Quantum dynamics of Φ^4 field theory in the two-particle point-irreducible formalism

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- J. Baacke, A.H., Phys. Rev. **D67** 105020 (2003) [hep-ph/0212312]
- J. Baacke, A.H., [hep-ph/0305220]

1. Introduction – Nonequilibrium quantum field theory

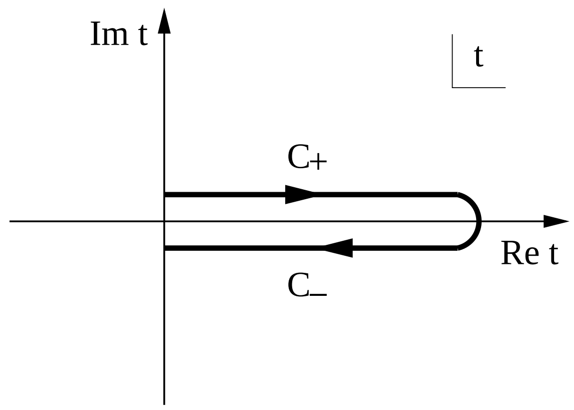
Motivation:

- Search for approximations beyond **Hartree** and the **leading order** in $1/N$ (large-N)
- Φ^4 theory as a toy model in QFT

Action for the Φ^4 model:

$$S[\Phi] = \int d^D x \left[\frac{1}{2} \partial_\mu \Phi(x) \partial^\mu \Phi(x) - \frac{1}{2} m^2 \Phi^2(x) - \frac{\lambda}{4!} \Phi^4(x) \right] \tag{1}$$

- Split Φ into classical expectation values $\phi = \langle \Phi \rangle$ and quantum parts via $\Phi(t, \mathbf{x}) = \phi(t) + \eta(t, \mathbf{x}) \rightarrow$ assume **spatially homogeneous** fields
- One needs **causal dynamics** $\rightarrow$ **closed time path** (CTP) formalism;
  J. Schwinger, J. Math. Phys. **2** 407-432(1961); L. V. Keldysch, Zh. Eksp. Teor. Fiz. **47** 1515-1527 (1964); E. Calzetta, B. L. Hu, Phys. Rev. **D37** 2878 (1988)



Approximation schemes

An **approximation** scheme for QFT **out of thermal equilibrium** has to be at least:

- non-perturbative ($\longrightarrow$ **resumming** perturbative diagrams)
- **energy conserving** ($\longrightarrow$ variational principle)
- **renormalizable** ($\longrightarrow$ compare to the equilibrium case)

A reasonable choice: the **two-particle irreducible** (2PI) effective action or Cornwall-Jackiw-Tomboulis (CJT) action! $\longrightarrow$ resummation of 2PI graphs.

J. M. Cornwall, R. Jackiw, E. Tomboulis, Phys. Rev. **D10**, 2428 (1974)

Remarks on the 2PI effective action:

The **Schwinger-Dyson** equation for the Greens function G is a **partial integro-differential equation** $\longrightarrow$ solution in general numerically involved

How to renormalize beyond the leading order large-N or Hartree approximation**?**

2. The two-particle point-irreducible (2PPI) effective action formalism

H. Verschelde, M. Coppens, Phys. Lett. **B287**, 133 (1992); M. Coppens, H. Verschelde, Z. Phys. **C58**, 319 (1993)

Idea: *local sources* $(K(x,y) \to K(x)\delta(x-y))$ for the 2PI effective action $\to$ 2PPI effective action.

$$\Gamma[\phi,\mathcal{M}^2] = \int d^D x \left[\frac{1}{2}\partial_\mu\phi(x)\,\partial^\mu\phi(x) - \frac{1}{2}\mathcal{M}^2(x)\phi^2(x) + \frac{\lambda}{12}\phi^4(x) + \frac{1}{2\lambda}\left(\mathcal{M}^2(x)-\mu^2\right)^2\right] + \Gamma^{2PPI}[\phi,\mathcal{M}^2] \quad (2)$$

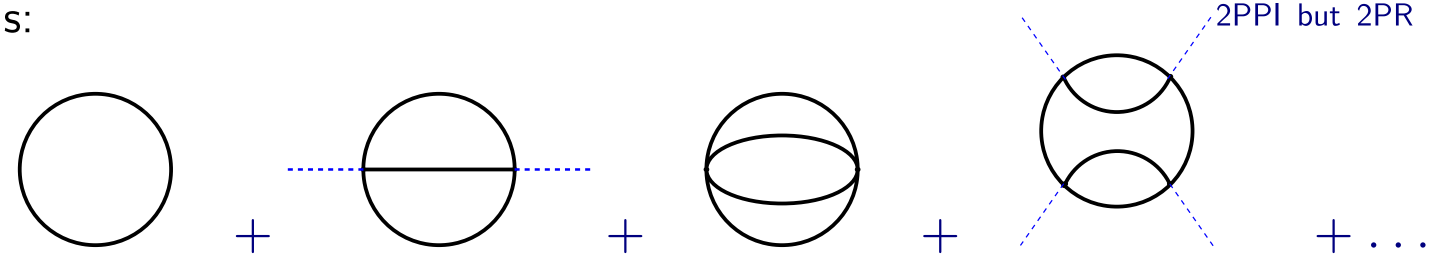
A diagram that does not decay if two lines joining at a point are cut is called 2PPI

Equations of motion:

$$\frac{\delta\Gamma[\phi,\mathcal{M}^2]}{\delta\mathcal{M}^2} = 0 \quad ; \quad \frac{\delta\Gamma[\phi,\mathcal{M}^2]}{\delta\phi} = 0 \quad (3)$$

Sum over all 2PPI diagrams:

$\Gamma^{2PPI}[\phi,\mathcal{M}^2] =$



Gap equation (Schwinger-Dyson equation) from $\frac{\delta\Gamma}{\delta\mathcal{M}^2} = 0$:

$$\mathcal{M}^2(t) = m^2 + \frac{\lambda}{2}\phi^2(t) + \frac{\lambda}{2}\Delta(t) \quad ; \quad \Delta(t) := -2\frac{\delta\Gamma^{2\mathrm{PPI}}[\phi,\mathcal{M}^2]}{\delta\mathcal{M}^2(t)} \tag{4}$$

Green's function (local equation):

$$G^{-1}(x,x') = i\left[\Box + \mathcal{M}^2(x)\right]\delta^D(x-x') \tag{5}$$

In momentum space $\longrightarrow$ **factorization** in **mode functions**

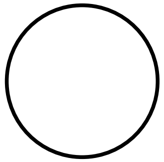
$$G_>(t,t';\mathbf{p}) = \frac{1}{2\omega_p}(2n_p+1)f(t;\mathbf{p})f^*(t';\mathbf{p}) \tag{6}$$

$$0 = \ddot{f}(t;\mathbf{p}) + \left[\mathbf{p}^2 + \mathcal{M}^2(t)\right]f(t;\mathbf{p}) \ , \tag{7}$$

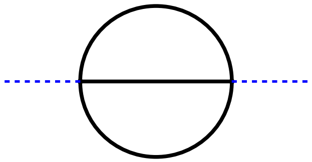
with $\omega_p = \sqrt{\mathbf{p}^2 + \mathcal{M}^2(0)}$.
The variational parameters are ϕ and $\mathcal{M}^2$ $\longrightarrow$ the Green's function G *is no variational parameter* of the theory.

Two-loop approximation

Two-loop approximation: $\Gamma^{2\text{PPI}} = \Gamma^{(1)} + \Gamma^{(2)}$ with

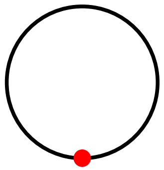


$$\Gamma^{(1)}[\mathcal{M}^2] = \frac{i}{2}\mathrm{Tr}\ln[G^{-1}(\mathcal{M}^2)] \tag{8}$$



$$\Gamma^{(2)}[\phi,\mathcal{M}^2] = i\frac{\lambda^2}{12}\int d^{D-1}x\,d^{D-1}x'\int_{\mathrm{CTP}} dt\,dt'\phi(t)\phi(t')G^3(t,t';\mathbf{x},\mathbf{x}')$$

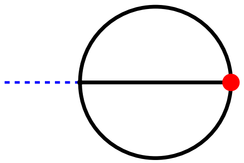
One-loop part of Δ:



$$\Delta^{(1)}(t) = -2\frac{\delta\Gamma^{(1)}[\phi,\mathcal{M}^2]}{\delta\mathcal{M}^2(t)} = \int\frac{d^{D-1}p}{(2\pi)^{D-1}}G_>(t,t;\mathbf{p}) \tag{9}$$

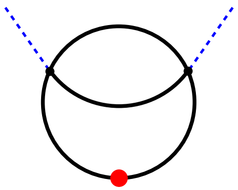
Two-loop contributions:

$$\mathcal{S}(t) = -\frac{\delta\Gamma^{(2)}[\phi,\mathcal{M}^2]}{\delta\phi(t)} \quad ; \quad \Delta^{(2)}(t) = -2\frac{\delta\Gamma^{(2)}[\phi,\mathcal{M}^2]}{\delta\mathcal{M}^2(t)} \tag{10}$$



$$\begin{aligned}\mathcal{S}(t) &= -i\frac{\lambda^2}{6}\int_0^t dt'\phi(t')\int\prod_{\ell=1}^{3}\left(\frac{d^{D-1}p_\ell}{(2\pi)^{D-1}}\right)(2\pi)^{D-1}\delta\left(\sum_{\ell=1}^{3}\mathbf{p}_\ell\right)\\ &\times\left[\prod_{\ell=1}^{3}G_>(t,t';\mathbf{p}_\ell)-\prod_{\ell=1}^{3}G_>(t',t;\mathbf{p}_\ell)\right]\end{aligned} \tag{11}$$

and



$$\begin{aligned}\Delta^{(2)}(t) &= -\lambda^2\int_0^t dt'\phi(t')\int_0^{t'} dt''\phi(t'')\int\prod_{\ell=1}^{3}\left(\frac{d^{D-1}p_\ell}{(2\pi)^{D-1}}\right)(2\pi)^{D-1}\delta\left(\sum_{\ell=1}^{3}\mathbf{p}_\ell\right)\\ &\times\left[G_>(t,t';\mathbf{p}_3)-G_>(t',t;\mathbf{p}_3)\right]\\ &\times\left[G_>(t',t'';\mathbf{p}_1)G_>(t',t'';\mathbf{p}_2)G_>(t,t'';\mathbf{p}_3)\right.\\ &\quad\left.-G_>(t'',t';\mathbf{p}_1)G_>(t'',t';\mathbf{p}_2)G_>(t'',t;\mathbf{p}_3)\right]\ .\end{aligned} \tag{12}$$

Equations of motion:

$$0 = \ddot{\phi}(t) + \mathcal{M}^2(t)\phi(t) - \frac{\lambda}{3}\phi^3(t) + \mathcal{S}(t) \tag{13}$$

$$\mathcal{M}^2(t) = m^2 + \frac{\lambda}{2}\left(\phi^2(t) + \Delta^{(1)}(t) + \Delta^{(2)}(t)\right) . \tag{14}$$

Remarks:

- **local mass term** $\mathcal{M}^2 \longrightarrow$ the **selfenergy** is local

- G can be factorized in mode functions $\longrightarrow$ the equations of motion are **ordinary differential equations**

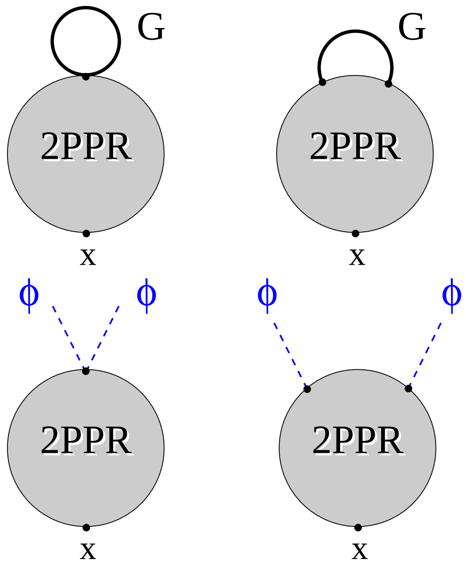
- there are memory effects ($\longrightarrow$ time integrations over the past of ϕ and G)
- there is some scattering of the quanta ($\longrightarrow$ two momentum integrations)
- **less powerful in resumming diagrams** (compared to the 2PI effective action)

- the problem of renormalization in $3+1$ dimensions has been solved in equilibrium QFT [explicit two-loop calculation at finite temperature: G. Smet et al., Phys. Rev. **D65** 045015 (2002)]

3. Renormalization in the 2PPI scheme

H. Verschelde, *Summation and renormalization of bubble graphs to all orders*, Phys. Lett. **B497** 165-171 (2001)

In $(3+1)$ dimensions:



$$\delta\mathcal{L} = \frac{\delta Z}{2}(\partial_\mu \Phi)^2 + \delta Z_2 \frac{m^2\Phi^2}{2} + \delta Z_\lambda \frac{\lambda \Phi^4}{4!} \qquad (15)$$

In a **mass independent subtraction** scheme $\delta Z_\lambda^{\mathrm{2PPR}} = \delta Z_2$;
$\Rightarrow$ **renormalized** effective mass:

$$\mathcal{M}_{\mathrm{R}}^2 = m^2(1 + \delta Z_2^{\mathrm{2PPR}}) + \frac{\lambda}{2}(1 + \delta Z_2)(\phi^2 + \Delta) \qquad (16)$$

In $(1+1)$ dimensions: Remove the primitive divergence with

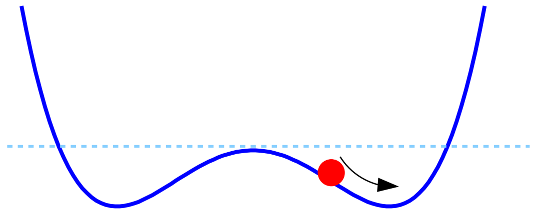
$$m^2 \to m^2 + \delta m^2 \quad \Rightarrow \quad \delta m^2 + \frac{\lambda}{2}\Delta^{(1)}(t) = \frac{\lambda}{8\pi}\ln\frac{|m^2|}{m_0^2} + \Delta_{\mathrm{fin}}^{(1)}(t) \qquad (17)$$

$$\Delta_{\mathrm{fin}}^{(1)}(t) = \int \frac{dp}{2\pi}\left[G_>(t,t;p) - \frac{1}{2\omega_p}\right] = \int \frac{dp}{2\pi 2\omega_p}\left[(2n_p+1)\,|f(t,p)|^2 - 1\right] \qquad (18)$$

Initial conditions

- Gaussian initial density matrix $\longrightarrow$ **Fock space**
  - nonzero value $\phi(0)$ for the mean field
  - $f(0,p) = 1,\ \dot{f}(0,p) = -i\omega_p$
  - initial ensemble n_p for the quanta $\longrightarrow$ Bogoliubov rotated Fock space; we will take a Bose-Einstein distribution with

$$n_p = \frac{1}{e^{\omega_p/T_0} - 1} \tag{19}$$



- at $t = 0$: quanta like free particles with a mass $m_0^2 = \mathcal{M}^2(0)$; m_0^2 has to be determined from the **self consistent** equation

$$m_0^2 = m^2 + \frac{\lambda}{2}\phi^2(0) + \frac{\lambda}{8\pi}\ln\frac{|m^2|}{m_0^2} + \frac{\lambda}{2}\int\frac{dp}{2\pi}\frac{n_p}{\omega_p} \tag{20}$$

$$\omega_p = \sqrt{p^2 + m_0^2} \tag{21}$$

4. Numerical simulations in the two-loop approximation and $(1+1)$ dimensions

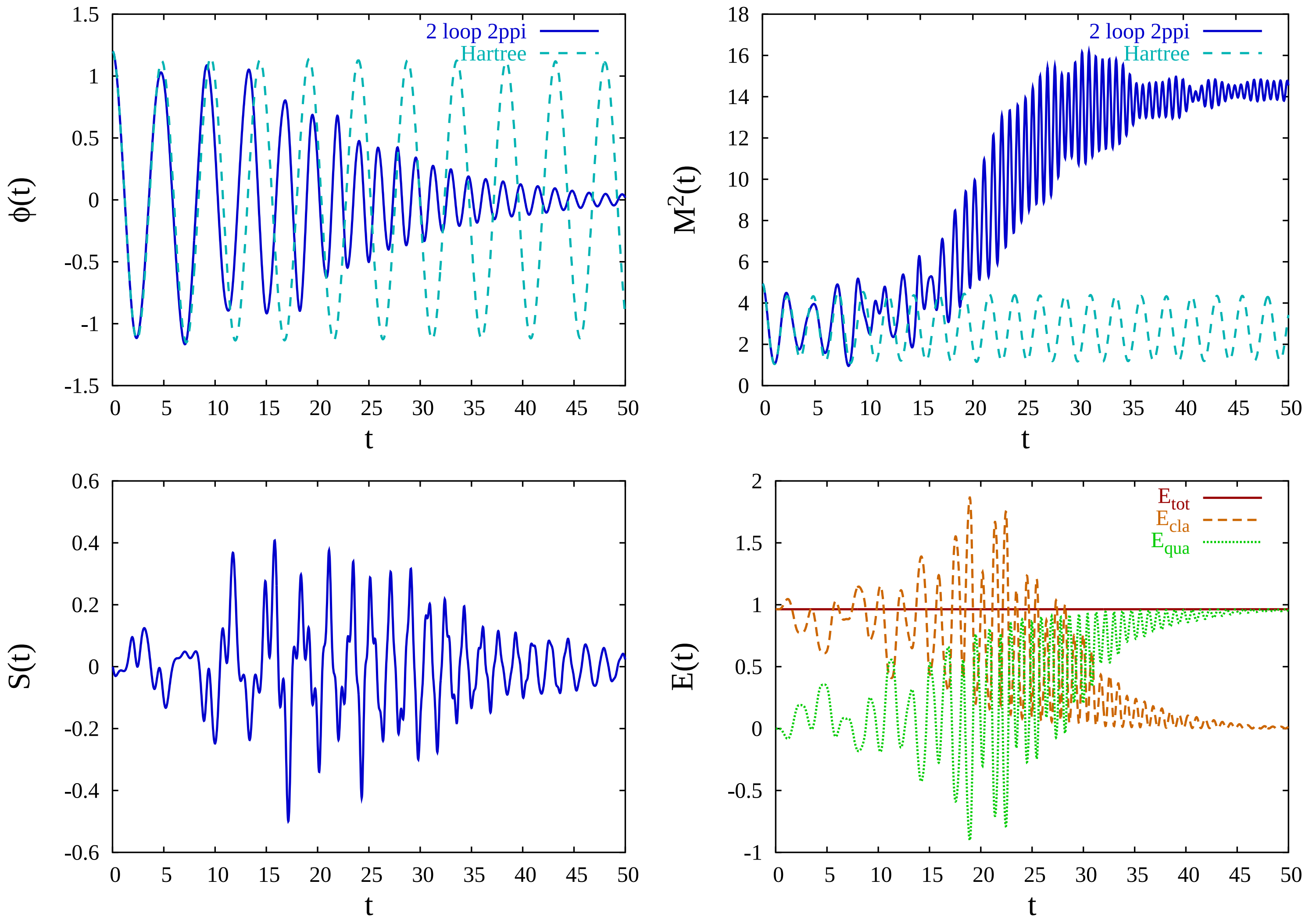
Numerical implementation:

- Runge-Kutta algorithm for the integration of the differential equations with stepsize $\Delta t = 0.001$—0.005
- Momentum discretization with $p_{\max} = 15$—20 and $dp \leq 0.05$
- Energy conserved to at least five significant digits
- Wronskians constant with a relative precision of 10^{-8}

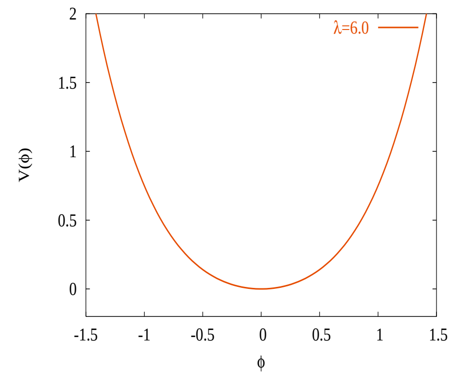
Results compared to:

1. the **Hartree** approximation (**one-loop 2PPI**)
2. For the double-well potential: the 2PI-next-to-leading-order-1/N (2PI-1/N) and the Bare Vertex Approximation (BVA); results from F. Cooper, J. Dawson and B. Mihaila, Phys. Rev. **D67** 056003 (2003)

Symmetric potential – dissipative dynamics



classical potential $V(\phi)$:

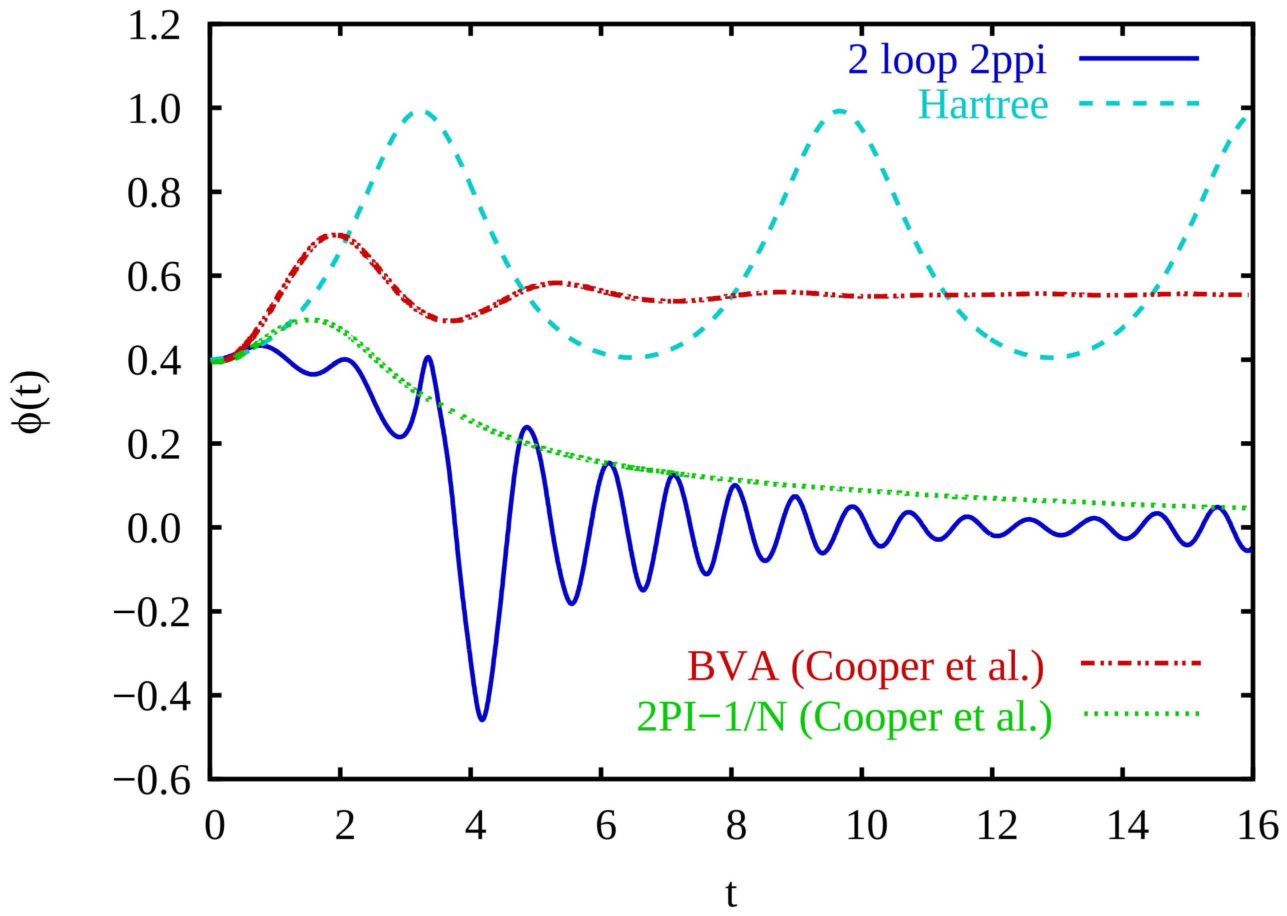


parameters:

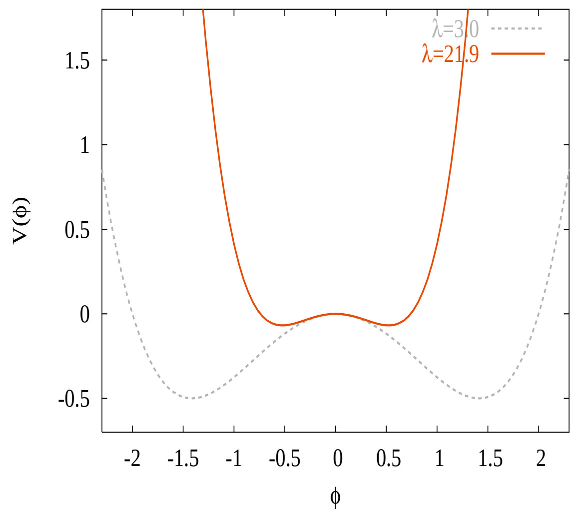
$\phi(0) = 1.2 \; T_0 = 0$

$\lambda = 6, \; m^2 = +1$

Double well potential: classical field $\phi(t)$ – larger coupling



classical potential $V(\phi)$:

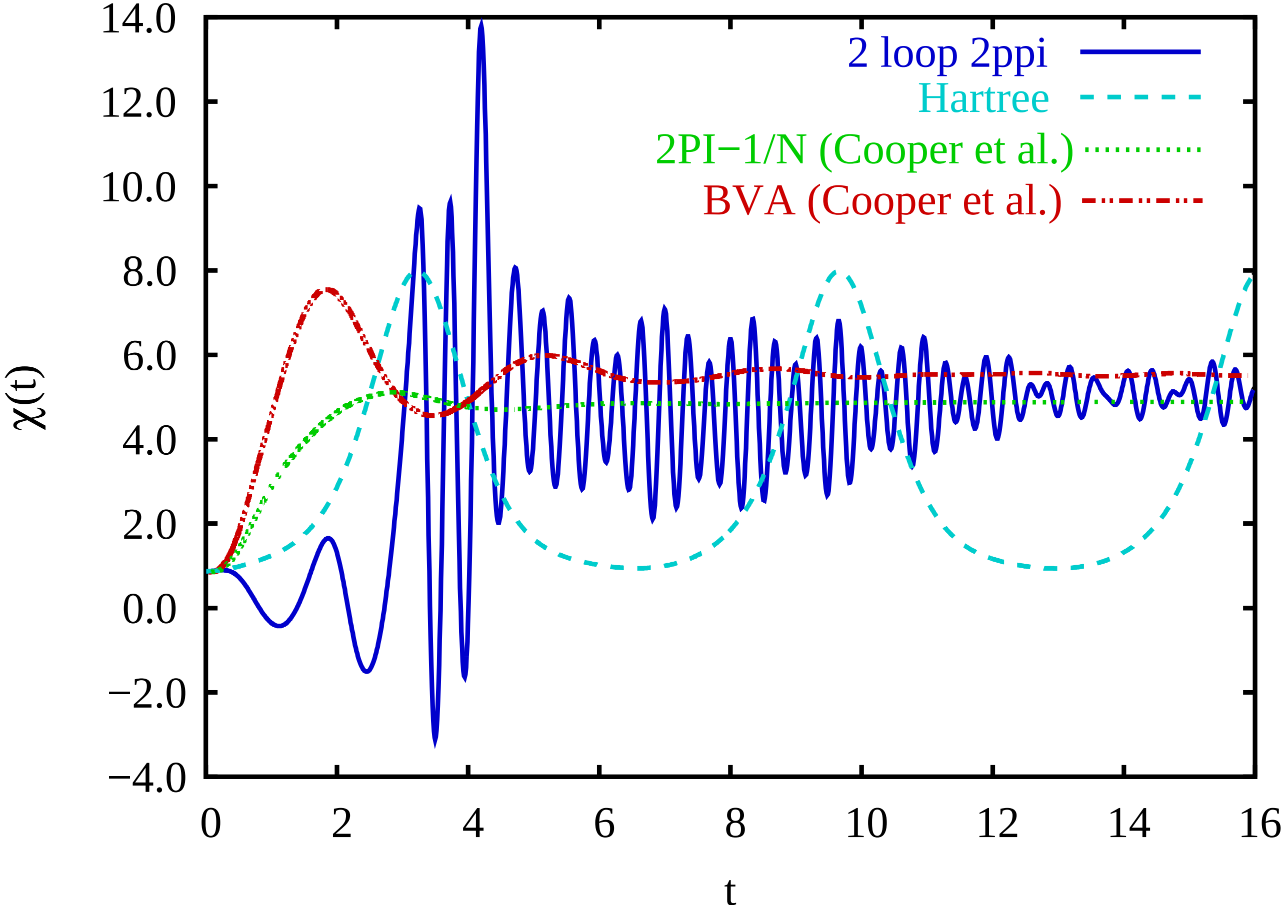


parameters:

$\phi(0) = 0.4$, $T_0 = 0.1$

$\lambda = 21.9$, $m^2 = -1$

Composite field $\chi(t)$ – larger coupling



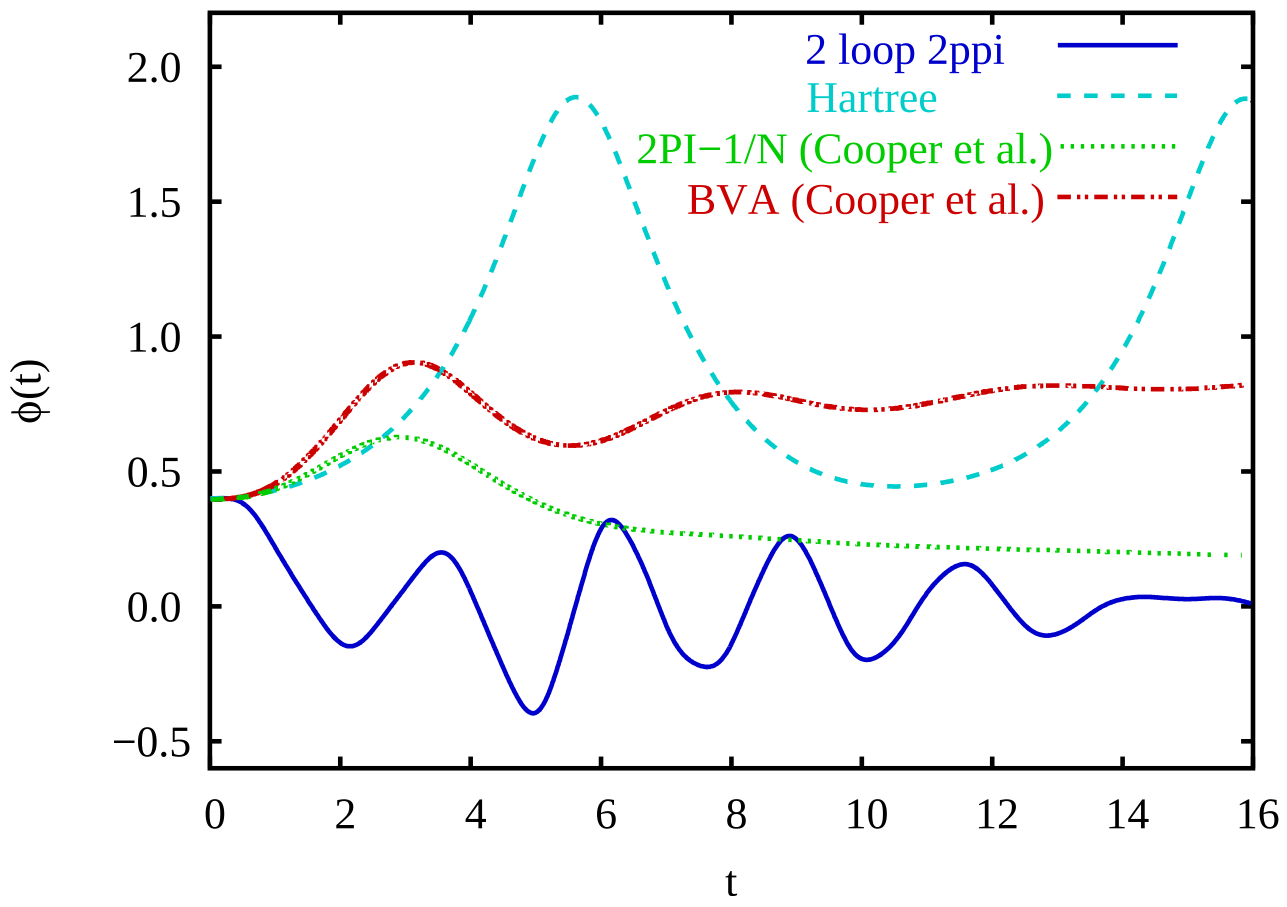
Define:

$$\chi(t) := m^2 + \frac{\lambda}{2}\phi^2(t) + \frac{\lambda}{2}\int \frac{dp}{2\pi} G(t,t;p) \tag{22}$$

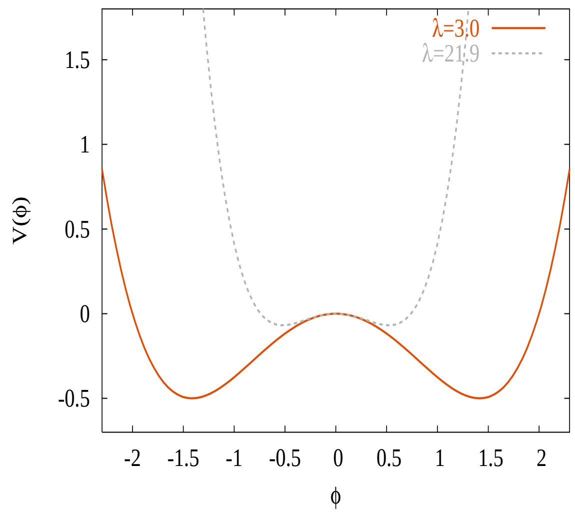
$$= m^2 + \frac{\lambda}{8\pi}\ln\frac{m_0^2}{|m^2|} + \frac{\lambda}{2}\left[\phi^2(t) + \Delta^{(1)}_{\text{fin}}(t)\right] \tag{23}$$

$$= \mathcal{M}^2(t) - \frac{\lambda}{2}\Delta^{(2)}(t)$$

Classical field $\phi(t)$ – smaller coupling



classical potential $V(\phi)$:

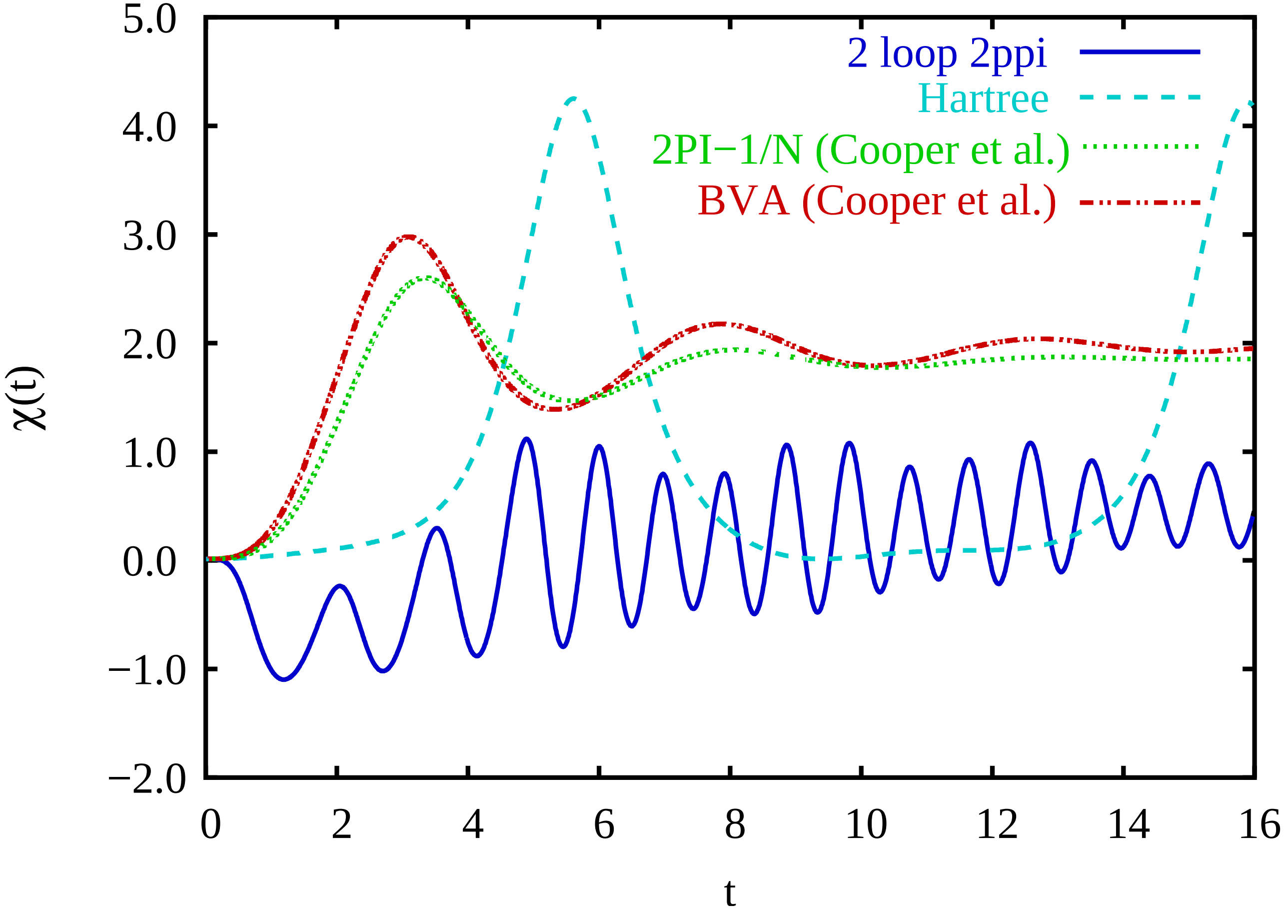


parameters:

$\phi(0) = 0.4,\ T_0 = 0.1$

$\lambda = 3.0,\ m^2 = -1$

Composite field $\chi(t)$ – smaller coupling



5. Summary

- We have presented the **generalization** of the **2PPI** scheme to **nonequilibrium** quantum field theory
- Compared to other methods computationally less involved framework for simulations beyond the large-N or Hartree approximation
- Numerical simulations in $1+1$ dimensions in the two-loop approximation displayed:
  - the **absence of symmetry breaking** as expected from the exact theory
  - **normal dissipative dynamics** for the mean field ϕ
  - this both cures (some) obvious deficits of the Hartree (and large-N) approximation