

Quantum dynamics of Φ^4 field theory in the two-particle point-irreducible formalism

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- J. Baacke, [A.H.](#), Phys. Rev. **D67** 105020 (2003) [hep-ph/0212312]
- J. Baacke, [A.H.](#), [hep-ph/0305220]

1. Introduction – Nonequilibrium quantum field theory

Motivation:

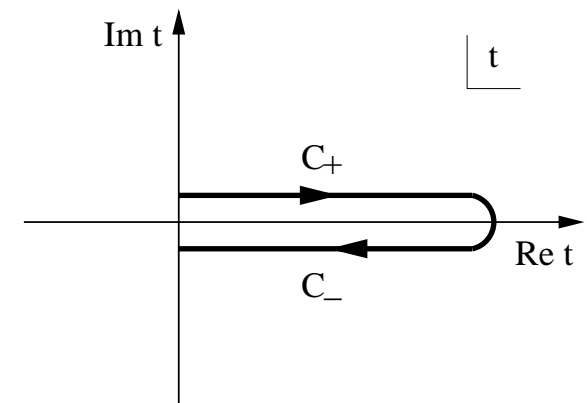
- Search for approximations beyond **Hartree** and the **leading order** in $1/N$ (large- N)
- Φ^4 theory as a toy model in QFT

Action for the Φ^4 model:

$$S[\Phi] = \int d^D x \left[\frac{1}{2} \partial_\mu \Phi(x) \partial^\mu \Phi(x) - \frac{1}{2} m^2 \Phi^2(x) - \frac{\lambda}{4!} \Phi^4(x) \right] \quad (1)$$

- Split Φ into classical expectation values $\phi = \langle \Phi \rangle$ and quantum parts via $\Phi(t, \mathbf{x}) = \phi(t) + \eta(t, \mathbf{x}) \rightarrow$ assume **spatially homogeneous** fields
- One needs **causal dynamics** \rightarrow **closed time path** (CTP) formalism;

J. Schwinger, J. Math. Phys. **2** 407-432(1961); L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47** 1515-1527 (1964); E. Calzetta, B. L. Hu, Phys. Rev. **D37** 2878 (1988)



Approximation schemes

An **approximation** scheme for QFT **out of thermal equilibrium** has to be at least:

- non-perturbative (\rightarrow **resumming** perturbative diagrams)
- **energy conserving** (\rightarrow variational principle)
- **renormalizable** (\rightarrow compare to the equilibrium case)

A reasonable choice: the **two-particle irreducible** (2PI) effective action or Cornwall-Jackiw-Tomboulis (CJT) action! \rightarrow resummation of 2PI graphs.

J. M. Cornwall, R. Jackiw, E. Tomboulis, Phys. Rev. **D10**, 2428 (1974)

Remarks on the 2PI effective action:



The **Schwinger-Dyson** equation for the Greens function G is a **partial integro-differential equation** \rightarrow solution in general numerically involved



How to renormalize beyond the leading order large- N or Hartree approximation?

2. The two-particle point-irreducible (2PPI) effective action formalism

H. Verschelde, M. Coppins, Phys. Lett. **B287**, 133 (1992); M. Coppins, H. Verschelde, Z. Phys. **C58**, 319 (1993)

Idea: *local sources* ($K(x, y) \rightarrow K(x)\delta(x - y)$) for the 2PI effective action \rightarrow 2PPI effective action.

$$\Gamma[\phi, \mathcal{M}^2] = \int d^D x \left[\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{2} \mathcal{M}^2(x) \phi^2(x) + \frac{\lambda}{12} \phi^4(x) + \frac{1}{2\lambda} \left(\mathcal{M}^2(x) - \mu^2 \right)^2 \right] + \Gamma^{2\text{PPI}}[\phi, \mathcal{M}^2] \quad (2)$$

A diagram that does not decay if two lines joining at a point are cut is called 2PPI

Equations of motion:

$$\frac{\delta \Gamma[\phi, \mathcal{M}^2]}{\delta \mathcal{M}^2} = 0 \quad ; \quad \frac{\delta \Gamma[\phi, \mathcal{M}^2]}{\delta \phi} = 0 \quad (3)$$

Sum over all 2PPI diagrams:

$$\Gamma^{2\text{PPI}}[\phi, \mathcal{M}^2] = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \dots$$

Gap equation (Schwinger-Dyson equation) from $\frac{\delta\Gamma}{\delta\mathcal{M}^2} = 0$:

$$\mathcal{M}^2(t) = m^2 + \frac{\lambda}{2}\phi^2(t) + \frac{\lambda}{2}\Delta(t) \quad ; \quad \Delta(t) := -2\frac{\delta\Gamma^{2\text{PPI}}[\phi, \mathcal{M}^2]}{\delta\mathcal{M}^2(t)} \quad (4)$$

Green's function (local equation):

$$G^{-1}(x, x') = i \left[\square + \mathcal{M}^2(x) \right] \delta^D(x - x') \quad (5)$$

In momentum space \rightarrow **factorization** in **mode functions**

$$G_{>}(t, t'; \mathbf{p}) = \frac{1}{2\omega_p} (2n_p + 1) f(t; \mathbf{p}) f^*(t'; \mathbf{p}) \quad (6)$$

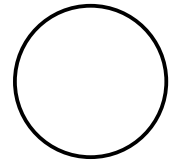
$$0 = \ddot{f}(t; \mathbf{p}) + \left[\mathbf{p}^2 + \mathcal{M}^2(t) \right] f(t; \mathbf{p}) , \quad (7)$$

with $\omega_p = \sqrt{\mathbf{p}^2 + \mathcal{M}^2(0)}$.

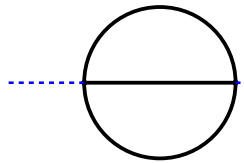
The variational parameters are ϕ and \mathcal{M}^2 \rightarrow the Green's function G is no variational parameter of the theory.

Two-loop approximation

Two-loop approximation: $\Gamma^{2\text{PPI}} = \Gamma^{(1)} + \Gamma^{(2)}$ with

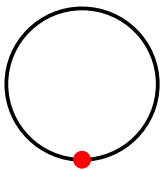


$$\Gamma^{(1)}[\mathcal{M}^2] = \frac{i}{2} \text{Tr} \ln[G^{-1}(\mathcal{M}^2)] \quad (8)$$



$$\Gamma^{(2)}[\phi, \mathcal{M}^2] = i \frac{\lambda^2}{12} \int d^{D-1}x d^{D-1}x' \int_{\text{CTP}} dt dt' \phi(t) \phi(t') G^3(t, t'; \mathbf{x}, \mathbf{x}')$$

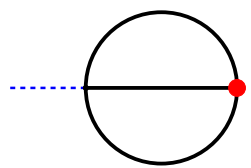
One-loop part of Δ :



$$\Delta^{(1)}(t) = -2 \frac{\delta \Gamma^{(1)}[\phi, \mathcal{M}^2]}{\delta \mathcal{M}^2(t)} = \int \frac{d^{D-1}p}{(2\pi)^{D-1}} G_{>}(t, t; \mathbf{p}) \quad (9)$$

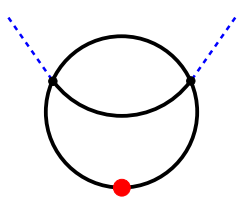
Two-loop contributions:

$$\mathcal{S}(t) = -\frac{\delta\Gamma^{(2)}[\phi, \mathcal{M}^2]}{\delta\phi(t)} \quad ; \quad \Delta^{(2)}(t) = -2\frac{\delta\Gamma^{(2)}[\phi, \mathcal{M}^2]}{\delta\mathcal{M}^2(t)} \quad (10)$$



$$\mathcal{S}(t) = -i\frac{\lambda^2}{6} \int_0^t dt' \phi(t') \int \prod_{\ell=1}^3 \left(\frac{d^{D-1}p_\ell}{(2\pi)^{D-1}} \right) (2\pi)^{D-1} \delta \left(\sum_{\ell=1}^3 \mathbf{p}_\ell \right) \\ \times \left[\prod_{\ell=1}^3 G_{>}(t, t'; \mathbf{p}_\ell) - \prod_{\ell=1}^3 G_{>}(t', t; \mathbf{p}_\ell) \right] \quad (11)$$

and



$$\Delta^{(2)}(t) = -\lambda^2 \int_0^t dt' \phi(t') \int_0^{t'} dt'' \phi(t'') \int \prod_{\ell=1}^3 \left(\frac{d^{D-1}p_\ell}{(2\pi)^{D-1}} \right) (2\pi)^{D-1} \delta \left(\sum_{\ell=1}^3 \mathbf{p}_\ell \right) \\ \times [G_{>}(t, t'; \mathbf{p}_3) - G_{>}(t', t; \mathbf{p}_3)] \\ \times [G_{>}(t', t''; \mathbf{p}_1)G_{>}(t', t''; \mathbf{p}_2)G_{>}(t, t''; \mathbf{p}_3) \\ - G_{>}(t'', t'; \mathbf{p}_1)G_{>}(t'', t'; \mathbf{p}_2)G_{>}(t'', t; \mathbf{p}_3)] \quad (12)$$

Equations of motion:

$$0 = \ddot{\phi}(t) + \mathcal{M}^2(t)\phi(t) - \frac{\lambda}{3}\phi^3(t) + \mathcal{S}(t) \quad (13)$$

$$\mathcal{M}^2(t) = m^2 + \frac{\lambda}{2} \left(\phi^2(t) + \Delta^{(1)}(t) + \Delta^{(2)}(t) \right) . \quad (14)$$

Remarks:

- **local mass** term $\mathcal{M}^2 \rightarrow$ the **selfenergy** is local
- G can be factorized in mode functions \rightarrow the equations of motion are **ordinary differential equations**
- there are memory effects (\rightarrow time integrations over the past of ϕ and G)
- there is some scattering of the quanta (\rightarrow two momentum integrations)
- **less powerful in resumming diagrams** (compared to the 2PI effective action)
- the problem of renormalization in $3 + 1$ dimensions has been solved in equilibrium QFT [explicit two-loop calculation at finite temperature: [G. Smet et al., Phys. Rev. D65 045015 \(2002\)](#)]



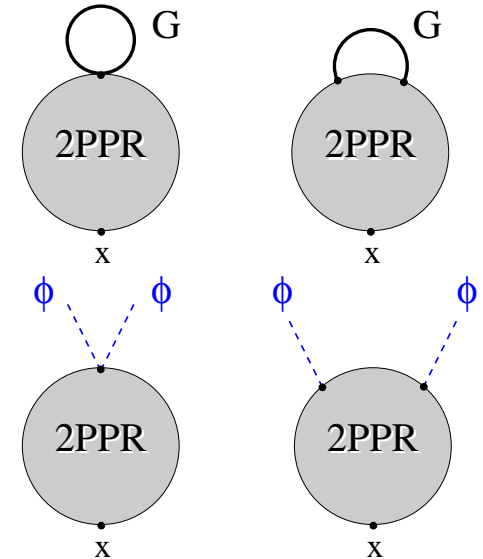
3. Renormalization in the 2PPR scheme

H. Verschelde, *Summation and renormalization of bubble graphs to all orders*, Phys. Lett. **B497** 165-171 (2001)
In (3 + 1) dimensions:

$$\delta\mathcal{L} = \frac{\delta Z}{2}(\partial_\mu\Phi)^2 + \delta Z_2\frac{m^2\Phi^2}{2} + \delta Z_\lambda\frac{\lambda\Phi^4}{4!} \quad (15)$$

In a **mass independent subtraction** scheme $\delta Z_\lambda^{2PPR} = \delta Z_2$;
 \Rightarrow **renormalized** effective mass:

$$\mathcal{M}_R^2 = m^2(1 + \delta Z_2^{2PPR}) + \frac{\lambda}{2}(1 + \delta Z_2)(\phi^2 + \Delta) \quad (16)$$



In (1 + 1) dimensions: Remove the primitive divergence with

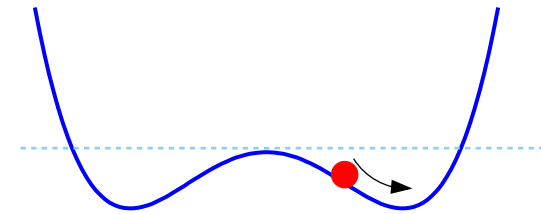
$$m^2 \rightarrow m^2 + \delta m^2 \quad \Rightarrow \quad \delta m^2 + \frac{\lambda}{2}\Delta^{(1)}(t) = \frac{\lambda}{8\pi} \ln \frac{|m^2|}{m_0^2} + \Delta_{\text{fin}}^{(1)}(t) \quad (17)$$

$$\Delta_{\text{fin}}^{(1)}(t) = \int \frac{dp}{2\pi} \left[G_{>}(t, t; p) - \frac{1}{2\omega_p} \right] = \int \frac{dp}{2\pi 2\omega_p} \left[(2n_p + 1) |f(t, p)|^2 - 1 \right] \quad (18)$$

Initial conditions

- Gaussian initial density matrix \rightarrow **Fock space**
 - nonzero value $\phi(0)$ for the mean field
 - $f(0, p) = 1$, $\dot{f}(0, p) = -i\omega_p$
 - initial ensemble n_p for the quanta \rightarrow Bogoliubov rotated Fock space; we will take a Bose-Einstein distribution with

$$n_p = \frac{1}{e^{\omega_p/T_0} - 1} \quad (19)$$



Double well potential

- at $t = 0$: quanta like free particles with a mass $m_0^2 = \mathcal{M}^2(0)$; m_0^2 has to be determined from the **self consistent** equation

$$m_0^2 = m^2 + \frac{\lambda}{2}\phi^2(0) + \frac{\lambda}{8\pi} \ln \frac{|m^2|}{m_0^2} + \frac{\lambda}{2} \int \frac{dp n_p}{2\pi \omega_p} \quad (20)$$

$$\omega_p = \sqrt{p^2 + m_0^2} \quad (21)$$

4. Numerical simulations in the two-loop approximation and $(1 + 1)$ dimensions

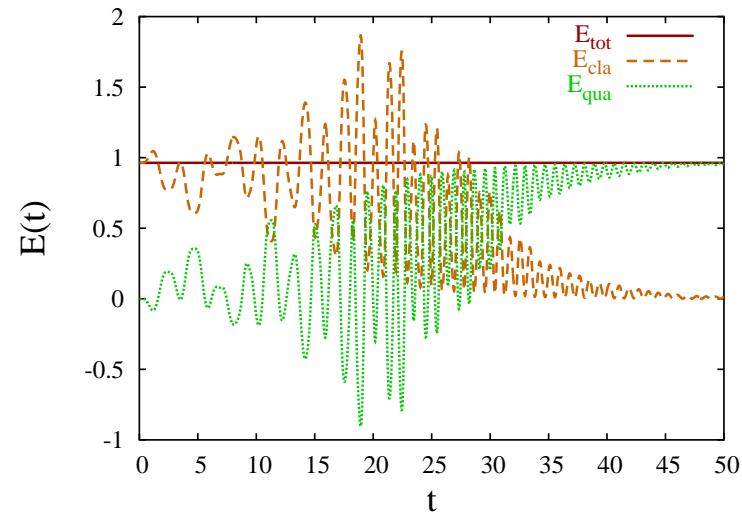
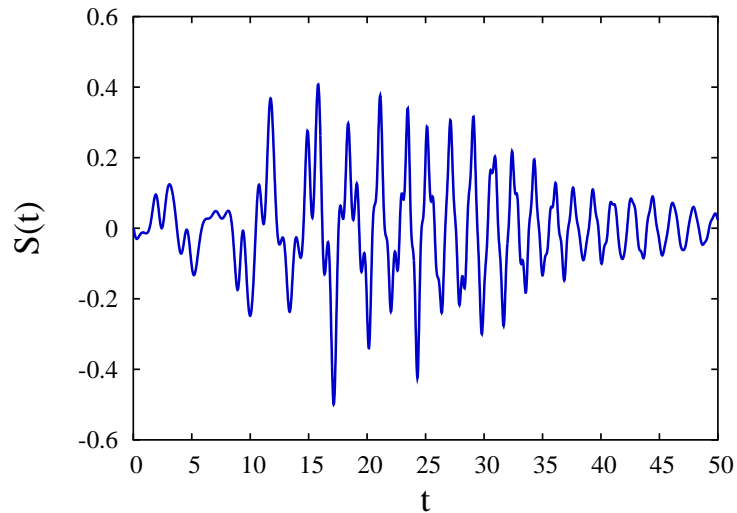
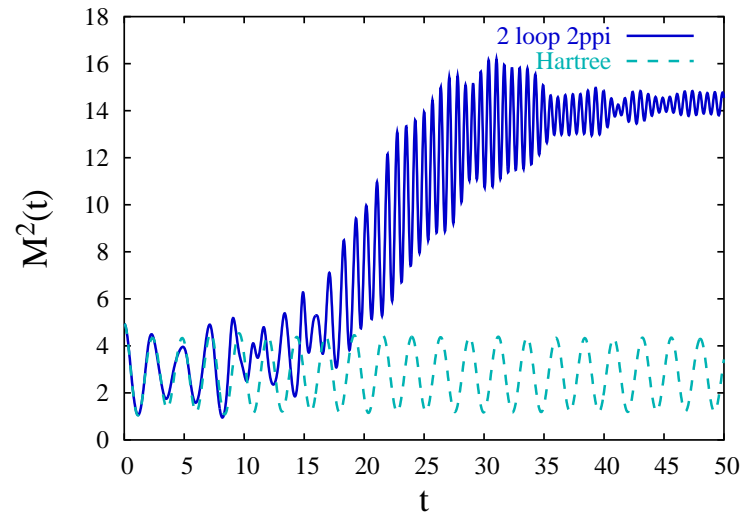
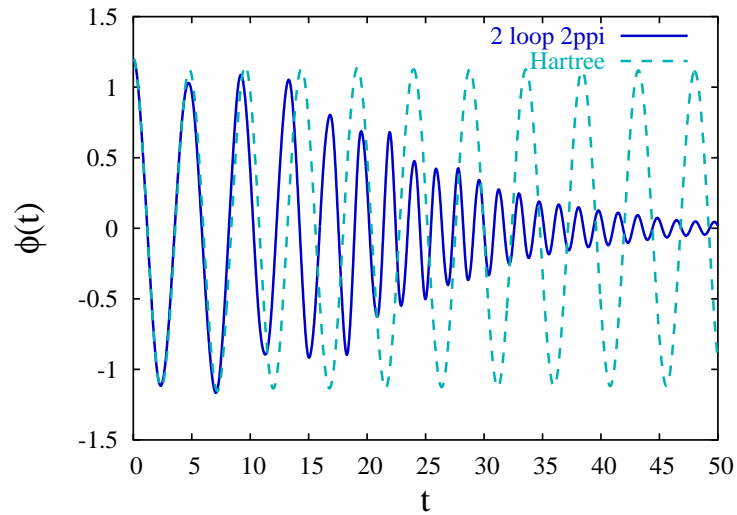
Numerical implementation:

- Runge-Kutta algorithm for the integration of the differential equations with stepsize $\Delta t = 0.001$ — 0.005
- Momentum discretization with $p_{\max} = 15$ — 20 and $dp \leq 0.05$
- Energy conserved to at least five significant digits
- Wronskians constant with a relative precision of 10^{-8}

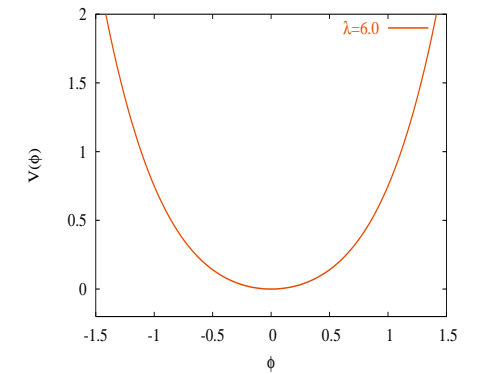
Results compared to:

1. the **Hartree** approximation (**one-loop 2PPI**)
2. For the double-well potential: the 2PI-next-to-leading-order-1/N (2PI-1/N) and the Bare Vertex Approximation (BVA); results from [F. Cooper, J. Dawson and B. Mihaila, Phys. Rev. **D67** 056003 \(2003\)](#)

Symmetric potential – dissipative dynamics



classical potential
 $V(\phi)$:

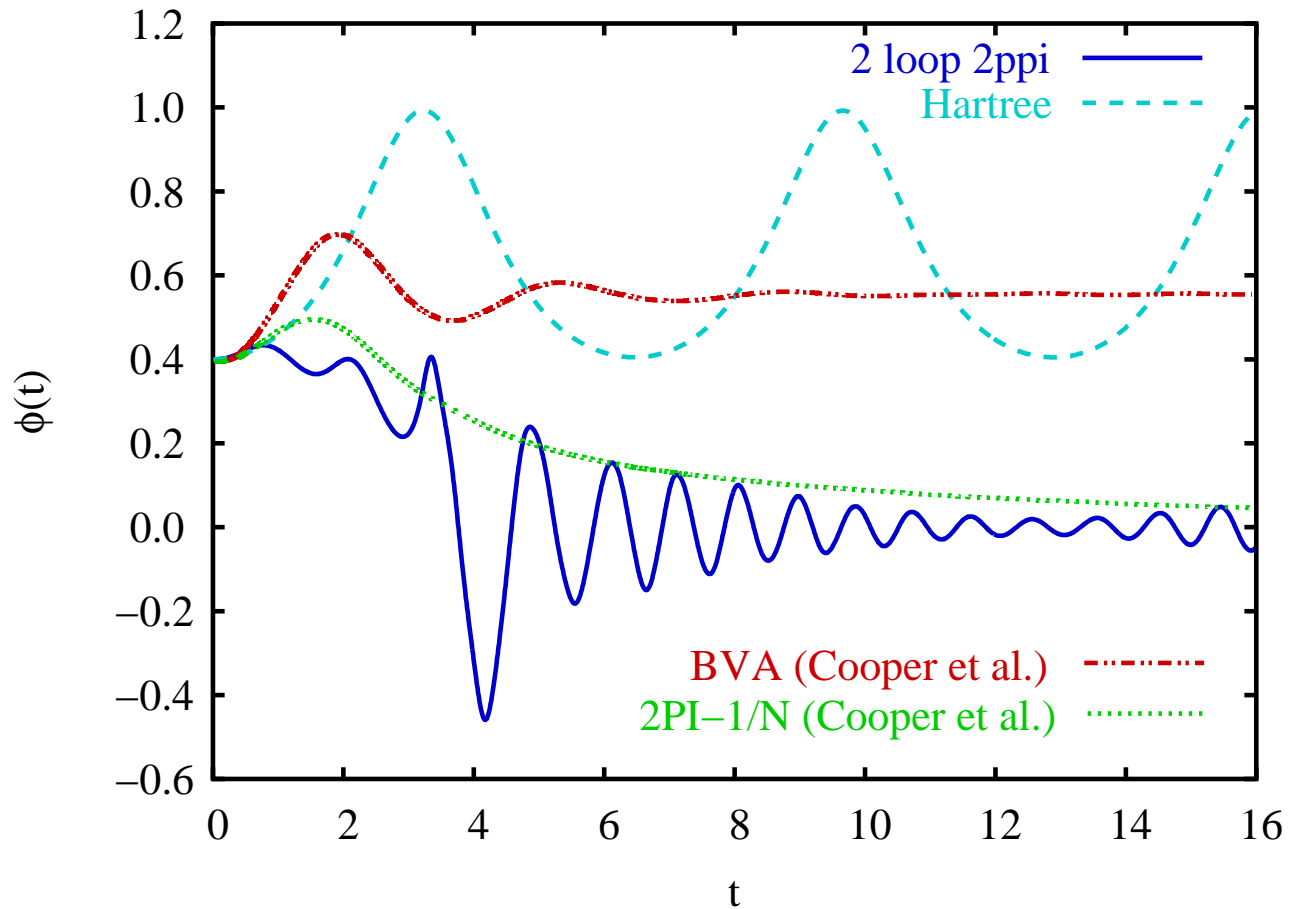


parameters:

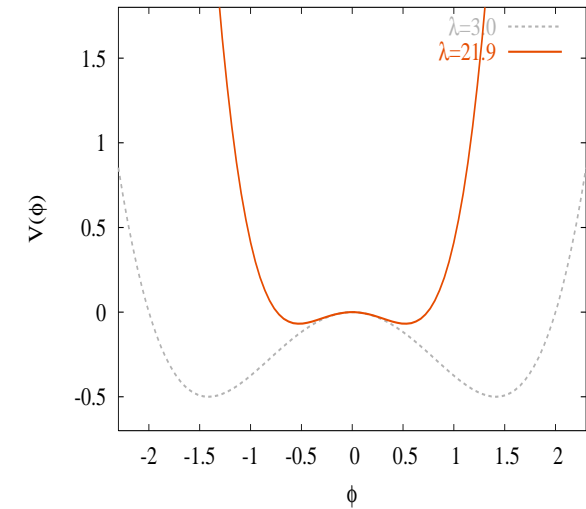
$$\phi(0) = 1.2 \quad T_0 = 0$$

$$\lambda = 6, \quad m^2 = +1$$

Double well potential: classical field $\phi(t)$ – larger coupling



classical potential $V(\phi)$:

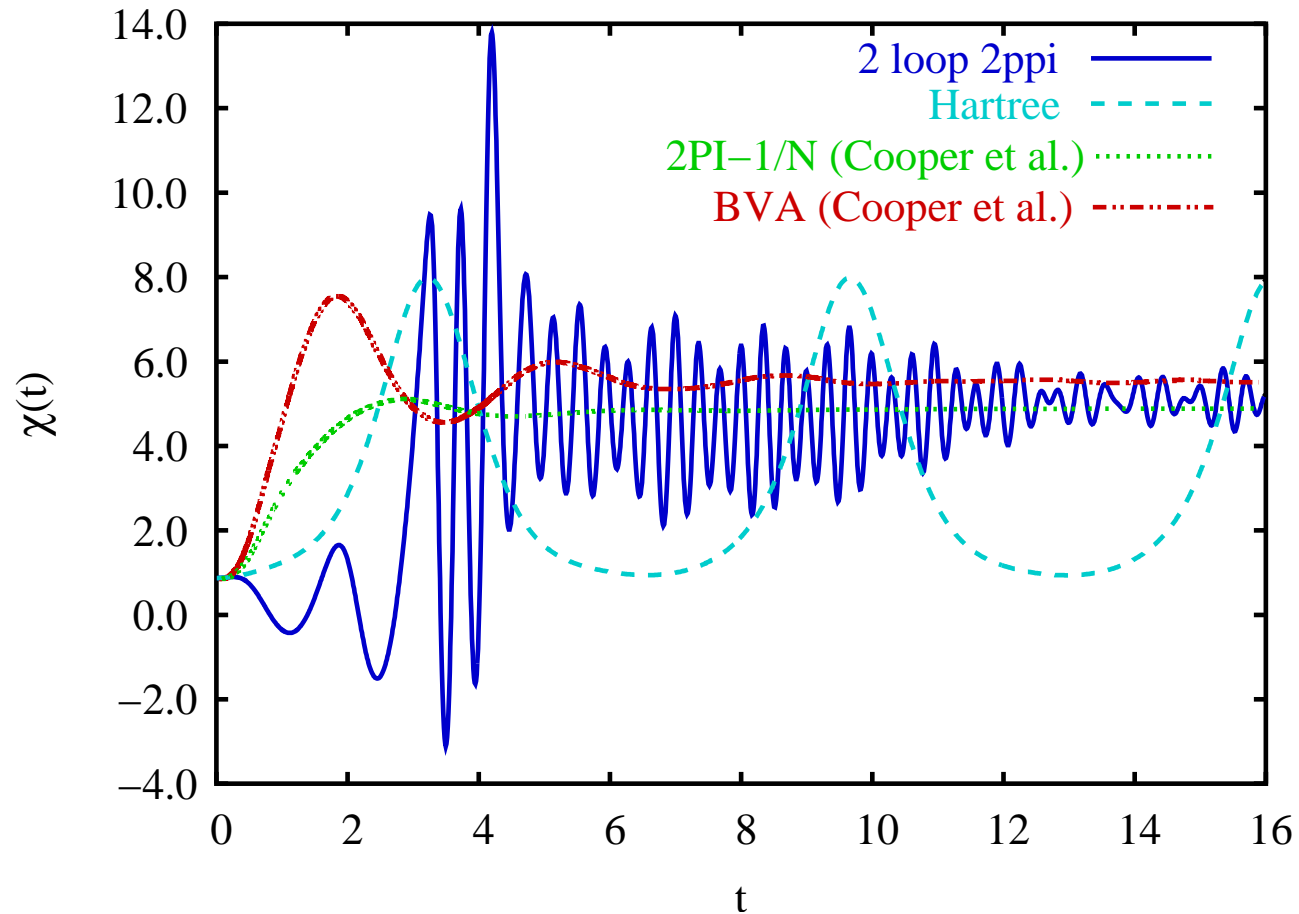


parameters:

$$\phi(0) = 0.4, T_0 = 0.1$$

$$\lambda = 21.9, m^2 = -1$$

Composite field $\chi(t)$ – larger coupling



Define:

$$\chi(t) := m^2 + \frac{\lambda}{2} \phi^2(t) \quad (22)$$

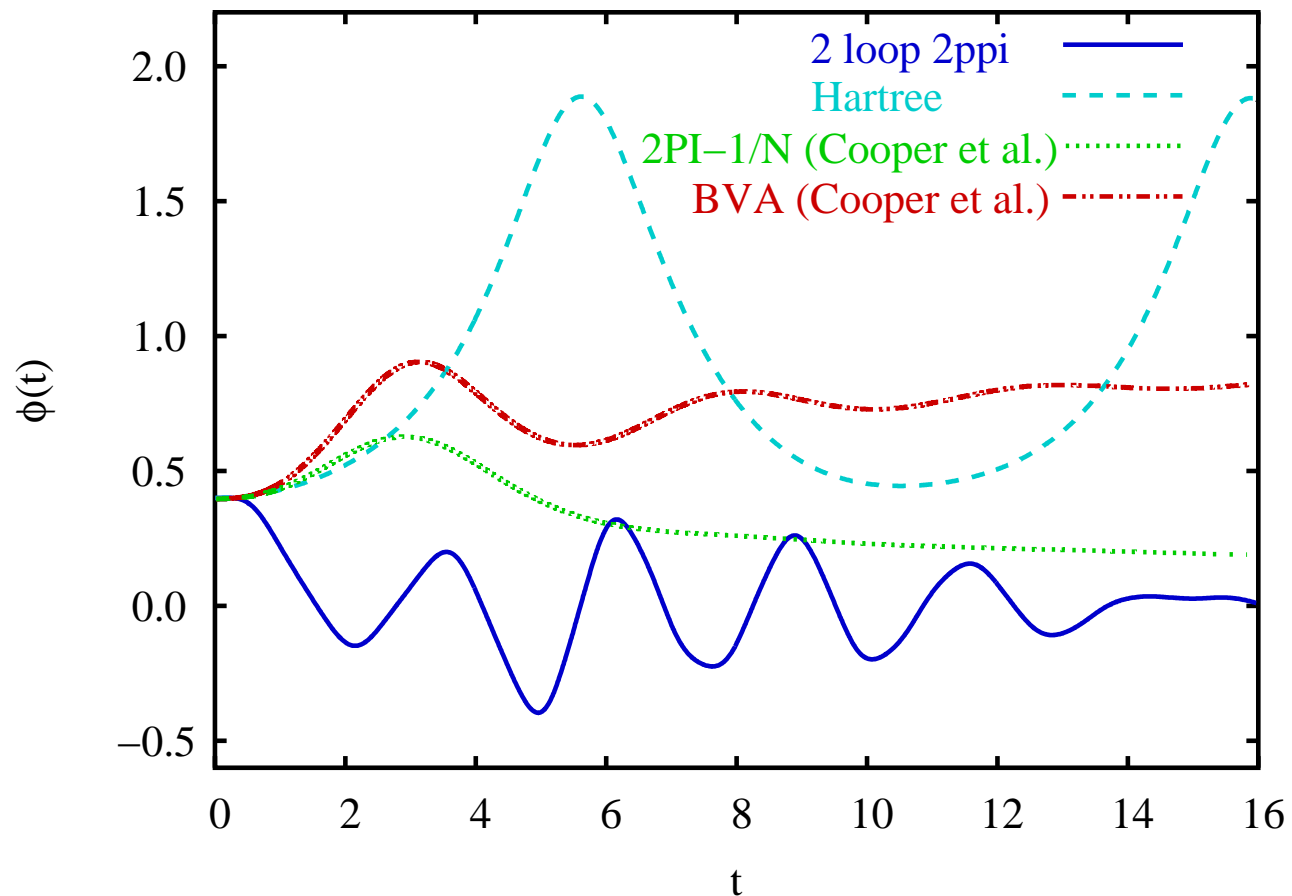
$$+ \frac{\lambda}{2} \int \frac{dp}{2\pi} G(t, t; p)$$

$$= m^2 + \frac{\lambda}{8\pi} \ln \frac{m_0^2}{|m^2|} \quad (23)$$

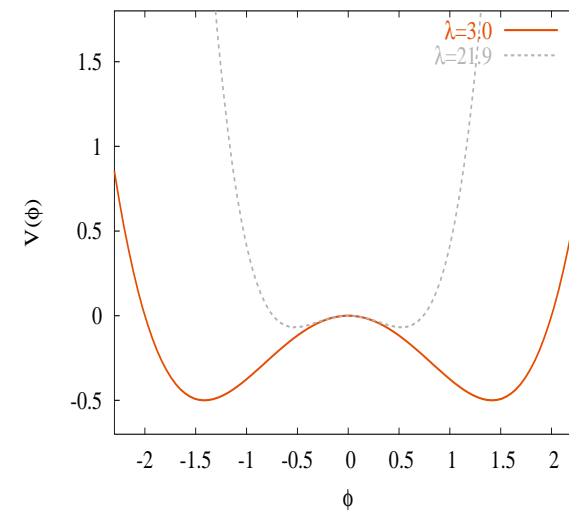
$$+ \frac{\lambda}{2} \left[\phi^2(t) + \Delta_{\text{fin}}^{(1)}(t) \right]$$

$$= \mathcal{M}^2(t) - \frac{\lambda}{2} \Delta^{(2)}(t)$$

Classical field $\phi(t)$ – smaller coupling



classical potential $V(\phi)$:

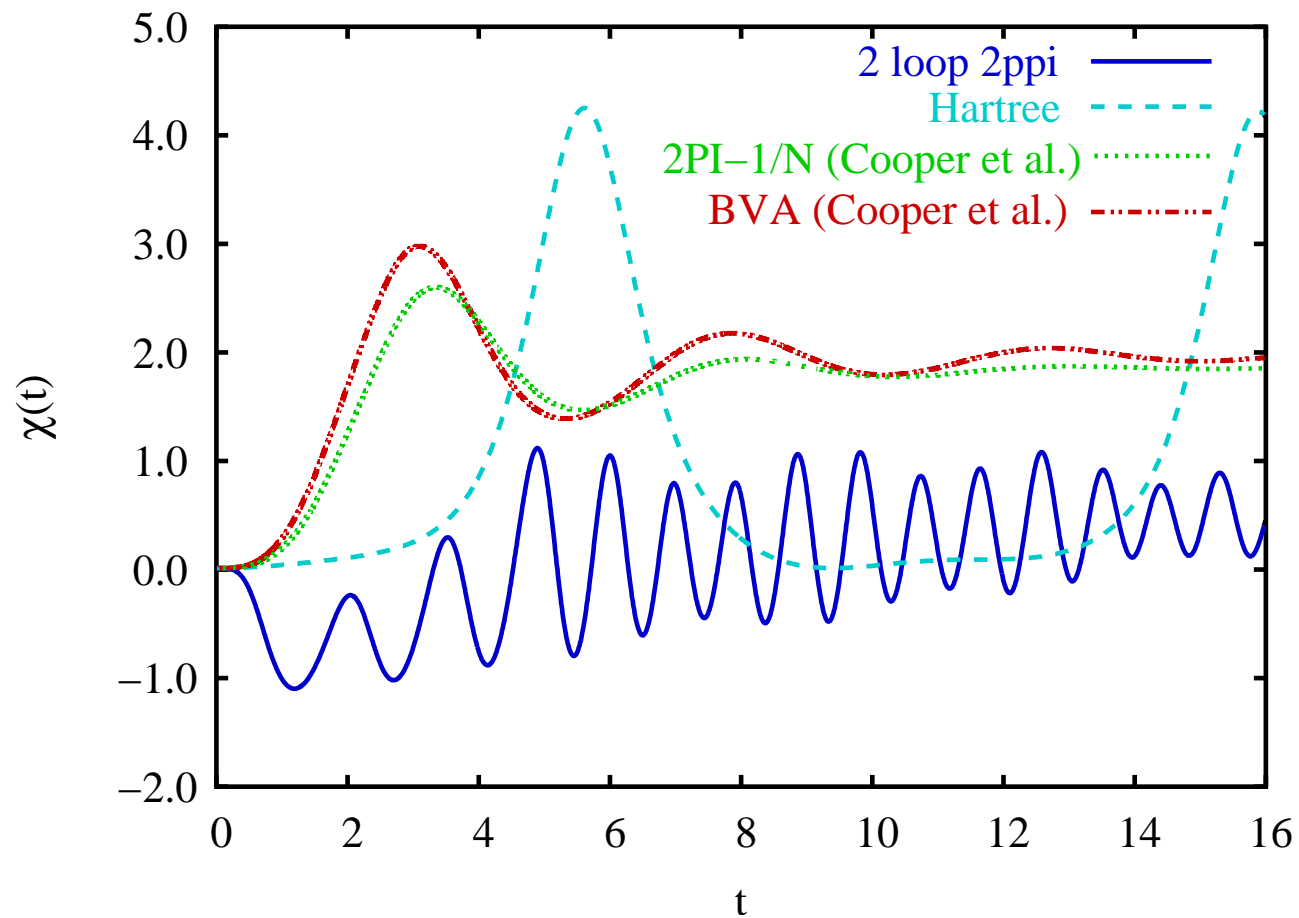


parameters:

$$\phi(0) = 0.4, T_0 = 0.1$$

$$\lambda = 3.0, m^2 = -1$$

Composite field $\chi(t)$ – smaller coupling



5. Summary

- We have presented the **generalization** of the **2PPI** scheme to **nonequilibrium** quantum field theory
- Compared to other methods computationally less involved framework for simulations beyond the large- N or Hartree approximation
- Numerical simulations in $1 + 1$ dimensions in the two-loop approximation displayed:
 - the **absence of symmetry breaking** as expected from the exact theory
 - **normal dissipative dynamics** for the mean field ϕ
 - this both cures (some) obvious deficits of the Hartree (and large- N) approximation