x) and T^{0j}(x) vanish in LR(x), while T^{ij}(x) is diagonal:

$$T^{\mu\nu}(\mathrm{x})\big|_{\mathrm{LR(x)}} = \begin{pmatrix} \epsilon(\mathrm{x}) & 0 & 0 & 0 \\ 0 & \mathcal{P}(\mathrm{x}) & 0 & 0 \\ 0 & 0 & \mathcal{P}(\mathrm{x}) & 0 \\ 0 & 0 & 0 & \mathcal{P}(\mathrm{x}) \end{pmatrix}$$

Relativistic perfect fluid dynamics

In perfect / ideal fluids:

... one can at each point x find a reference frame LR(x), such that an observer at rest in LR(x) sees the instantaneous local properties of the fluid as isotropic.

The 4-velocity u(x) of that observer with respect to an observer at rest in another reference frame $\mathcal{R}$ defines the flow 4-velocity of the fluid w.r.t. $\mathcal{R}$.

That is, the flow 4-velocity u(x) — which is timelike and normalized to -1, i.e. $[u(x)]^2 = -1$ — has in the "local rest frame" LR(x) at x the Minkowski components

$$u^\mu(\mathrm{x})\big|_{\mathrm{LR(x)}} = \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix}$$

As is now clear, the metric with signature (-,+,+,+) will be used.

Relativistic perfect fluid dynamics

In perfect / ideal fluids:

... one can at each point x find a reference frame LR(x), such that an observer at rest in LR(x) sees the instantaneous local properties of the fluid as isotropic.

The 4-velocity u(x) of that observer with respect to an observer at rest in another reference frame $\mathcal{R}$ defines the flow 4-velocity of the fluid w.r.t. $\mathcal{R}$.

Denoting by $\vec{\mathrm{v}}(\mathrm{x})$ the corresponding 3-velocity w.r.t. $\mathcal{R}$ and by $\gamma(\mathrm{x})$ the associated Lorentz factor, the components of the 4-velocity in $\mathcal{R}$ read

$$u^\mu(\mathrm{x})\big|_{\mathcal{R}} = \begin{pmatrix} \gamma(\mathrm{x}) \\ \gamma(\mathrm{x})\vec{\mathrm{v}}(\mathrm{x}) \end{pmatrix} \quad \text{with} \quad \gamma(\mathrm{x}) \equiv \frac{1}{\sqrt{1-\vec{\mathrm{v}}(\mathrm{x})^2}}$$

Relativistic perfect fluid dynamics

In the local frame LR(x) the quantum number 4-current $N_a(x)$ and energy-momentum tensor T(x), have the simple expressions

$$N_a^\mu(x)\big|_{\rm LR(x)} = \begin{pmatrix} n_a(x) \\ \vec{0} \end{pmatrix} \qquad T^{\mu\nu}(x)\big|_{\rm LR(x)} = \begin{pmatrix} \epsilon(x) & 0 & 0 & 0 \\ 0 & \mathcal{P}(x) & 0 & 0 \\ 0 & 0 & \mathcal{P}(x) & 0 \\ 0 & 0 & 0 & \mathcal{P}(x) \end{pmatrix}$$

involving local densities $n_a(x)$ for quantum number a & $\epsilon(x)$ for energy, and pressure $\mathcal{P}(x)$.

In an arbitrary reference frame and system of coordinates, they are given by

$$N_a^\mu(x) = n_a(x) u^\mu(x)$$

$$T^{\mu\nu}(x) = \big[\epsilon(x) + \mathcal{P}(x)\big] u^\mu(x) u^\nu(x) + \mathcal{P}(x) g^{\mu\nu}(x)$$

(identities between two 4-vectors / tensors valid in one reference frame, thus in any.)

Relativistic perfect fluid dynamics

The components of the quantum number 4-current $N_a(x)$ and energy-momentum tensor $T(x)$ of a perfect fluid are given by

$$N_a^\mu(x) = n_a(x)u^\mu(x)$$

$$T^{\mu\nu}(x) = \big[\epsilon(x) + \mathcal{P}(x)\big]u^\mu(x)u^\nu(x) + \mathcal{P}(x)g^{\mu\nu}(x)$$

in terms of the flow 4-velocity $u(x)$.

Introducing the tensor $\Delta^{\mu\nu}(x) \equiv g^{\mu\nu}(x) + u^\mu(x)u^\nu(x)$, which projects on the 3-space orthogonal to the 4-velocity [check!], the latter may be recast as

$$T^{\mu\nu}(x) = \epsilon(x)u^\mu(x)u^\nu(x) + \mathcal{P}(x)\Delta^{\mu\nu}(x)$$

Relativistic perfect fluid dynamics

The components of the quantum number 4-current $N_a(x)$ and energy-momentum tensor $T(x)$ of a perfect fluid are given by

$$N_a^\mu(x) = n_a(x) u^\mu(x)$$

$$T^{\mu\nu}(x) = \epsilon(x) u^\mu(x) u^\nu(x) + \mathcal{P}(x) \Delta^{\mu\nu}(x)$$

in terms of the flow 4-velocity $u(x)$ and the projector orthogonal to it.

One easily checks [exercise!] the identities

$$n_a(x) = \frac{N_a^\mu(x) u_\mu(x)}{u^\nu(x) u_\nu(x)} \quad , \quad \epsilon(x) = u_\mu(x) T^{\mu\nu}(x) u_\nu(x) \ , \ \mathcal{P}(x) = \frac{1}{3} \Delta_{\mu\nu}(x) T^{\mu\nu}(x)$$

valid in reference frame and system of coordinates.

These relations show that $n_a(x)$, $\epsilon(x)$, $\mathcal{P}(x)$ are scalar fields.

Relativistic perfect fluid dynamics

dynamical variables and equations

In a perfect relativistic fluid the conserved currents are:

- N_f quantum number 4-currents $N_a(x)$, with components $N_a^\mu(x)$, whose conservation equation read

$$\partial_\mu N_a^\mu(x) = 0 \qquad \text{with} \qquad N_a^\mu(x) = n_a(x) u^\mu(x)$$

- the energy-momentum tensor $T(x)$, with components $T^{\mu\nu}(x)$, whose conservation equation reads

$$\partial_\mu T^{\mu\nu}(x) = 0 \qquad \text{with} \qquad T^{\mu\nu}(x) = \epsilon(x) u^\mu(x) u^\nu(x) + \mathcal{P}(x) \Delta^{\mu\nu}(x)$$

5 (= $\epsilon(x)$, $\mathcal{P}(x)$, only 3 components of $u(x)$) + N_f fields, 4 + N_f equations

☛ equation of state to close the system of equations!

Relativistic perfect fluid dynamics

a quick example

Consider a flow with small velocity.

To first order in the latter:

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & (\epsilon+\mathcal{P})\mathsf{v}^1 & (\epsilon+\mathcal{P})\mathsf{v}^2 & (\epsilon+\mathcal{P})\mathsf{v}^3 \\ (\epsilon+\mathcal{P})\mathsf{v}^1 & \mathcal{P} & 0 & 0 \\ (\epsilon+\mathcal{P})\mathsf{v}^2 & 0 & \mathcal{P} & 0 \\ (\epsilon+\mathcal{P})\mathsf{v}^3 & 0 & 0 & \mathcal{P} \end{pmatrix}$$

so that the energy-momentum conservation equation reads

$$\partial_\mu T^{\mu 0} = 0 \qquad \partial_t\epsilon + \vec{\nabla}\cdot\left[(\epsilon+\mathcal{P})\vec{\mathsf{v}}\right] = 0 \tag{2}$$

$$\partial_\mu T^{\mu j} = 0 \qquad \partial_t\left[(\epsilon+\mathcal{P})\vec{\mathsf{v}}\right] + \vec{\nabla}\mathcal{P} = \vec{0} \tag{3}$$

Relativistic perfect fluid dynamics

a quick example

Consider small adiabatic perturbation (sound wave!) of a uniform and steady flow:

$$\begin{cases} \epsilon(\mathsf{x}) = \epsilon_0 + \delta\epsilon(\mathsf{x}) \\ \mathcal{P}(\mathsf{x}) = \mathcal{P}_0 + \delta\mathcal{P}(\mathsf{x}) \end{cases}$$

Linearizing the equations of motion (2,3) yields:

- from (2): $$\partial_t(\delta\epsilon) + (\epsilon_0 + \mathcal{P}_0)\vec{\nabla}\cdot\vec{\mathsf{v}} = 0 \qquad (4)$$
- from (3): $$(\epsilon_0 + \mathcal{P}_0)\partial_t\vec{\mathsf{v}} + \vec{\nabla}\delta\mathcal{P} = \vec{0} \qquad (5)$$

Defining $c_s^2 \equiv \left(\frac{\partial\mathcal{P}}{\partial\epsilon}\right)_{|\frac{s}{n}}$, eqs. (4) & (5) lead to

$$\partial_t^2(\delta\epsilon) - c_s^2\vec{\nabla}^2(\delta\epsilon) = 0$$

wave equation, c_s speed of sound.

☛ equation of state $\mathcal{P}(\epsilon)$!

Relativistic perfect fluid dynamics

"equation of state"

Since the speed of sound c_s is given by the relation

$$c_s^2 \equiv \left(\frac{\partial \mathcal{P}}{\partial \epsilon}\right)_{|\frac{s}{n}}$$

the equation of state when there are no relevant conserved quantum number is precisely the relation between pressure and energy density.

Simple examples (for analytical calculations) are

- for massless particles (living in 1+3 dimensions): $\epsilon = 3\mathcal{P}$
- massive particles (in cosmology): $\mathcal{P} = 0$ ("dust")
- "vacuum" (in cosmology): $\mathcal{P} = -\epsilon$

For heavy-ion collisions, compact astrophysical objects ($n \neq 0$), precision cosmology: more complicated forms!

Elements of fluid dynamics for the modeling of heavy-ion collisions

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  - an example of solution

- Application of perfect relativistic fluid dynamics to the description of high-energy nucleus-nucleus collisions

- Beyond perfect relativistic fluid dynamics

Relativistic perfect fluid dynamics

an example of solution

The dynamical equations of perfect relativistic fluid read

- N_f times $\partial_\mu N_a^\mu(\mathsf{x}) = 0$ with $N_a^\mu(\mathsf{x}) = n_a(\mathsf{x}) u^\mu(\mathsf{x})$
- and $\partial_\mu T^{\mu\nu}(\mathsf{x}) = 0$ with $T^{\mu\nu}(\mathsf{x}) = \epsilon(\mathsf{x}) u^\mu(\mathsf{x}) u^\nu(\mathsf{x}) + \mathcal{P}(\mathsf{x}) \Delta^{\mu\nu}(\mathsf{x})$

Projecting the energy-momentum conservation equation parallel and perpendicular to the 4-velocity yields:

$$u^\mu(\mathsf{x}) \partial_\mu \epsilon(\mathsf{x}) + \big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big] \partial_\mu u^\mu(\mathsf{x}) = 0$$

$$\big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big] u^\mu(\mathsf{x}) \partial_\mu u^\nu(\mathsf{x}) + \Delta^{\mu\nu}(\mathsf{x}) \partial_\mu \mathcal{P}(\mathsf{x}) = 0$$

Convenient notation: $\nabla^\nu \equiv \Delta^{\mu\nu}(\mathsf{x}) \partial_\mu$

Rem.: in non-Minkowski coordinates, replace partial by covariant derivatives: $\partial_\mu \to \mathrm{d}_\mu$

Relativistic perfect fluid dynamics

an example of solution

The dynamical equations of perfect relativistic fluid read

- N_f times $\partial_\mu N_a^\mu(\mathsf{x}) = 0$ with $N_a^\mu(\mathsf{x}) = n_a(\mathsf{x}) u^\mu(\mathsf{x})$
- and $\partial_\mu T^{\mu\nu}(\mathsf{x}) = 0$ with $T^{\mu\nu}(\mathsf{x}) = \epsilon(\mathsf{x}) u^\mu(\mathsf{x}) u^\nu(\mathsf{x}) + \mathcal{P}(\mathsf{x}) \Delta^{\mu\nu}(\mathsf{x})$

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$$u^\mu(\mathsf{x})\partial_\mu \epsilon(\mathsf{x}) + \left[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\right] \partial_\mu u^\mu(\mathsf{x}) = 0$$

$$\left[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\right] u^\mu(\mathsf{x}) \partial_\mu u^\nu(\mathsf{x}) + \nabla^\nu \mathcal{P}(\mathsf{x}) = 0$$

Convenient notation: $\nabla^\nu \equiv \Delta^{\mu\nu}(\mathsf{x})\partial_\mu$

Rem.: in non-Minkowski coordinates, replace partial by covariant derivatives: $\partial_\mu \to \mathrm{d}_\mu$

An example of relativistic flow: "Bjorken flow"

Quantum-number-free perfect fluid, ☛ $n_a(x) = 0$

in one-dimensional motion along the z-axis with the 3-velocity $\mathrm{v}^z = \frac{z}{t}$

for $z < ct$ and $t > t_0$.

First(?) discussed by R.Hwa (1974); made ~~popular~~ famous by J.D.Bjorken (1983)

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Highly relativistic nucleus-nucleus collisions: The central rapidity region

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KEK scanned document ; ADS Abstract Service; Fermilab Library Server (fulltext available); Phys. Rev. D Server; Link to Fulltext
Detailed record - Cited by 2316 records 1000+ as of June 7, 2015

Bjorken flow

One-dimensional flow along the z-axis with the 3-velocity $\mathrm{v}^z = \frac{z}{t}$ for $z < ct$ and $t > t_0$.

☞ 4-velocity* $u^\mu(\mathrm{x}) = \begin{pmatrix} \gamma(\mathrm{x}) \\ \gamma(\mathrm{x})\mathrm{v}^z(\mathrm{x}) \end{pmatrix}$ with $\gamma(\mathrm{x}) = \frac{1}{\sqrt{1-\mathrm{v}^z(\mathrm{x})^2}}$

$$= \frac{t}{\sqrt{t^2 - z^2}}$$

*only the non-trivial components are shown

Bjorken flow

One-dimensional flow along the z-axis with the 3-velocity $\mathrm{v}^z = \dfrac{z}{t}$ for $z < ct$ and $t > t_0$.

☞ 4-velocity* $u^\mu(\mathrm{x}) = \begin{pmatrix} \dfrac{t}{\sqrt{t^2 - z^2}} \\ \dfrac{z}{\sqrt{t^2 - z^2}} \end{pmatrix}$

*only the non-trivial components are shown

Bjorken flow

One-dimensional flow along the z-axis with the 3-velocity $\mathrm{v}^z = \frac{z}{t}$ for $z < ct$ and $t > t_0$.

☞ 4-velocity* $u^\mu(\mathrm{x}) = \begin{pmatrix} \frac{t}{\sqrt{t^2 - z^2}} \\ \frac{z}{\sqrt{t^2 - z^2}} \end{pmatrix} = \begin{pmatrix} \cosh\varsigma \\ \sinh\varsigma \end{pmatrix}$

Milne coordinates $\begin{cases} \tau \equiv \sqrt{t^2 - z^2} \\ \varsigma \equiv \frac{1}{2}\log\frac{t+z}{t-z} \end{cases} \Leftrightarrow \begin{cases} t = \tau\cosh\varsigma \\ z = \tau\sinh\varsigma \end{cases}$

*only the non-trivial components are shown

Bjorken flow

One-dimensional flow along the z-axis with the 3-velocity $\mathrm{v}^z = \dfrac{z}{t}$ for $z < ct$ and $t > t_0$.

☞ 4-velocity* $u^\mu(\mathrm{x}) = \begin{pmatrix} \dfrac{t}{\sqrt{t^2 - z^2}} \\ \dfrac{z}{\sqrt{t^2 - z^2}} \end{pmatrix} = \begin{pmatrix} \cosh\varsigma \\ \sinh\varsigma \end{pmatrix}$

Milne coordinates $\begin{cases} \tau \equiv \sqrt{t^2 - z^2} \\ \varsigma \equiv \dfrac{1}{2}\log\dfrac{t+z}{t-z} \end{cases} \Leftrightarrow \begin{cases} t = \tau\cosh\varsigma \\ z = \tau\sinh\varsigma \end{cases}$

Two possibilities:

- work directly in Milne coordinates $(x^{0'}, x^{1'}) = (\tau, \varsigma)$ ☛ $u^{\mu'}(\mathrm{x}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- stay with Minkowski coordinates ☛ simple derivatives

*only the non-trivial components are shown

Bjorken flow

$$\begin{cases} \tau \equiv \sqrt{t^2 - z^2} \\ \varsigma \equiv \frac{1}{2}\log\frac{t+z}{t-z} \end{cases} \quad \Leftrightarrow \quad \begin{cases} t = \tau\cosh\varsigma \\ z = \tau\sinh\varsigma \end{cases}$$

One easily shows [check!] that the equation of motion "along u(x)"

$$u^\mu(\mathsf{x})\partial_\mu\epsilon(\mathsf{x}) + \left[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\right]\partial_\mu u^\mu(\mathsf{x}) = 0$$

becomes

$$\partial_\tau\epsilon(\mathsf{x}) + \frac{\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})}{\tau} = 0$$

or equivalently $$\partial_\tau\left[\tau\epsilon(\mathsf{x})\right] = -\mathcal{P}(\mathsf{x})$$

which simply relates the change in the energy in a comoving volume (proportional to τ) to the work of pressure forces...

Bjorken flow

$$\begin{cases} \tau \equiv \sqrt{t^2 - z^2} \\ \varsigma \equiv \frac{1}{2}\log\frac{t+z}{t-z} \end{cases} \quad \Leftrightarrow \quad \begin{cases} t = \tau\cosh\varsigma \\ z = \tau\sinh\varsigma \end{cases}$$

In turn, the equation of motion "perpendicular to u(x)"

$$\big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]u^\mu(\mathsf{x})\partial_\mu u^\nu(\mathsf{x}) + \nabla^\nu\mathcal{P}(\mathsf{x}) = 0$$

yields [check!]

$$\partial_\varsigma\mathcal{P}(\mathsf{x}) = 0$$

i.e. pressure — and, invoking the equation of state, energy density — is independent of space-time rapidity.

☞ boost invariance!

Bjorken flow

Coming back to
$$\partial_\tau \epsilon(\mathsf{x}) + \frac{\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})}{\tau} = 0$$

inserting the equation of state $\mathcal{P}(\mathsf{x}) = c_s(\mathsf{x})^2 \epsilon(\mathsf{x})$ yields

$$\partial_\tau \epsilon(\mathsf{x}) + \left[1 + c_s(\mathsf{x})^2\right] \frac{\epsilon(\mathsf{x})}{\tau} = 0$$

If the speed of sound is constant, this leads to

$$\epsilon(\mathsf{x}) \propto \frac{1}{\tau^{1+c_s^2}} \qquad \mathcal{P}(\mathsf{x}) \propto \frac{1}{\tau^{1+c_s^2}}$$

Bjorken flow

In turn, the entropy conservation equation $\partial_\mu[s(\mathsf{x})u^\mu(\mathsf{x})] = 0$ becomes

$$\partial_\tau s(\mathsf{x}) + \frac{s(\mathsf{x})}{\tau} = 0$$

leading at once to

$$s(\mathsf{x}) \propto \frac{1}{\tau}$$

Using now $\epsilon(\mathsf{x}) \propto \frac{1}{\tau^{1+c_s^2}}$, $\mathcal{P}(\mathsf{x}) \propto \frac{1}{\tau^{1+c_s^2}}$ and the relation $\epsilon + \mathcal{P} = Ts$, one arrives at

$$T(\mathsf{x}) \propto \frac{1}{\tau^{c_s^2}}$$

Bjorken flow

We have found $\epsilon(\mathrm{x}) \propto \frac{1}{\tau^{1+c_s^2}}$, $\mathcal{P}(\mathrm{x}) \propto \frac{1}{\tau^{1+c_s^2}}$, $s(\mathrm{x}) \propto \frac{1}{\tau}$

and indirectly $T(\mathrm{x}) \propto \frac{1}{\tau^{c_s^2}}$

For an ultrarelativistic gas $\epsilon \propto T^4$ (Stefan–Boltzmann!), $\mathcal{P} \propto T^4$

$s \propto T^3$ (remember $\epsilon + \mathcal{P} = Ts$) and $c_s^2 = \frac{1}{3}$ ... Everything is OK!