# Methods for measuring collective anisotropic flow in heavy-ion collisions

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# Methods for measuring collective anisotropic flow

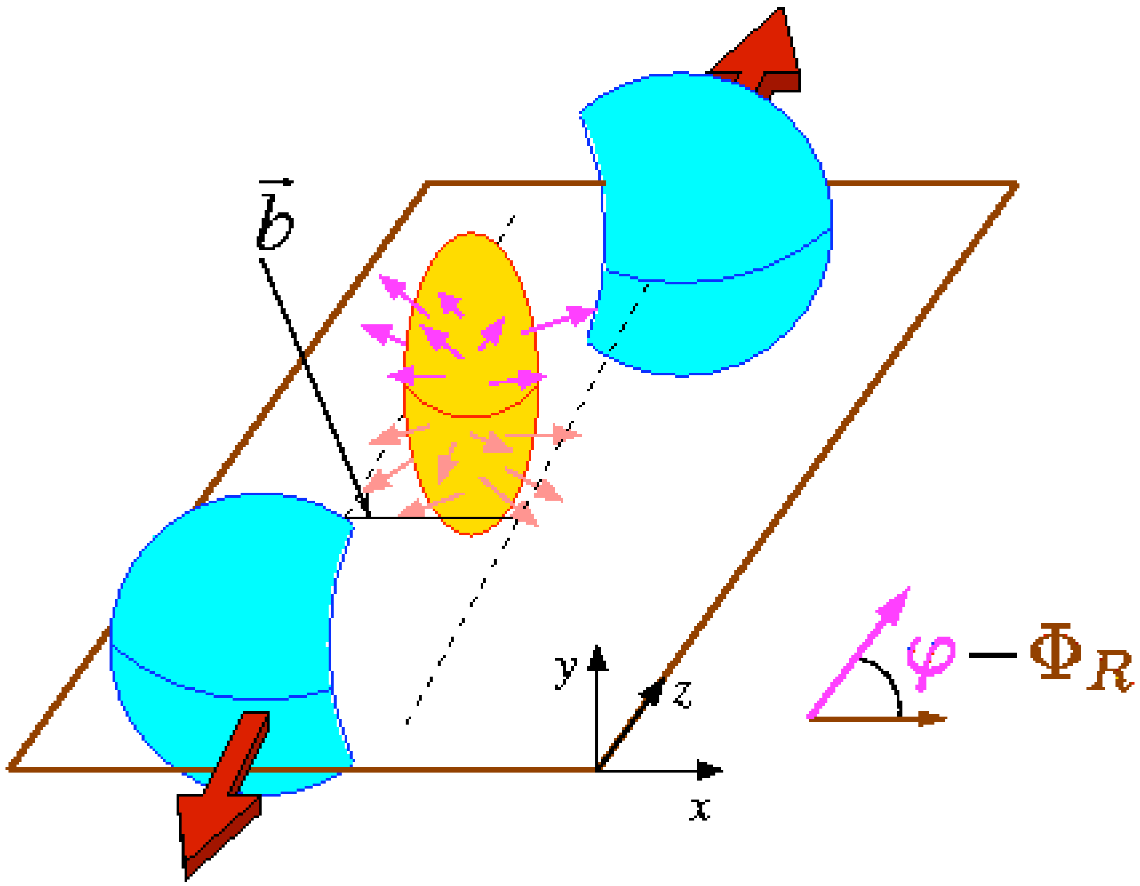
- The "standard" event-plane based method
  - intuitive... but plagued by unwanted correlations
- Multiparticle-cumulant method
  - remedies the problem faced by the standard approach, at the price of larger statistical uncertainties
- "Lee-Yang zeroes" method
  - even less intuitive than the cumulants, yet faster and with similar performance

Not mentioned here (among others):

- How to measure the fluctuations of anisotropic flow
- Acceptance issues: my detector covers $2\pi$ in azimuth!

# Anisotropic (collective) flow

Consider a non-central collision:




anisotropy of the source (in the plane transverse to the beam)

⇒ anisotropic pressure gradients (larger along the impact parameter)
push

⇒ anisotropic fluid velocities
anisotropic emission of particles:

"anisotropic collective flow"

$$E\frac{\mathrm{d}N}{\mathrm{d}^3\mathbf{p}} \propto \frac{\mathrm{d}N}{p_T\mathrm{d}p_T\mathrm{d}y}\left[1 + 2v_1\cos(\varphi - \Phi_R) + 2v_2\cos 2(\varphi - \Phi_R) + \cdots\right]$$

More particles along the impact parameter ($\varphi - \Phi_R = 0$ or 180°) than perpendicular to it ☞ "elliptic flow" $v_2 \equiv \langle\cos 2(\varphi - \Phi_R)\rangle$.

average over particles

# Anisotropic (collective) flow

$$E\frac{\mathrm{d}N}{\mathrm{d}^3\mathbf{p}} \propto \frac{\mathrm{d}N}{p_T\,\mathrm{d}p_T\,\mathrm{d}y}\left[1 + 2v_1\cos(\varphi - \Phi_R) + 2v_2\cos 2(\varphi - \Phi_R) + \cdots\right]$$

"Flow", $v_n$ do not imply fluid dynamics...

(Transverse) anisotropy of the source in a non-central collision

$\Rightarrow$ the amount of matter seen by a high-$p_T$ particle traversing the medium is anisotropic (shorter path along the impact parameter)

$\Rightarrow$ anisotropic jet quenching ("with respect to the reaction plane"): anisotropic distribution of high-$p_T$ particles

which is best characterized in terms of Fourier harmonics $v_n$ (detector independent; more robust in Monte-Carlo computations)

# Measuring anisotropic flow

At first sight, a straightforward procedure:

① Determine the reaction plane (= plane spanned by the beam axis and the impact parameter): azimuth $\Phi_R$ in the lab. frame;

② Compute the Fourier coefficients $v_n \equiv \langle \cos n(\varphi - \Phi_R) \rangle$, using the particle azimuths.

Note: if parity is conserved, symmetry with respect to the reaction plane ⇒ sin terms in the Fourier expansion vanish: $v_n = \left\langle \mathrm{e}^{\mathrm{i}n(\varphi-\Phi_R)} \right\rangle$.

**BUT!!!**

The impact parameter is NOT measured (neither its size, nor its direction).

Even worse (?), $\Phi_R$ varies from event to event.

☞ need to **estimate** the reaction plane: “event plane”

# Event-plane method

Principle:

① Estimate the event plane: azimuth $\Psi_R$ in the lab. frame;

② Compute Fourier coefficients $v_n^{\rm obs.} \equiv \langle\cos n(\varphi - \Psi_R)\rangle$ from the particle azimuths and the event plane;

$$v_n^{\rm obs.} \equiv \langle\cos n(\varphi - \Psi_R)\rangle \neq v_n \equiv \langle\cos n(\varphi - \Phi_R)\rangle$$

③ Correct the "observed" coefficients $v_n^{\rm obs.}$ to account for the difference between event plane and reaction plane.

# Event-plane method

① Estimate the event plane: azimuth $\Psi_R$ in the lab. frame.

Only way to do it: use the azimuths of the particles!
(idea: the impact parameter selects a preferred direction in the transverse plane — it breaks the isotropy; if the transverse momenta of the particles seem to favour some direction, then this direction has some relation to the impact parameter!)

Define the "event flow vector": $\mathbf{Q} \equiv \sum_j \mathbf{p_T}_j \equiv |\mathbf{Q}| e^{i\Psi_R}$, with $\mathbf{p_T}_j = p_{T\,j} e^{i\phi_j}$

sum over all particles → $\sum_j$

P.Danielewicz, G.Odyniec, PLB **157** (1985) 146

Generalize, using "arbitrary" weights: $\mathbf{Q} \equiv \sum_j w(j)\, e^{i\varphi} \equiv |\mathbf{Q}| e^{i\Psi_R}$

In the following, I shall use unit weights $w(j) = 1$

# Event-plane method

① Estimate the event plane: azimuth $\Psi_R$ in the lab. frame.

Issue: at ultrarelativistic energies, $\langle p_x \rangle$ is very small around midrapidity, where (most of) the detectors sit: the event flow vector is small.

Generalize even further: $\mathbf{Q}_n \equiv \sum_j e^{in\varphi_j} \equiv |\mathbf{Q}_n| e^{in\Psi_n}$

"second-order event-plane": $\Psi_2$

J.-Y.Ollitrault, PRD **48** (1993) 1132

- Uncertainty on $\Psi_2$ smaller than that on $\Psi_1$
- $\Psi_2$ only defined up to π (vs. 2π for $\Psi_1$): information lost

☞ can say something about "in-plane" vs. "out-of-plane", but cannot distinguish between + or - directions along the $x$ axis

# Event-plane method

② Compute Fourier coefficients $v_n^{\rm obs.} \equiv \langle \cos n(\varphi - \Psi_n) \rangle$ from the

- In each event, extract $\Psi_n$ and compute $\cos n(\varphi - \Psi_n)$ for all particles (or, say, for all protons) in the event; average over these particles;

- Do the same thing for the next event... and average over events!

- One complication: the particle whose flow you're after (azimuth $\varphi_k$) was used in the estimation of the event-plane: $\mathbf{Q}_n \equiv \sum_{\text{all } j} \mathrm{e}^{\mathrm{i}n\varphi_j}$

⇒ need to avoid the trivial "autocorrelation" of particle $\varphi_k$ with itself:

$$\mathbf{Q}'_n \equiv \sum_{j \neq k} \mathrm{e}^{\mathrm{i}n\varphi_j} \equiv |\mathbf{Q}'_n| \mathrm{e}^{\mathrm{i}n\Psi'_n}$$

- A refinement: one can compute $v_{mn}^{\rm obs.} \equiv \langle \cos n(m\varphi - \Psi_n) \rangle$, to obtain higher flow harmonics.

# Event-plane method

③ Correct the "observed" coefficients $v_n^{\rm obs.}$ to account for the difference between event plane and reaction plane.

One is after $v_n \equiv \left\langle {\rm e}^{{\rm i}n(\varphi-\Phi_R)}\right\rangle$, yet has measured $v_n^{\rm obs.} \equiv \left\langle {\rm e}^{{\rm i}n(\varphi-\Psi_n)}\right\rangle$

$$v_n^{\rm obs.} = \langle \cos n(\varphi - \Phi_R + \Phi_R - \psi_N)\rangle = \underbrace{\langle \cos n(\varphi - \Phi_R)\rangle}_{v_n} \underbrace{\langle \cos n(\Phi_R - \psi_N)\rangle}_{\equiv\, \Delta\Phi}$$

$\Delta\Phi$ uncertainty in the reaction plane determination: results from the competition between flow (which tends to align $\Psi_n$ along $\Phi_R$ ) and statistical fluctuations (whose relative size decreases like $1/\sqrt{N}$ ).

☞ can be computed (cf. next slide), to get $v_n \equiv \dfrac{v_n^{\rm obs.}}{\langle \cos \Delta\Phi\rangle}$

"event-plane resolution"

# Event-plane method

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$$v_n^{\rm obs.} = \langle \cos n(\varphi - \Phi_R + \Phi_R - \psi_N)\rangle \stackrel{!}{=} \underbrace{\langle \cos n(\varphi - \Phi_R)\rangle}_{v_n} \langle \cos \underbrace{n(\Phi_R - \psi_N)}_{\equiv \Delta\Phi}\rangle$$

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"event-plane resolution"

BUT there is a huge assumption here, namely that all correlations in the system are due to anisotropic flow.

# Event-plane method

$\Delta\Phi$ uncertainty in the reaction plane determination:
one can show (central limit theorem... and some work!) that

$$\langle\cos\Delta\Phi\rangle = \frac{\sqrt{\pi}}{2}\chi_n\, e^{-\chi_n^2/2}\left[I_0\left(\frac{\chi_n^2}{2}\right) + I_1\left(\frac{\chi_n^2}{2}\right)\right], \qquad (1)$$

where $\chi_n$ is the so-called "resolution parameter", which characterizes the relative magnitudes of flow and statistical fluctuations.

$$\chi_n \approx v_n\sqrt{N}$$

J.-Y.Ollitrault, nucl-ex/9711003

$\chi_n$ can be extracted from the data!

- Split an event into two "subevents" (assumed to be equivalent!), with "subevent flow vectors" $\mathbf{Q_a} \equiv e^{in\Psi_a}$, $\mathbf{Q_b} \equiv e^{in\Psi_b}$.
- Measure $\sqrt{\langle\cos(\Psi_a - \Psi_b)\rangle} = \langle\cos\Delta\Phi_{\rm sub.}\rangle$: resolution for the subevents
- Use Eq.(1) to deduce the resolution parameter for the subevent $\chi_{\rm sub.}$
- Say that $\chi_n$ for the whole event is $\sqrt{2}\times\chi_{\rm sub.}$

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assumes that all correlations are due to flow

# Flow from 2-particle correlations

Basically, the event-plane method relies on a study of two-particle correlations.
The core assumption is that these 2-body correlations are only due to flow, i.e., to the correlation of each particle to the reaction plane:

$$\langle \cos n(\varphi_1 - \varphi_2) \rangle = \langle \cos n(\varphi_1 - \Phi_R) \rangle \langle \cos n(\Phi_R - \varphi_2) \rangle = v_n^2$$

What if the assumption is wrong?

# Flow from 2-particle correlations

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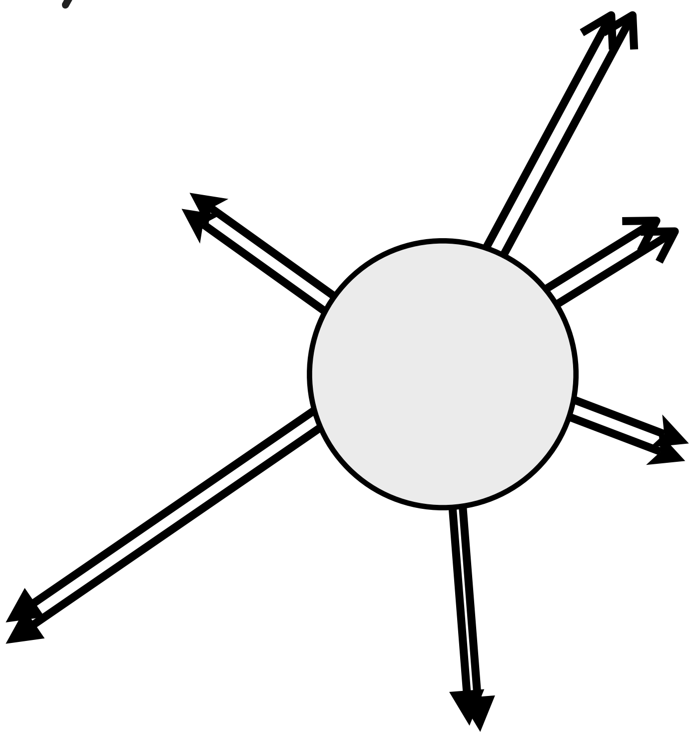
What if the assumption is wrong?

Toy model: collisions without flow, but with particles emitted by pairs



- $N$/2 correlated pairs for which cos = 1;
- $N(N$-1)/2 pairs in total

☞ probability 1/($N$-1) that an arbitrary pair be correlated:

$$\langle \cos n(\varphi_1 - \varphi_2) \rangle = 1/(N-1) \neq v_n{}^2$$

# "Nonflow" correlations

- Quantum-statistics effects
- Resonance decays
- Momentum conservation
- (Mini)jets
- Strong & Coulomb interaction
- ...

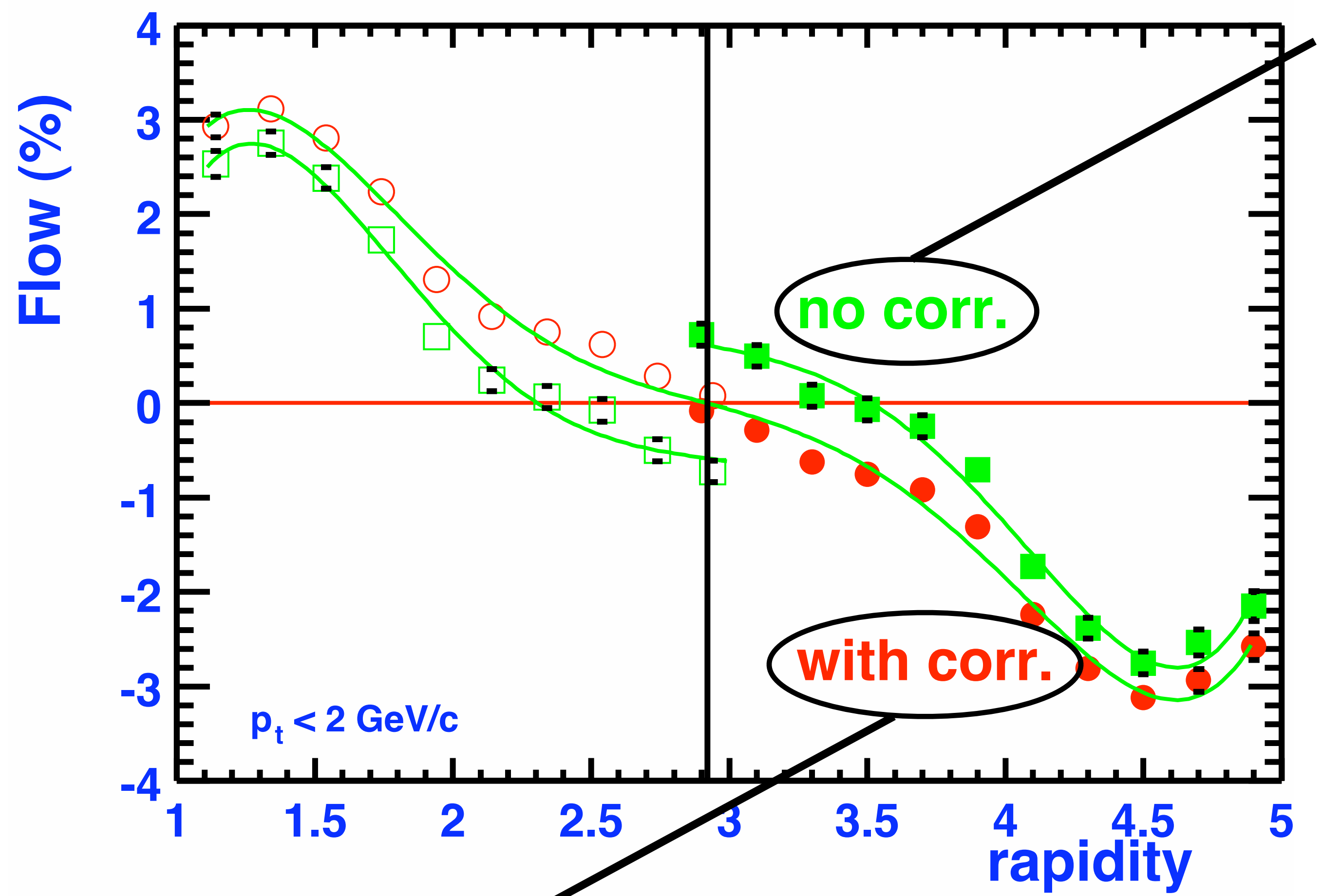
Can possibly be mistaken as correlations due to anisotropic flow, endangering the flow reconstruction.

"Nonflow" effects

One possible way(?) to remedy the problem:
Compute / estimate the effect of the correlations, and subtract it so as to isolate the flow signal.

# Nonflow correlations vs. standard flow mesurements

NA49 pions, mininimum bias, 158A GeV




value obtained without correcting for $p_T$ conservation

value including the correction for momentum conservation

N.B., P.M.Dinh, J.-Y.Ollitrault, A.M.Poskanzer, S.A.Voloshin, PRC **66** (2002) 014901

# Nonflow correlations vs. anisotropic flow mesurements

Correcting the standard method to take into account the possible sources of nonflow correlations is an intuitive approach.
But is it safe? NO: you do not know all sources of correlations.
Can one do better? YES!

- At the two-particle level: competition between flow effects of order $(v_n)^2$ and nonflow correlations of order $1/N$ :
the approach is safe if $v_n \gg 1/N^{1/2}$
- Imagine we performed a study of four-particle correlations:
  - the contribution of flow is of order $(v_n)^4$
  - for combinatorial reasons, the probability that 4 particles are all correlated together is of order $1/N^3$

☞ will allow one to measure flow if $v_n \gg 1/N^{3/4}$: improved sensitivity
Similarly, the bias from nonflow effects is a priori smaller

# Four-particle correlations

OK, since going to four-particle correlations seems to be a good idea, let's do it!

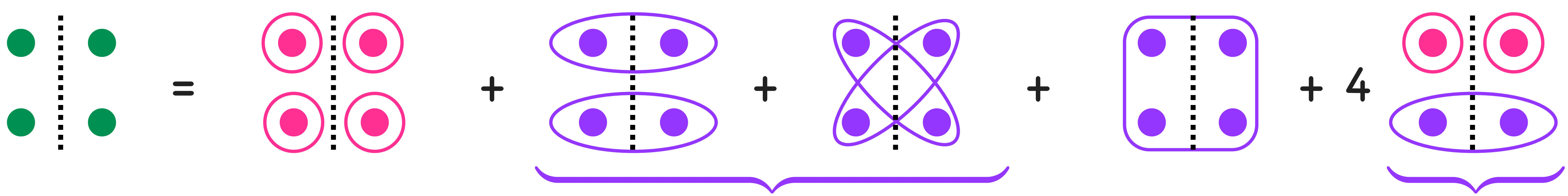
Take 4 arbitrary particles, compute $\left\langle e^{in(\varphi_1+\varphi_2-\varphi_3-\varphi_4)}\right\rangle$...

- Each particle is individually correlated by anisotropic flow to the reaction plane: term $(v_n)^4$

- All four particles are correlated together: term $1/N^3$

- Particle 1 is correlated to particle 3, particle 2 to particle 4 (or 1+4 & 2+3): term $1/N \times 1/N$

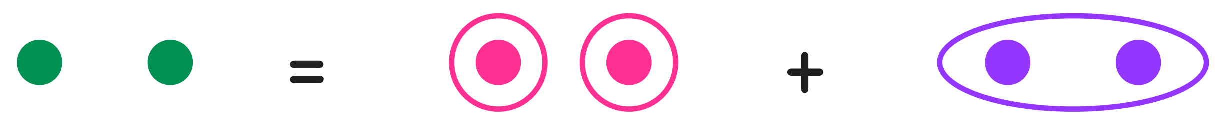
... which spoils the interest of the measurement



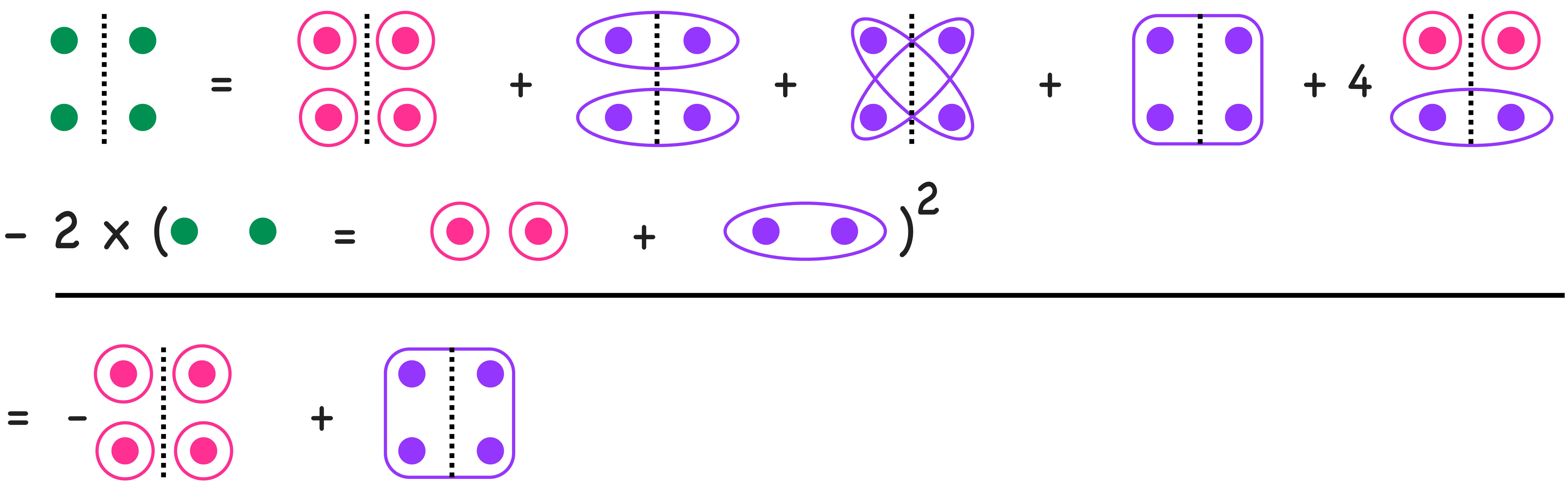

these terms are a nuisance!

# Four-particle cumulants

Yet there is still hope!
Of course we don't want to compute the two-particle correlations.
Yet we have already encountered them... when studying two-particle distributions:




So if both 2- and 4-particle distributions have been measured:



# Four-particle cumulants

$$\left\langle e^{in(\varphi_1+\varphi_2-\varphi_3-\varphi_4)}\right\rangle$$
$$-\left\langle e^{in(\varphi_1-\varphi_3)}\right\rangle\left\langle e^{in(\varphi_2-\varphi_4)}\right\rangle - \left\langle e^{in(\varphi_1-\varphi_4)}\right\rangle\left\langle e^{in(\varphi_2-\varphi_3)}\right\rangle$$
$= -v_n{}^4 +$ genuine 4-particle correlations

= 4-particle cumulant!

☞ new recipe: in a given event, consider all quadruplets of particles, compute the corresponding the cumulant.
Average over the quadruplets in the event; average over events.

The final 4-particle cumulant $c_n\{4\}$ receives contributions:

- from flow: $-v_n{}^4$
- from "nonflow" effects: $\mathcal{O}(1/N^3)$ (obvious for short-range effects, non-trivial for the effect of total momentum conservation)

define flow estimate: $v_n\{4\} \equiv [-c_n\{4\}]^{1/4}$

N.B., P.M.Dinh, J.-Y.Ollitrault, PRC **63** (2001) 054904, **64** (2001) 054901

# Cumulant method of flow analysis

A flow measurement using cumulants proceeds in 2 steps:

- First, one constructs the (4-particle) cumulant by letting particles 1, 2, 3 & 4 be any of the detected particles.

☞ gives an estimate of $v_n$ averaged over the whole phase space: "integrated flow" $v_n\{4\} \equiv [-c_n\{4\}]^{1/4}$

(equivalent of the event-plane determination; but here, no $\Psi_n$)

- In a second step, particles 1 are restricted to being only a given particle type in some corner of phase space (say 400 MeV protons at midrapidity), while particles 2, 3 & 4 can still be any particle in the event (except particle 1!).

☞ yields a cumulant $d_n\{4\}$ from which one deduces an estimate of the proton "differential" flow $d_n\{4\} \equiv -v'_n\{4\}\, v_n\{4\}^3$

Actually, one can obtain the proton higher-harmonics $v'_{mn}\{3+m\}$.

# Cumulant method of flow analysis

"Consider all quadruplets of particles, compute the corresponding the cumulant, average over the quadruplets in the event, then over events."
This sounds tedious!

Trick ☺:

① Introduce the generating function (of a complex variable $z$)

$$G_n(z) = \prod_{k=1}^{N} \left(1 + z^* \mathrm{e}^{\mathrm{i}n\varphi_k} + z\,\mathrm{e}^{-\mathrm{i}n\varphi_k}\right)$$

where the product runs over all (detected) particles in the event.

Why?

② Average $G_n(z)$ over many events:

$$\langle G_n(z)\rangle = 1 + |z|^2 \left\langle \sum_{j\neq k} \mathrm{e}^{\mathrm{i}n(\varphi_j-\varphi_k)} \right\rangle + \frac{|z|^4}{4} \left\langle \sum_{j,k,l,m} \mathrm{e}^{\mathrm{i}n(\varphi_j+\varphi_k-\varphi_l-\varphi_m)} \right\rangle + \cdots$$

$G_n$ generates multiparticle distributions (including all combinations)

# Cumulant method of flow analysis

③ To obtain the cumulants... take the logarithm (check!)

$$\ln\langle G_n(z)\rangle = N^2|z|^2 c_n\{2\} + \frac{N^4|z|^4}{4} c_n\{4\} + \cdots$$

This generates all cumulants at once!

④ Take a piece of paper, and compute the contribution of anisotropic flow to the cumulants.

modified Bessel function

In the presence of flow only $\ln\langle G_n(z)\rangle = \ln I_0(2Nv_n|z|)$, which you identify with the measured generating function: each power of $|z|^2$ yields an identity, which defines a flow estimate.

$$v_n\{2\}^2 \equiv c_n\{2\}, \quad v_n\{4\}^4 \equiv -c_n\{4\}, \quad v_n\{6\}^6 \equiv \frac{c_n\{6\}}{4}, \quad v_n\{8\}^8 \equiv -\frac{c_n\{8\}}{33}$$

Bonus! you get several estimates at once

⑤ Post the paper on nucl-ex; gather citations!

# Cumulant method of flow analysis

Principle: when going to cumulants of higher and higher order, the relative contribution of flow to the cumulant increases ( $\propto {v_n}^{2k}$ ) while that of nonflow correlations decreases $\propto 1/N^{2k-1}$

☞ systematic error on flow estimate decreases ($v_n\{2k\} \to v_n$)

On the other hand, the statistical uncertainty increases...
(typically, for a resolution parameter $\chi_n = 1$ the uncertainty on $v_n\{4\}$ is twice that on $v_n\{2\}$, while those on $v_n\{2k \geq 4\}$ are all similar).

You cannot have everything!

# An unsatisfactory issue...

(at least to a theorist's mind)

Anisotropic flow is a collective effect: (almost) all particles show a correlation to the impact-parameter direction.

Yet we measure it with correlations involving only a small number of particles (2, 4, 6, 8) out of several hundreds/thousands.

Aren't we missing something?

Idea: could we do "infinite-order" cumulants?

YES!

# An elegant solution!

We are interested in infinite-order cumulants, i.e., in the asymptotic behaviour of the coefficients in the power-series expansion of the cumulant generating function $\ln\langle G_n(z)\rangle$

This behaviour is entirely determined by the "first zero" (the closest to the origin) of $\langle G_n(z)\rangle$ in the complex plane.

(Think of $\ln\left(1-\frac{z}{z_0}\right)=\sum_{k=1}^{+\infty}\frac{z^k}{k\,{z_0}^k}$ : $z_0$ controls large-order coefficients)

First zero? When there is flow $\langle G_n(z)\rangle = I_0(2Nv_n|z|)$

The first zero lies at $z_0=\frac{2.405\mathrm{i}}{Nv_n}$:

measure the generating function, find its first zero, you obtain a flow estimate! (denoted by $v_n\{\infty\}$)

That's all, Folks!

R.S.Bhalerao, N.B., J.-Y.Ollitrault, NPA **727** (2003) 373

# A nice analogy

When there is flow $\langle G_n(z)\rangle = I_0(2Nv_n|z|)$, first zero lies at $z_0 = \frac{2.405\mathrm{i}}{Nv_n}$
The first zero comes closer to 0 as the system size increases.

On the other hand, if there is no flow — only short-range correlations (and momentum conservation) — $\langle G_n(z)\rangle$ factorizes: the position of the first zero does not change when the system size increases.

Does that remind you of something?

# A nice analogy

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C.N.Yang & T.D.Lee, PR **87** (1952) 404: a theory of phase transitions

- Grand partition function (fixed $T$, $V$) $\mathcal{Q}(\mu) = \sum_{N=0}^{+\infty} Z_N \mathrm{e}^{\mu N/kT}$
- Take a reference value $\mu_c$, define $z \equiv (\mu-\mu_c)/kT$
- Let $\mathcal{G}(z) \equiv \dfrac{\mathcal{Q}(\mu)}{\mathcal{Q}(\mu_c)} = \sum_{N=0}^{+\infty} P_N \mathrm{e}^{zN}$ ← probability to have $N$ particles at $\mu = \mu_c$
- Let the system size $V$ increase:
  - if no phase transition, the zeroes of $\mathcal{G}$ are unchanged;
  - if phase transition at $\mu = \mu_c$, the zeroes come closer to the origin.

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  - if phase transition at $\mu = \mu_c$, the zeroes come closer to the origin.

phase transition → long-range correlations, collective behaviour

# Methods for measuring collective anisotropic flow

A wealth of methods to measure flow...

- The "standard" event-plane based method: $v_n\{\mathrm{EP}_2\}$, ($v_n\{\mathrm{EP}_1\}$)
- Multiparticle-cumulant method: $v_n\{4\}$, $v_n\{6\}$, ...
- "Lee-Yang zeroes" method: $v_n\{\infty\}$

Two-particle methods can be plagued by nonflow effects.

This problem is solved in the cumulant and Lee-Yang zeroes methods; but at the cost of larger statistical uncertainties.
(the same in cumulant and Lee-Yang zeroes, about twice larger than in a two-particle measurement if $\chi_n \geq 1$).
The new methods are less intuitive. Yet determining the "integrated flow" gives access to everything you could dream of with a $\Psi_n$.