O(N) linear sigma model beyond the Hartree approximation at finite temperature

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Quantum Fields in and out of Equilibrium

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Jürgen Baacke and S.M., *Phys.Rev.* **D**67,085006 (2003)

Lagrangian and path integral

Classical action of O(N) linear sigma model in 3+1 dimensions

$$\mathcal{S}[\Phi] = \int d^4x \, \mathscr{L}[\Phi] = \int d^4x \left[\frac{1}{2}\partial_\mu \vec{\Phi} \cdot \partial^\mu \vec{\Phi} - \frac{\lambda}{4}\left(\vec{\Phi} \cdot \vec{\Phi} - v^2\right)^2\right]$$

where $\vec{\Phi} = (\Phi_1, \Phi_2, \ldots, \Phi_N)$

Generating functional (now of O(1) model for simplicity) with local source $J(x)$
and bilocal source $K(x,y)$

$$\mathcal{Z}[J,K] = \int \mathscr{D}\Phi \; \exp\left(\frac{i}{\hbar}\int d^4x \left\{\mathscr{L}[\Phi] + J\Phi + \frac{1}{2}\int d^4y \, \Phi(x)\, K(x,y)\Phi(y)\right\}\right)$$

Schwinger's generating functional

$$\mathcal{W}[J,K] = \frac{\hbar}{i}\log \mathcal{Z}[J,K]$$

2PI Effective Action

- Decompose quantum field (operator)

$$\Phi(x) = \phi(x) + \hbar^{1/2}\,\varphi(x)$$

- $\phi(x)$ is expectation value of a single field operator

$$\frac{\delta \mathcal{W}}{\delta J(x)} = \langle \Phi(x) \rangle = \phi(x)$$

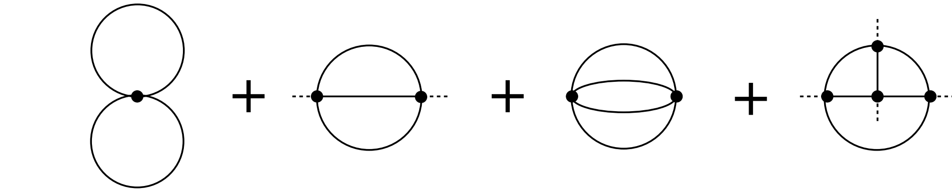
- $G(x, y)$ is (part of) the expectation value of the *bilocal composite operator* $\Phi(x)\Phi(y)$

$$2\,\frac{\delta \mathcal{W}}{\delta K(x, y)} = \langle \Phi(x)\,\Phi(y) \rangle = \phi(x)\,\phi(y) + \hbar\,G(x, y)$$

- Double Legendre transform of $\mathcal{W}[J, K]$ yields *2PI (CJT) effective action* $\Gamma^{\mathrm{2PI}}[\phi, G]$

$$\Gamma^{\mathrm{2PI}}[\phi, G] = \mathcal{S}[\phi] + \Gamma^{\mathrm{2PI}}_{1-\mathrm{loop}}[\phi, G] + \Gamma^{\mathrm{2PI}}_{2}[\phi, G]$$

- Graphical representation of 2PI effective action

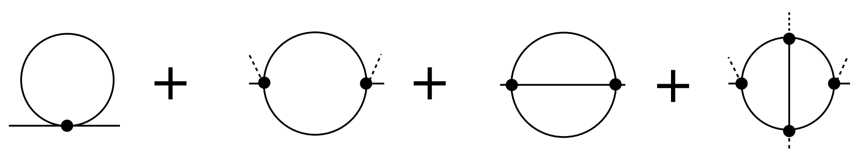
$\Gamma_2^{2PI} =$ 

- here a line represents the Green function $G(x, y)$ with a 2PR *non-local* self–energy Σ
- $\delta\Gamma^{2PI}/\delta G = 0$ yields Dyson–Schwinger equation (resummation of 2PR graphs)

$$iG^{-1} = i\mathcal{D}^{-1}[\phi] + i\Sigma[\phi, G]$$

- with self energy defined by

$$\Sigma[\phi, G] = 2\,\frac{i}{\hbar}\,\frac{\delta\Gamma_2^{2PI}[\phi, G]}{\delta G}$$

$\Sigma(k) =$ 

2PPI Effective Action

H. Verschelde *et al.*

- Make *bilocal* source *local* $K(x,y) \to K(x,x) = K(x)$
- Expectation values

$$\frac{\delta \mathcal{W}}{\delta J(x)} = \langle \Phi(x) \rangle = \phi(x)$$

$$2\,\frac{\delta \mathcal{W}}{\delta K(x)} = \langle \Phi(x)\Phi(x) \rangle = \phi^2(x) + \hbar\, G(x,x)$$

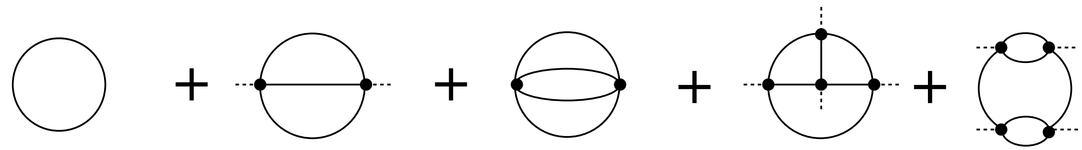
- Define (connected part of) expectation value of *local composite operator* $\Phi^2(x)$

$$\Delta(x) \equiv G(x,x)$$

- Double Legendre transform of $\mathcal{W}[J,K]$ yields *2-particle point-irreducible* *(2PPI) effective action* $\Gamma^{\mathrm{2PPI}}[\phi,\Delta]$

$$\Gamma^{\mathrm{2PPI}}[\phi,\Delta] = \mathcal{S}[\phi] + \Gamma^{\mathrm{2PPI}}_{\mathrm{class}}[\Delta] + \Gamma^{\mathrm{2PPI}}_{q}[\phi,\Delta]$$

- Graphical representation of 2PPI effective action ← subset of 2PI

$\Gamma_q^{2PPI} =$ 

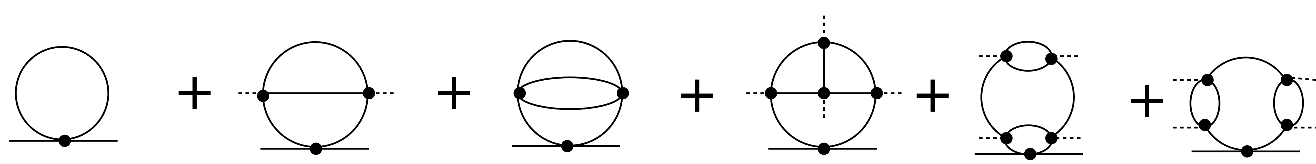
- here a line represents a Green function $\mathcal{G}$ with a 2PPR *local* self–energy or an effective mass

$$\mathcal{G}^{-1}(k) = \left[k^2 + \mathcal{M}^2\right] \ , \quad \mathcal{M}^2 = \lambda(3\phi^2 - v^2) + 3\hbar\lambda\Delta$$

$\Rightarrow$ $\Gamma[\phi, \Delta] \rightarrow \Gamma[\phi, \mathcal{M}^2]$ more practical

- $\delta\Gamma_q^{2PPI}/\delta\mathcal{M}^2$ yields explicit form of Δ as a function of ϕ and $\mathcal{M}^2$

$$\Delta(\phi, \mathcal{M}^2) = 2\,\frac{\delta\Gamma_q^{2PPI}[\phi, \mathcal{M}^2]}{\delta\mathcal{M}^2}$$

$\Delta =$ 

2PPI Effective Potential

- Effective potential

$$V_{\rm eff}^{\rm 2PPI}[\phi, \mathcal{M}^2] = V_{\rm class}^{\rm 2PPI}[\phi, \mathcal{M}^2] + V_q^{\rm 2PPI}[\phi, \mathcal{M}^2]$$

- Gap equation

$$\frac{\partial V_{\rm eff}^{\rm 2PPI}}{\partial \mathcal{M}^2} = 0 \quad \Rightarrow \quad \mathcal{M}^2 = \lambda(3\phi^2 - v^2) + 3\hbar\lambda\Delta(\phi, \mathcal{M}^2)$$

  solution of this equation is $\mathcal{M}^2 = \mathcal{M}^2(\phi)$

- $\longrightarrow$ Later we plot 1PI effective potential

$$V_{\rm eff}^{\rm 1PI}[\phi] = V_{\rm eff}^{\rm 2PPI}[\phi, \mathcal{M}^2(\phi)]$$

- Vacuum expectation value by tadpole condition

$$\frac{\partial V_{\rm eff}^{\rm 1PI}[\phi]}{\partial \phi} = \frac{d V_{\rm eff}^{\rm 2PPI}[\phi, \mathcal{M}^2]}{d\phi} = \frac{\partial V_{\rm eff}^{\rm 2PPI}}{\partial \phi} + \frac{\partial V_{\rm eff}^{\rm 2PPI}}{\partial \mathcal{M}^2}\frac{\partial \mathcal{M}^2(\phi)}{\partial \phi} = 0$$

O(N) linear sigma model

- Explicit form of 2PPI effective action

$$\Gamma^{2\mathrm{PPI}}[\phi, \mathcal{M}^2_{ij}] = \mathcal{S}[\phi] + \frac{\lambda}{4}\hbar(\Delta_{ii}\Delta_{jj} + 2\,\Delta_{ij}\Delta_{ij}) + \Gamma_q^{2\mathrm{PPI}}[\phi, \mathcal{M}^2_{ij}]$$

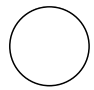
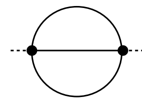
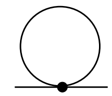
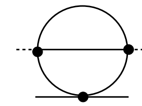
- For convenience define O(N)–symmetric effective mass $\mathcal{M}^2_{ij}$ and self–energy Δ_{ij}

$$\mathcal{M}^2_{ij} = \frac{\phi_i\phi_j}{\phi^2}\mathcal{M}^2_\sigma + \left(\delta_{ij} - \frac{\phi_i\phi_j}{\phi^2}\right)\mathcal{M}^2_\pi$$

- Sigma and pion mass parameters follow immediately

$$\begin{aligned} \mathcal{M}^2_\sigma &= \lambda\left[3\phi^2 - v^2 + 3\hbar\Delta_\sigma + (N-1)\hbar\Delta_\pi\right] \\ \mathcal{M}^2_\pi &= \lambda\left[\phi^2 - v^2 + \hbar\Delta_\sigma + (N+1)\hbar\Delta_\pi\right] \end{aligned}$$

2PPI 2-loop approximation

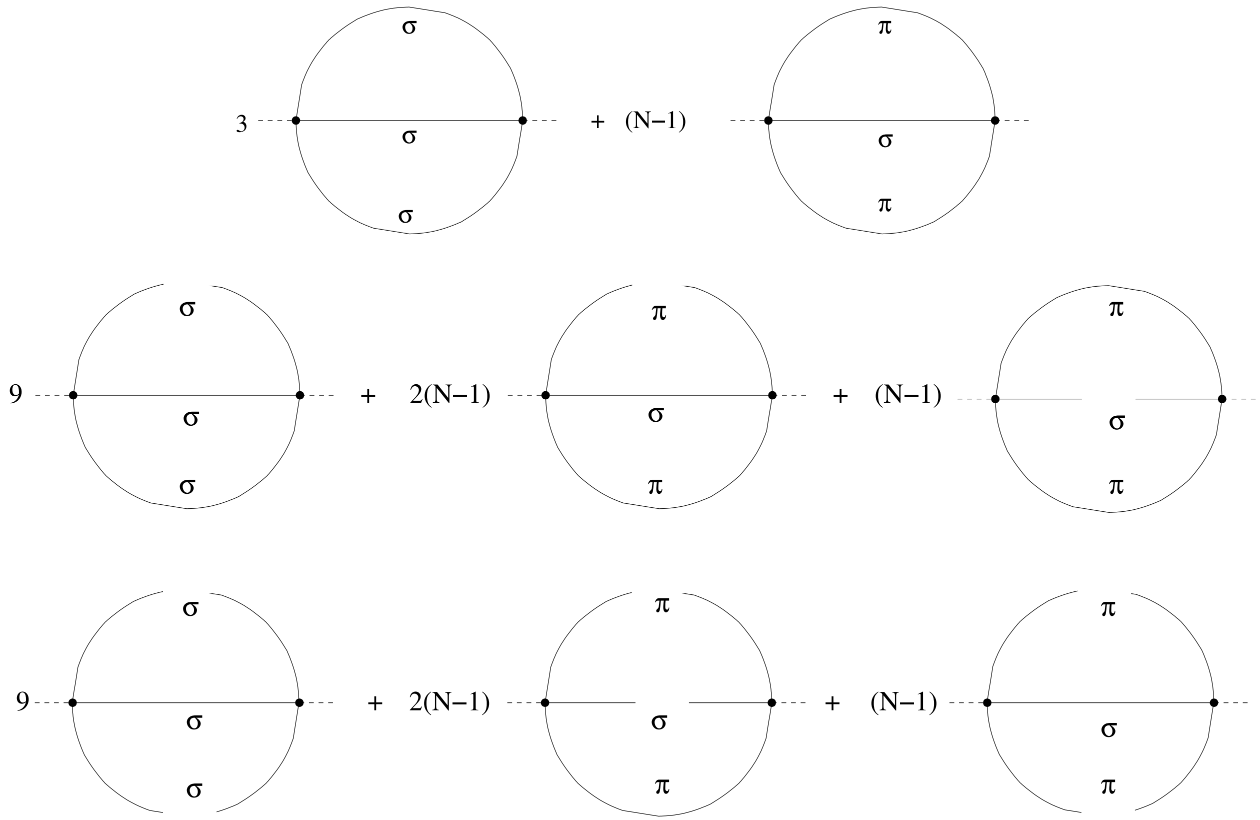
$\Gamma_q^{2PPI} =$  $+$  $\qquad \Delta =$  $+$ 

$$\Gamma_q^{2-\text{loop}} = \text{“ln det”} + \Gamma_{\text{sunset}}$$

$$\begin{aligned}
V_{\text{eff}} &= V_{\text{eff}}^{\text{class}} + V_{\text{eff}}^{1-\text{loop}} + V_{\text{eff}}^{\text{sunset}} \\
V_{\text{eff}}^{\text{class}} &= \frac{1}{2}\mathcal{M}_\sigma^2\phi^2 - \frac{\lambda}{2}\phi^4 - \frac{v^2}{2\lambda(N+2)}\left[\mathcal{M}_\sigma^2 + (N-1)\mathcal{M}_\pi^2\right] \\
&\quad -\frac{1}{8\lambda(N+2)}\left[(N+1)\mathcal{M}_\sigma^4 + 3(N-1)\mathcal{M}_\pi^4 - 2(N-1)\mathcal{M}_\sigma^2\mathcal{M}_\pi^2 + 2N\lambda^2 v^4\right] \\
V_{\text{eff}}^{1-\text{loop}} &= \frac{\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\left\{\log(k^2+\mathcal{M}_\sigma^2) + (N-1)\log(k^2+\mathcal{M}_\pi^2)\right\} \\
&\quad +\hbar T\int\frac{d^3k}{(2\pi)^3}\left\{\log\left[1-e^{-E_\sigma(\mathbf{k})/T}\right] + (N-1)\log\left[1-e^{-E_\pi(\mathbf{k})/T}\right]\right\}
\end{aligned}$$

Graphical representation of graphs in $V_{\mathrm{eff}}^{\mathrm{sunset}}$ at finite temperature

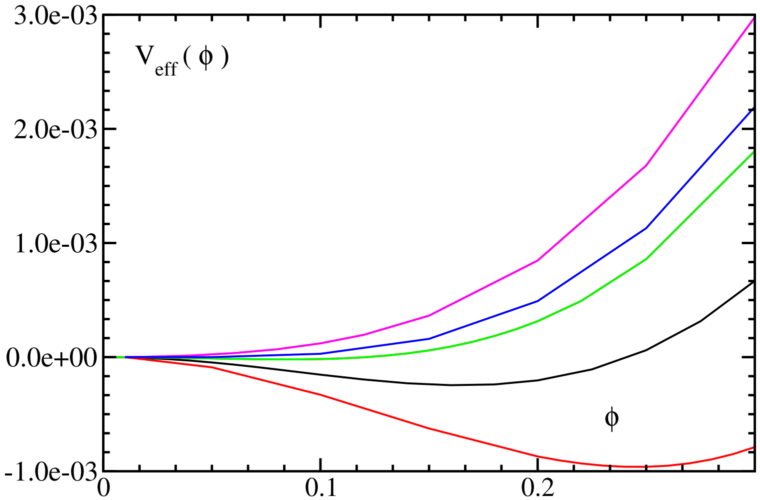
Propagator consists of a zero-temperature part (solid line) and a thermal part (interrupted line)

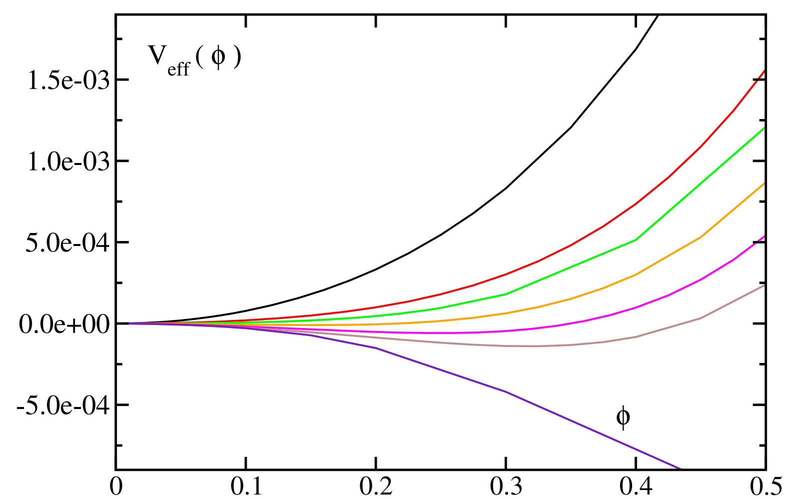


2PPI 2-loop effective potential

Fix ϕ and find the maximum of $V_{\text{eff}}(\mathcal{M}_\sigma, \mathcal{M}_\pi; \phi)$ with parameters $N = 4$ and $\mu^2 = v^2$

$\Rightarrow$ effective potential now exhibits a second-order phase transition



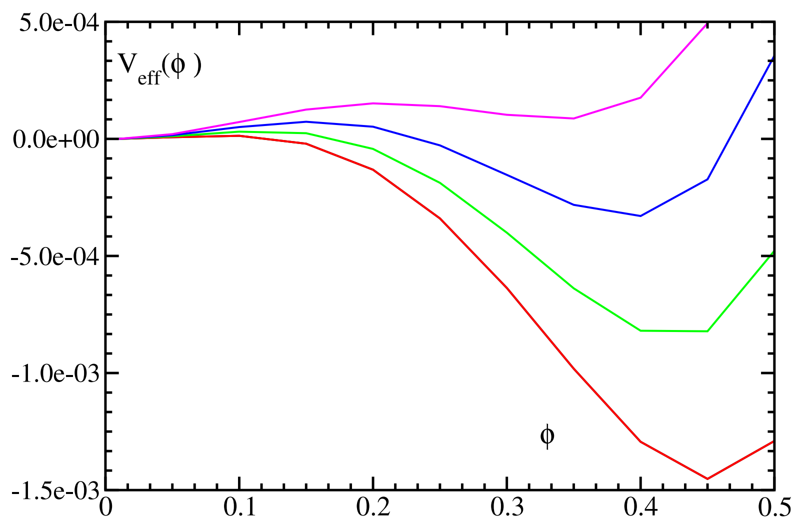


$T/v = 1.62, 1.66, 1.69, 1.70, 1.72$
$\lambda = 1$

$T/v = 1.2, 1.4, 1.425, 1.45, 1.475, 1.5, 1.6$
$\lambda = 0.1$

Hartree approximation

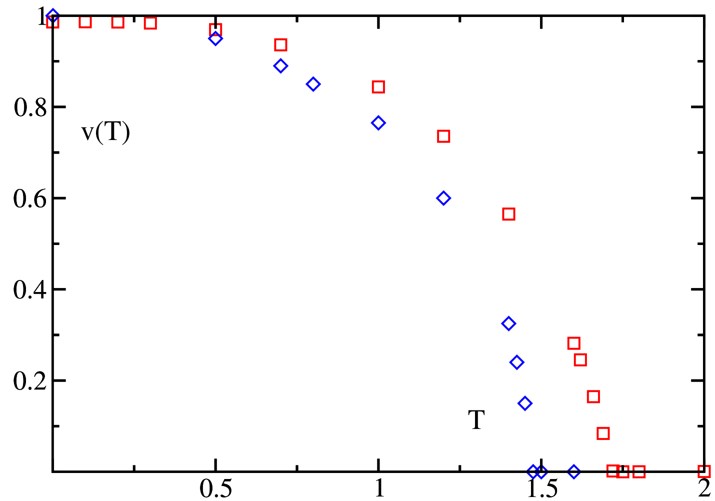
⇒ effective potential (real part) exhibits first-order phase transition:



$T/v = 1.46, 1.47, 1.48, 1.49$ and $\lambda = 1, N = 4$

Temperature-dependent behavior of VEV

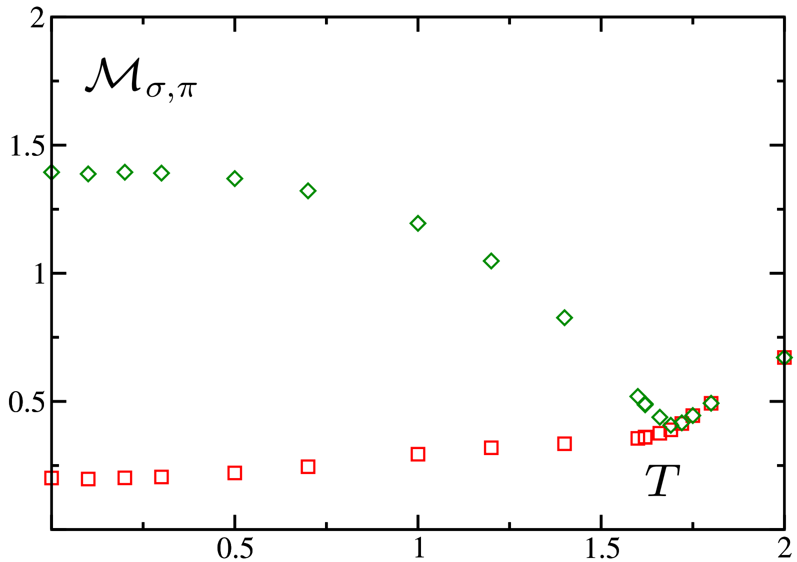
$v(T)$ = minimum of the effective potential for varying T, $N = 4$ and $\mu^2 = v^2$

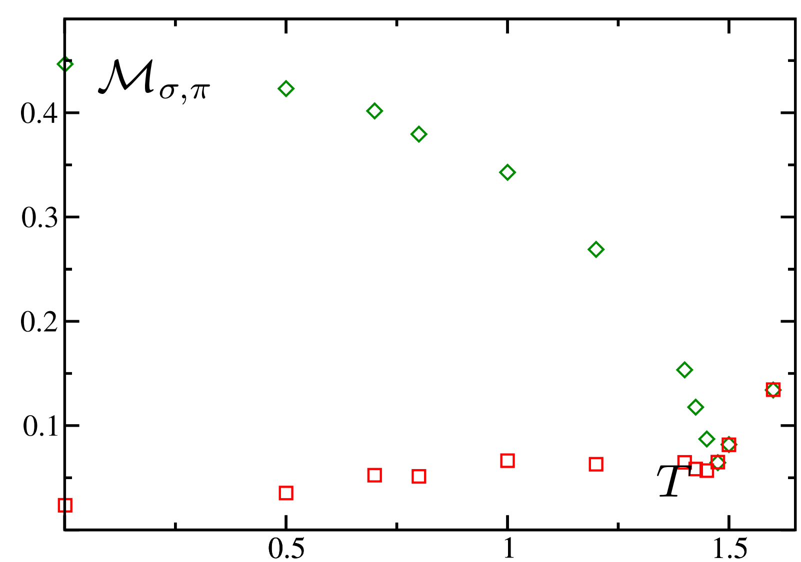


$\lambda = 1$ (squares), $\lambda = 0.1$ (diamonds)

Continuous transition from $v(T < T_c) \neq 0$ to $v(T \geq T_c) = 0$

Variational mass parameters





- Both mass parameters are the same in the (restored) symmetric phase
- Pion mass (parameter) is never equal to zero
- Mass parameters $\mathcal{M}_{\sigma,\pi} \neq$ physical masses

Physical pion mass

- The physical pion mass is always zero if the potential is O(N)-symmetric

$$M^2_{ij,\mathrm{phys}} = \frac{\phi_i\phi_j}{\phi^2} M^2_{\sigma,\mathrm{phys}} + \left(\delta_{ij} - \frac{\phi_i\phi_j}{\phi^2}\right) M^2_{\pi,\mathrm{phys}}$$

- But also

$$M^2_{ij,\mathrm{phys}} = \left.\frac{\partial^2 V(\vec{\phi}^2)}{\partial\phi_i\partial\phi_j}\right|_{\vec{\phi}=\vec{\phi}_0} = 2\,\delta_{ij}\,V'(\vec{\phi}^2)\Big|_{\vec{\phi}=\vec{\phi}_0} + 4\phi_i\phi_j\,V''(\vec{\phi}^2)\Big|_{\vec{\phi}=\vec{\phi}_0}$$

- Since $\vec{\phi}_0 = \{v(T), 0, \ldots, 0\}$ and $M^2_{\pi,\mathrm{phys}}$ is obtained by derivatives perpendicular to $\vec{\phi}_0$

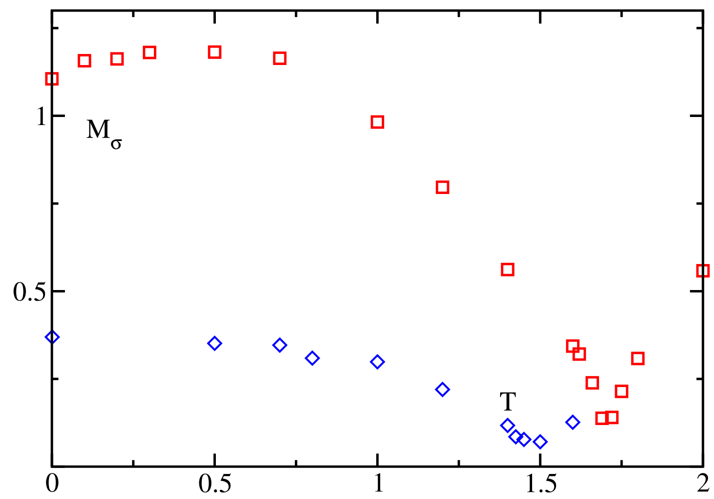
$$\Rightarrow M^2_{\pi,\mathrm{phys}} = 2\,V'(\vec{\phi}_0^2) = 0$$

$\Rightarrow$ No violation of Nambu–Goldstone theorem

Physical sigma mass

$$M^2_{\sigma,\mathrm{phys}}(T) = \left.\frac{\partial^2 V^{\mathrm{1PI}}_{\mathrm{eff}}(\phi)}{\partial \phi^2}\right|_{\phi=v(T)}$$

Calculated numerically (by interpolation) of the effective potential



$\lambda = 1$ (squares), $\lambda = 0.1$ (diamonds)

Sketch of renormalization procedure

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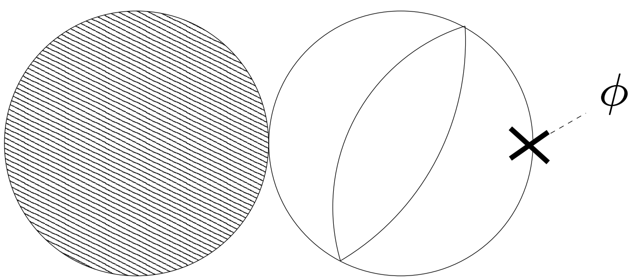
- Counter term Lagrangian (again for $N = 1$)

$$\delta\mathscr{L} = \frac{1}{2}\delta Z_0\,(\partial_\mu\phi)^2 + \delta Z_2\,\frac{1}{2}m^2\phi^2 + \delta Z_\lambda\,\frac{\lambda}{4}\phi^4 + \delta E_{\mathrm{vac}}\ , \quad m^2 = -\lambda v^2$$

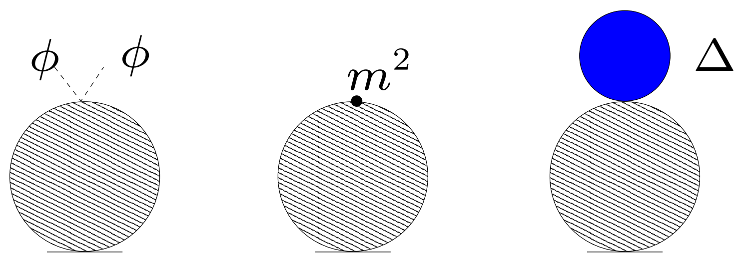
- Renormalize equations $\delta\Gamma/\delta\phi = 0$ and $\delta\Gamma/\delta\mathcal{M}^2 = 0$ on 1PI level

$$\frac{\delta}{\delta\phi}\Gamma_q^{\mathrm{1PI}}[\phi] = \frac{\partial}{\partial\phi}\Gamma_q^{\mathrm{2PPI}}[\phi,\mathcal{M}^2] + 6\lambda\phi\,\frac{\partial}{\partial\mathcal{M}^2}\Gamma_q^{\mathrm{2PPI}}[\phi,\mathcal{M}^2]$$



2PPR part 2PPI rest (earmarked)

- Renormalize bubble subgraphs at first



$$3\lambda\,\delta Z_\lambda^{\rm 2PPR}\,\phi^2 + \delta Z_2^{\rm 2PPR}\,m^2 + 3\lambda\,\delta Z_\lambda^{\rm 2PPR}\,\hbar\Delta$$

- In a mass–independent renormalization scheme (MS, e.g.)

$$\delta Z_\lambda^{\rm 2PPR} = \delta Z_2$$

- (Bubble–)renormalized effective mass

$$\mathcal{M}_R^2 = Z_2^{\rm 2PPR}\,m^2 + 3Z_\lambda^{\rm 2PPR}\,\lambda(\phi^2 + \hbar\Delta)$$

- Renormalized composite operator

$$\Delta_R = \langle\Phi^2\rangle_{{\rm conn},R} = Z_2\,\Delta + \delta Z_2\,\phi^2 + \delta Z_\lambda^{\rm 2PPR}\frac{m^2}{3\lambda}$$

- Divergences left in 2PPI remainder

$$\frac{\delta}{\delta\phi}\Gamma^{\rm 1PI}_{q,{\rm BR}}[\phi] = \frac{\partial}{\partial\phi}\Gamma_q^{\rm 2PPI}[\phi,\mathcal{M}_R^2] + 6\lambda\phi\,\frac{\partial}{\partial\mathcal{M}_R^2}\Gamma_q^{\rm 2PPI}[\phi,\mathcal{M}^2]$$

- mass renormalization (subgraph γ) by

$$\delta Z_2(\gamma)\,\frac{1}{2}m^2\phi^2$$

- Subgraphs $\gamma' \equiv \gamma\Big|_{\phi^2\to\Delta_R}$

$$\delta Z_\lambda^{\rm 2PPR}(\gamma')\,\frac{3}{2}\lambda(\phi^2+\Delta_R)\phi^2$$

- Using $\delta Z_\lambda^{\rm 2PPR} = \delta Z_2$

$$\delta Z_2(\gamma)\,\frac{1}{2}\mathcal{M}_R^2\phi^2$$

- Rest of 2PPI remainder renormalized by wavefunction counterterm δZ_0
- Since

$$\frac{\partial\Gamma_{q,R}^{\rm 1PI}[\phi,m^2]}{\partial m^2} = \frac{\partial\Gamma_{q,R}^{\rm 2PPI}[\phi,\mathcal{M}_R^2]}{\partial\mathcal{M}_R^2}\frac{\partial\mathcal{M}_R^2}{\partial m^2}$$

- and from the path integral one obtains

$$\frac{\partial\Gamma_R^{\rm 1PI}[\phi,m^2]}{\partial m^2} = \frac{1}{2}Z_2(\phi^2+\Delta) + \frac{\partial}{\partial m^2}\,\delta E_{\rm vac}$$

- it shows that in a mass–independent scheme $\delta E_{\rm vac} = \frac{m^4}{2}\delta\zeta$ and $\delta\zeta = \delta Z_2^{\rm 2PPR}/\lambda$

$$\Rightarrow \delta E_{\rm vac} = \frac{m^4}{12\lambda}\,\delta Z_2^{\rm 2PPR}$$

- and hence

$$\frac{\partial\Gamma^{\rm 2PPI}_{q,R}[\phi,\mathcal{M}_R^2]}{\partial\mathcal{M}_R^2} = \frac{1}{2}\left(Z_2\Delta + \delta Z_2\,\phi^2 + \delta Z_2^{\rm 2PPR}\,\frac{m^2}{3\lambda}\right) = \frac{\Delta_R}{2}$$

$\Rightarrow$ gap equation is renormalized

Concluding remarks

- 2PPI effective action
  - is a subset of 2PI CJT formalism
  - resums local (bubble–shaped) graphs
  - self–energy corrections only local
  - less complicated than 2PI but fewer graphs resummed
- Effective potential (real part) of O(N) model at finite temperature
  - first-order phase transition in Hartree (1-loop 2PPI) approximation
  - phase transition of second order in 2-loop approximation
  - physical sigma mass obtained by derivative of 1PI effective potential
- Renormalization with standard counterterms δZ_0, $\delta Z_2\, m^2 = \delta m^2$ and $\delta\lambda$ is possible
  $\longrightarrow$ in practice: counter term for effective mass $\delta\mathcal{M}^2$