Far-from-Equilibrium Quantum Fields

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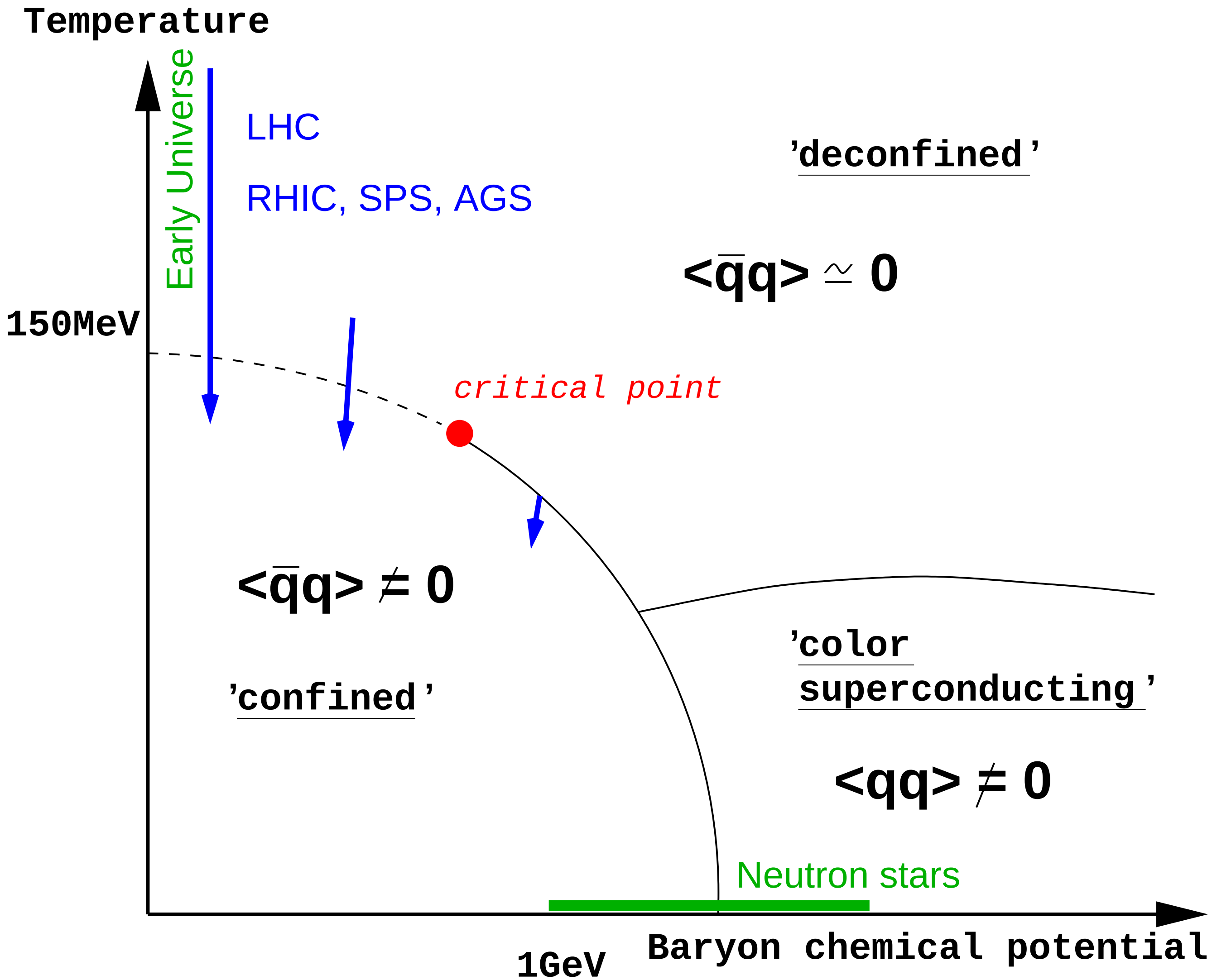
Motivation

Cartoon of the *thermal equilibrium* phase diagram of strongly interacting matter:



Current experiments at RHIC aim for a Quark-Gluon Plasma produced *in a transient, nonequilibrium state*

$\rightsquigarrow$ Initial value problem with *far-from-equilibrium initial conditions*

$\rightsquigarrow$ Description of *nonequilibrium dynamics/thermalization in quantum field theory* without relying on

- assumption of small departures from equilibrium?
- assumption of small fluctuations?

Prominent examples where strong departures from equilibrium with large fluctuations become relevant:

1. "Color Glas Condensate" as initial condition for heavy-ion collisions

$\rightsquigarrow$ characterized by large densities $\sim 1/\alpha_s$ with characteristic scale Q_s

2. "QCD critical point"

$\rightsquigarrow$ characterized by large correlation length/critical slowing down: the long wavelength modes fall out of equilibrium once cooling exceeds relaxation rate: $T(t)/\dot{T}(t) \ll t_{\mathrm{relaxation}}(T) \overset{T \to T_c}{\longrightarrow}$ "∞"

Similarly strong motivation from Early Universe/Condensed Matter physics for far-from-equilibrium dynamics in QFT from 'first principles'

Compare with 1.,2.:

1'. "Preheating" at the end of inflation in the Early Universe

$\rightsquigarrow$ nonperturbatively large particle number densities ($\sim 1/\lambda$) with a characteristic momentum scale (we'll come to this in detail)

2'. "Bose-Einstein condensates" in the laboratory

$\rightsquigarrow$ characterized by diverging correlation length/critical slowing down near second-order phase transition

$\rightsquigarrow$ $\boxed{\text{Required nonequilibrium techniques have most diverse applications}}$

Few ingredients for far-from-equilibrium quantum fields:

Not small departures from equilibrium:

Initial density matrix $\mathcal{D} \not\approx \mathcal{D}^{(\mathrm{eq})} \overset{\mathrm{e.g.}}{\sim} e^{-\beta H}$ or, equivalently, nonequilibrium initial one-point function $\mathrm{Tr}\{\mathcal{D}(t_0)\varphi(t_0,\mathbf{x})\}$, two-point function $\mathrm{Tr}\{\mathcal{D}(t_0)\varphi(t_0,\mathbf{x})\varphi(t_0,\mathbf{y})\}$, ...

$\rightsquigarrow$ e.g. no (non-)linear response, no fluctuation-dissipation relation

No other dynamics than the one dictated by the underlying quantum field theory:

$\rightsquigarrow$ Dynamics can be obtained from an **effective action** Γ (Legendre transform of the generating functional for connected Green's functions)

$\rightsquigarrow$ powerful nonperturbative approximation schemes available

$\rightsquigarrow$ preserves in particular energy conservation, time reversibility (here: no coupling to external heat-bath, no coarse-graining)

Two-particle irreducible generating functional

Quantum field theory with classical action $S[\varphi]$. Consider first a real, N–component scalar field φ_a $(a = 1, \ldots, N)$ with

$$S[\varphi] = \int_x \left(\frac{1}{2} \partial^\mu \varphi_a(x) \partial_\mu \varphi_a(x) - \frac{m^2}{2} \varphi_a(x) \varphi_a(x) - \frac{\lambda}{4! N} \left(\varphi_a(x) \varphi_a(x) \right)^2 \right)$$

($\int_x \equiv \int_{\mathcal{C}} \mathrm{d}x^0 \int \mathrm{d}^d x$ with $x = (x^0, \mathbf{x})$ and real (imaginary) time path $\mathcal{C}$)

Generating functional $W[J, R]$ for Green's functions with *two* source terms $\sim J_a(x)$ and $\sim R_{ab}(x, y)$:

$$\begin{aligned} Z[J, R] &= \exp\left(iW[J, R]\right) \\ &= \int \mathscr{D}\varphi \, \exp\left(i \left[S[\varphi] + \int_x J_a(x) \varphi_a(x) + \frac{1}{2} \int_{xy} R_{ab}(x, y) \varphi_a(x) \varphi_b(y) \right] \right) \end{aligned}$$

Define the macroscopic field ϕ_a and connected two-point function G_{ab}:

$$\frac{\delta W[J, R]}{\delta J_a(x)} = \phi_a(x) \quad , \quad \frac{\delta W[J, R]}{\delta R_{ab}(x, y)} = \frac{1}{2} \Big(\phi_a(x) \phi_b(y) + G_{ab}(x, y) \Big)$$

Effective action

$\rightsquigarrow$ Legendre transform of $W[J,R]$ with respect to source terms:

$$\Gamma[\phi,G] = W[J,R] - \int_x \frac{\delta W[J,R]}{\delta J_a(x)} J_a(x) - \int_{xy} \frac{\delta W[J,R]}{\delta R_{ab}(x,y)} R_{ab}(x,y)$$

$$= W[J,R] - \int_x \phi_a(x) J_a(x) - \frac{1}{2}\int_{xy} \phi_a(x) R_{ab}(x,y)\phi_b(y) - \frac{1}{2}\mathsf{Tr}\, G\, R$$

<u>One observes</u>:

$$\frac{\delta\Gamma[\phi,G]}{\delta\phi_a(x)} = -J_a(x) - \int_y R_{ab}(x,y)\phi_b(y)$$

$$\frac{\delta\Gamma[\phi,G]}{\delta G_{ab}(x,y)} = -\frac{1}{2} R_{ab}(x,y)$$

$\rightsquigarrow$ Equations of motion for ϕ and G for $J=0$ and $R=0$

Write the exact $\Gamma[\phi, G]$ as 'one-loop' type expression + 'rest':

$$\Gamma[\phi, G] = S[\phi] + \frac{i}{2}\mathrm{Tr}\ln G^{-1} + \frac{i}{2}\mathrm{Tr}\, G_0^{-1}(\phi)\, G + \Gamma_2[\phi, G]$$

(a) Consider first the 'one-loop' type expression, i.e. $\Gamma_2 \equiv 0$:

$$\frac{\delta\Gamma[\phi, G]}{\delta G} \;=\; -\frac{i}{2}G^{-1} + \frac{i}{2}G_0^{-1} = -\frac{1}{2}R$$

$$\Rightarrow \qquad G^{-1} \;=\; G_0^{-1} - iR$$

For $R \to 0$ one recovers the 'standard' *one-particle irreducible* effective action $\Gamma[\phi]$ to one-loop order:

$$\Gamma[\phi] = \Gamma[\phi, G(\phi)] = S[\phi] + \frac{i}{2}\mathrm{Tr}\ln G_0^{-1}(\phi) + \mathrm{const}$$

with $\mathrm{Tr}\, G_0^{-1} G_0 = \mathrm{Tr}\, \mathbf{1} = \mathrm{const}$

(b) To get an understanding of the 'rest' term $\Gamma_2[\phi, G]$ vary w.r.t. G:

$$\Rightarrow \quad G^{-1}_{ab}(x,y) \;=\; G^{-1}_{0,ab}(x,y;\phi) - iR_{ab}(x,y) - \Sigma_{ab}(x,y;\phi,G)$$

This is the *Schwinger-Dyson equation* for the propagator with

$$\boxed{\Sigma_{ab}(x,y;\phi,G) \equiv 2i\,\frac{\delta\Gamma_2[\phi,G]}{\delta G_{ab}(x,y)}}$$

$\rightsquigarrow$ $\Sigma_{ab}(x,y;\phi,G)$ corresponds to the proper self-energy, to which only *one-particle irreducible* diagrams contribute

$\rightsquigarrow$ $\Gamma_2[\phi,G]$ can only contain contributions from *two–particle irreducible* (2PI) diagrams

Check: Suppose a two–particle *reducible* contribution to Γ_2 exists, schematically:

$$\Gamma_2 \sim \tilde{\Gamma} G G \tilde{\Gamma}' \qquad \rightsquigarrow \qquad \Sigma \sim \tilde{\Gamma} G \tilde{\Gamma}'$$

The latter is one-particle *reducible* and cannot occur for the proper self-energy!

Summary: 2PI generating functional

Scalar quantum field theory with classical action S.

2PI effective action:

$$\Gamma[\phi, G] = S[\phi] + \frac{i}{2}\mathrm{Tr}\ln G^{-1} + \frac{i}{2}\mathrm{Tr}\, G_0^{-1}(\phi)\, G + \Gamma_2[\phi, G]$$

- Parametrized by macroscopic field: $\phi(x) = \langle \Phi(x) \rangle$ and
- exact connected propagator: $G(x,y) = \langle T\Phi(x)\Phi(y)\rangle - \phi(x)\phi(y)$
- $\Gamma_2[\phi, G]$ contains only 2PI diagrams

Equations of motion: (1) $\dfrac{\delta\Gamma[\phi, G]}{\delta\phi(x)} = 0$, (2) $\dfrac{\delta\Gamma[\phi, G]}{\delta G(x,y)} = 0$

in absence of sources. Simple relation to 1PI effective action ($R = 0$):

$$\Gamma[\phi] \equiv \Gamma[\phi, G(\phi)]\,, \qquad \left.\frac{\delta\Gamma[\phi, G]}{\delta G(x,y)}\right|_{G=G(\phi)} = 0$$

2PI loop expansion

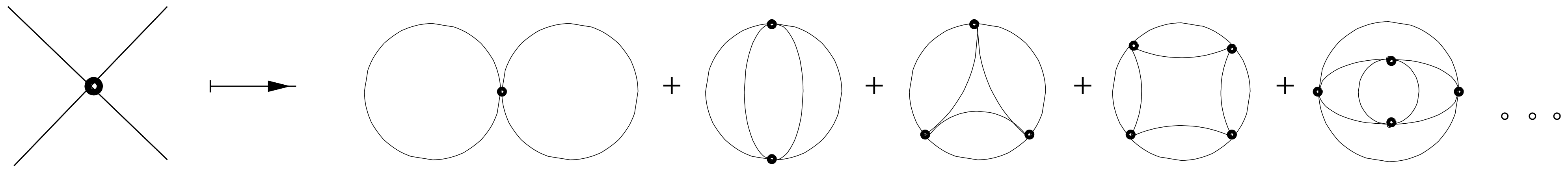
...proceeds along same lines as for 1PI effective action. Only difference:

- full propagator G is associated to propagator lines of a diagram
- only 2PI graphs are kept

$$\Gamma_2[\phi, G] = \Gamma_2^{(2\text{loop})}[\phi, G] + \Gamma_2^{(3\text{loop})}[\phi, G] + \Gamma_2^{(4\text{loop})}[\phi, G] + \ldots$$

<u>Example:</u> $\lambda\varphi^4/4!$ interaction for $N = 1$, $\Gamma[\phi = 0, G] \equiv \Gamma[G]$

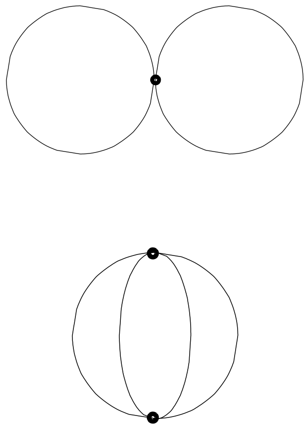
Topologically distinct diagrams up to five loops: (only five graphs)



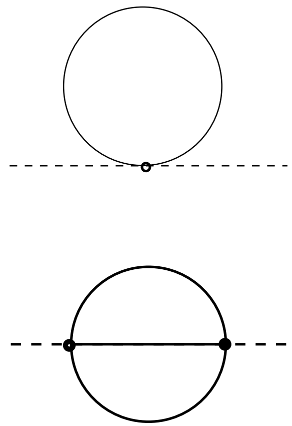
$$\Gamma_2[G] = -\frac{\lambda}{8}\int_x G^2(x,x) \quad + \quad \frac{i\lambda^2}{48}\int_{xy} G^4(x,y) \quad + \ldots$$

$(-i)\,3\;(-i\lambda/4!) \qquad (-i)\,4!\,(1/2)\,(-i\lambda/4!)^2$

2PI effective action, $\Gamma_2[G]$: Self-energy $\Sigma \sim \delta\Gamma_2/\delta G$:



$\rightsquigarrow$

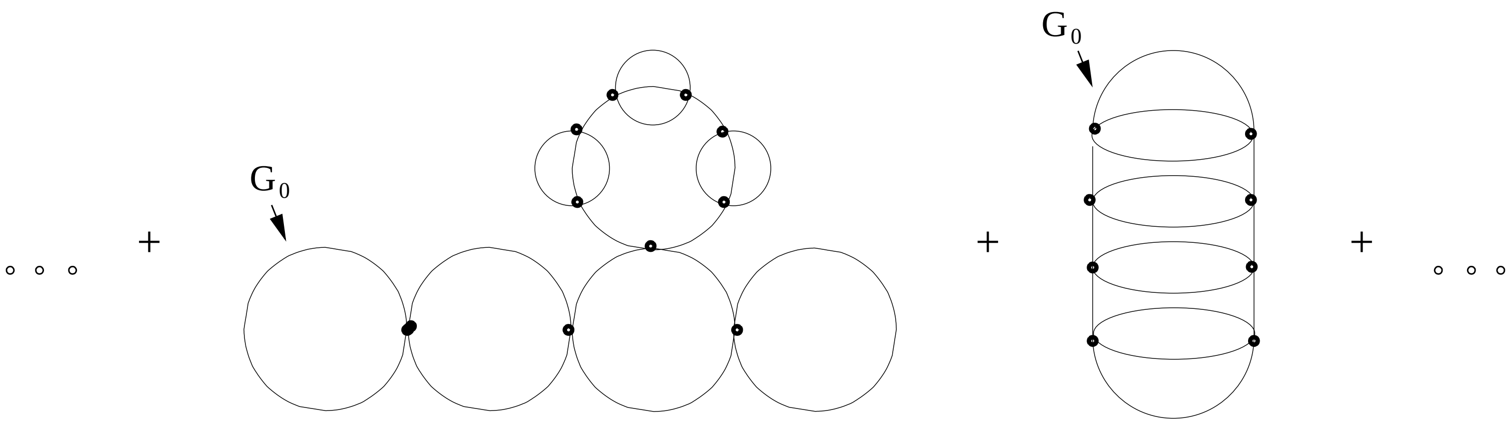


Comparison with 1PI effective action at stationary point,

$\delta\Gamma[G]/\delta G = 0 \Rightarrow G^{-1} = G_0^{-1} - \Sigma$:

$$G = G_0 + G_0\,\Sigma\,G_0 + G_0\,\Sigma\,G_0\,\Sigma\,G_0 + \ldots$$

$\rightsquigarrow$ Infinite series for finite 2PI loop order: daisy, super-daisy, ladders, ...



Limitation for 2PI loop expansion: Even for small coupling the expansion can become unreliable in the presence of large fluctuations

Prominent examples:

1. Phase transitions of the second order, with renormalized mass:

$$\lim_{T \to T_c} m_R(T) = 0$$

$\rightsquigarrow$ strong vertex corrections:

$$\lim_{T \to T_c} \frac{\lambda_R(T)\, T}{m_R(T)} = \text{const}$$

E.g. high temperature ϕ^4-theory in $4d$: 2PI 2-loop approximation gives continuous phase transition, however, not sensitive to universality class (classical exponents, 3-loop seems not to improve this)

2. Large particle number densities $n \sim 1/\lambda$

$\rightsquigarrow$ all loop diagrams contribute at same order in λ ! (we'll come to this)

Nonperturbative 2PI $1/N$ expansion can deal with 1., 2.!

2PI $1/N$–expansion

Real scalar fields $\varphi_a(x)$, $a = 1, \ldots, N$ with $\lambda(\varphi_a\varphi_a)^2/(4!N)$ interaction

$\rightsquigarrow \quad S[\phi] \sim N$ with $\mathrm{tr}\,(\phi\phi) \equiv \phi_a\phi_a \sim N$

2PI effective action $\Gamma[\phi, G]$:

- parametrized by two fields ϕ_a and G_{ab}
- Γ is singlet under $O(N)$ rotations $\longrightarrow$ even number of ϕ–fields
- irreducible $O(N)$–invariants: $((\phi\phi)_{ab} \equiv \phi_a\phi_b)$

$$\mathrm{tr}(\phi\phi), \qquad \mathrm{tr}(G^n)\,, \qquad\qquad \mathrm{tr}(\phi\phi G^n)$$

with $1 \leq n \leq N$ (no more independent invariants than fields)

$\rightsquigarrow$ each irreducible $O(N)$–invariant (trace over field indices) contributes a factor of N

$\rightsquigarrow$ each vertex provides a factor of $1/N$

$$\Gamma_2[\phi,G] = \Gamma_2^{\rm LO}[\phi,G] + \Gamma_2^{\rm NLO}[\phi,G] + \Gamma_2^{\rm NNLO}[\phi,G] + \ldots$$

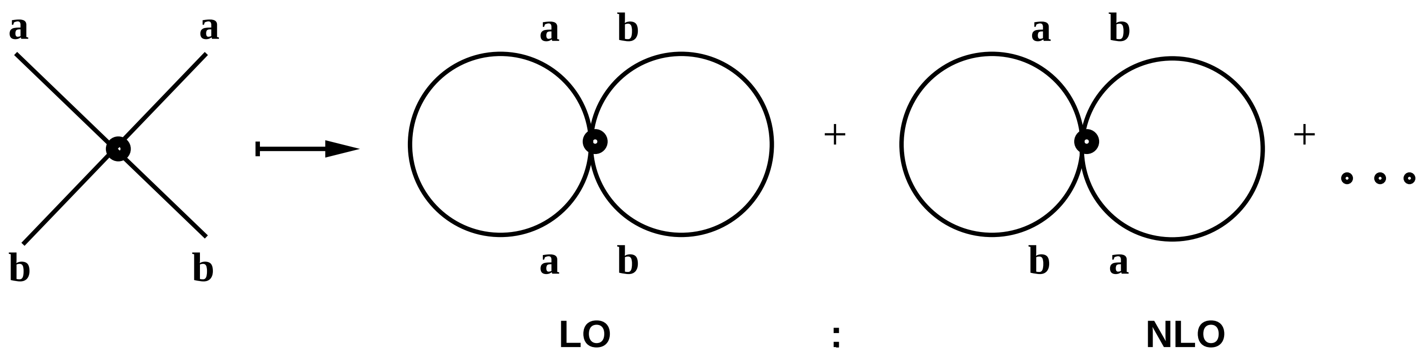
$$\sim N^1 \qquad\qquad \sim N^0 \qquad\qquad \sim N^{-1}$$

$\rightsquigarrow$ $\boxed{\text{Non-perturbative expansion of } \Gamma[\phi,G] \text{ in powers of } 1/N}$

Consider first $\Gamma_2[\phi=0,G] \equiv \Gamma_2[G]$:

LO contribution (only one graph)

$$\Gamma_2^{\rm LO}[G] = -\frac{\lambda}{4!N}\int_x G_{aa}(x,x)G_{bb}(x,x)$$

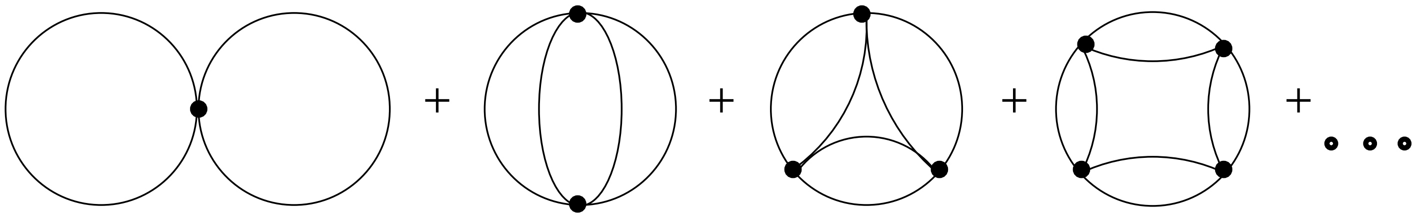


$$(\text{tr}G)^2/N \sim N \qquad\qquad \text{tr}G^2/N \sim N^0$$

NLO contribution (infinite series, can be summed)

$$\Gamma_2^{\rm NLO}[G] = \frac{i}{2} \mathsf{Tr} \ln[\, \mathbf{B}(G)\,]$$

$$\mathbf{B}(x,y;G) = \delta(x-y) + i \frac{\lambda}{6N}\, G_{ab}(x,y) G_{ab}(x,y)\,.$$



$$\mathsf{Tr} \ln[\, \mathbf{B}(G)\,] = \int_x \left(i \frac{\lambda}{6N} G_{ab}(x,x) G_{ab}(x,x) \right) \qquad , \mathsf{tr} G^2/N \sim N^0$$

$$- \frac{1}{2} \int_{xy} \left(i \frac{\lambda}{6N} G_{ab}(x,y) G_{ab}(x,y) \right)$$

$$\left(i \frac{\lambda}{6N}\, G_{a'b'}(y,x) G_{a'b'}(y,x) \right) \qquad , (\mathsf{tr} G^2/N)^2 \sim N^0$$

$$+ \quad \ldots \qquad , (\mathsf{tr} G^2/N)^n \sim N^0 \;, \; n \geq 3$$

2PI $1/N$ expansion with $\phi \neq 0$: $\varphi_a(x) \to \phi_a(x) + \varphi_a(x)$

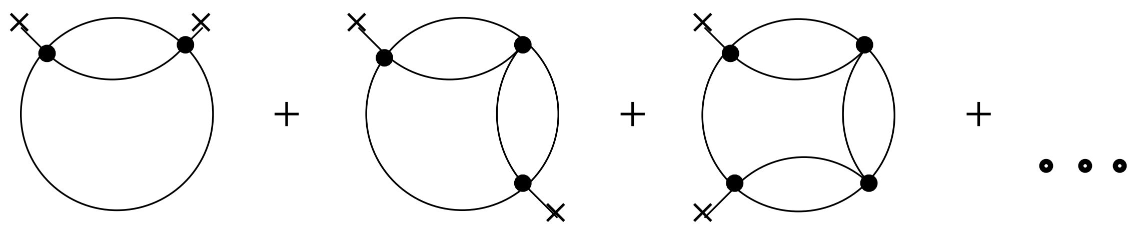
$$S_{\text{int}}[\phi,\varphi] = -\int_x \frac{\lambda}{6N}\phi_a(x)\varphi_a(x)\varphi_b(x)\varphi_b(x) - \int_x \frac{\lambda}{4!N}\Big(\varphi_a(x)\varphi_a(x)\Big)^2$$

LO contribution (ϕ–independent)

$$\Gamma_2^{\text{LO}}[\phi,G] = \Gamma_2^{\text{LO}}[G]$$

NLO contribution (infinite series, can be summed)

$$\begin{aligned}\Gamma_2^{\text{NLO}}[\phi,G] &= \Gamma_2^{\text{NLO}}[\phi\equiv 0,G] \\ &+ \frac{i\lambda}{6N}\int_{xy}\mathbf{I}(x,y;G)\,\phi_a(x)\,G_{ab}(x,y)\,\phi_b(y) \qquad \sim N^0 \\ \mathbf{I}(x,y;G) &= \frac{\lambda}{6N}G_{ab}(x,y)G_{ab}(x,y) \\ &- i\,\frac{\lambda}{6N}\int_z \mathbf{I}(x,z;G)G_{ab}(z,y)G_{ab}(z,y) \qquad \sim N^0\end{aligned}$$



Nonequilibrium generating functional

Out of equilibrium:

- Given initial density matrix $\mathcal{D}(0) \not\approx \mathcal{D}^{(\mathrm{eq})} \overset{\text{e.g.}}{\sim} e^{-\beta H}$
- or, equivalently, given nonequilibrium initial one-point function $\mathrm{Tr}\{\mathcal{D}(0)\Phi(0,\mathbf{x})\}$, two-point function $\mathrm{Tr}\{\mathcal{D}(0)\Phi(0,\mathbf{x})\Phi(0,\mathbf{y})\}$, three-point function etc.

$\boxed{\text{Question of nonequilibrium QFT: } n\text{-point functions for } t > 0}$

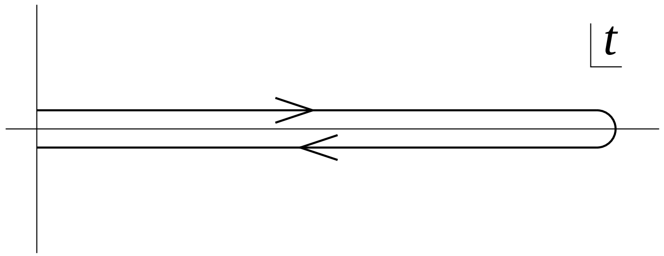
Generating functional:

$$Z[J,R;\mathcal{D}] \;=\; \mathrm{Tr}\left\{\mathcal{D}(0)\, T e^{i\left(\int_x J(x)\Phi(x)+\frac{1}{2}\int_{xy} R(x,y)\Phi(x)\Phi(y)\right)}\right\}$$

can be used to obtain *nonequilibrium Green's functions*, e.g.:

$$\begin{aligned}\mathrm{Tr}\{\mathcal{D}(0)\, T\Phi(x)\Phi(y)\} &\equiv \langle T\Phi(x)\Phi(y)\rangle \\ &= \left.\frac{\delta^2 Z[J,R;\mathcal{D}]}{i\delta J(x)\, i\delta J(y)}\right|_{J=R=0}\end{aligned}$$

To find a **path integral representation** consider $\int_x \equiv \int_{\mathcal{C}} dx^0 \int \mathrm{d}^d x$ for a finite, *closed* real-time contour $\mathcal{C}$:



$\leadsto$ T denotes contour time ordering: usual time ordering along the forward piece $\mathcal{C}^+$ and antitemporal ordering on the backward piece C^-. (Note that any time on C^- is considered later than any time on $\mathcal{C}^+$.)

1. Write the trace as

$$Z[J,R;\mathcal{D}] \;=\; \int \mathrm{d}\varphi^{(1)}(\mathbf{x})\mathrm{d}\varphi^{(2)}(\mathbf{x})\,\langle\varphi^{(1)}|\mathcal{D}(0)|\varphi^{(2)}\rangle$$
$$\langle\varphi^{(2)}|T e^{i\left(\int_x J(x)\Phi(x)+\frac{1}{2}\int_{xy}R(x,y)\Phi(x)\Phi(y)\right)}|\varphi^{(1)}\rangle$$

with $\Phi(t=0,\mathbf{x})|\varphi^{(i)}\rangle = \varphi^{(i)}(\mathbf{x})|\varphi^{(i)}\rangle$, $i=1,2$

2. For the source-dependent matrix element one has in complete analogy to the vacuum/equilibrium case:

$$\langle\varphi^{(2)}|Te^{i\left(\int_x J(x)\Phi(x)+\frac{1}{2}\int_{xy}R(x,y)\Phi(x)\Phi(y)\right)}|\varphi^{(1)}\rangle$$

$$= \int\limits_{\substack{\varphi(0^+,\mathbf{x})=\varphi^{(1)}(\mathbf{x})\\ \varphi(0^-,\mathbf{x})=\varphi^{(2)}(\mathbf{x})}} \mathscr{D}'\varphi\, e^{i\left(S[\varphi]+\int_x J(x)\varphi(x)+\frac{1}{2}\int_{xy}R(x,y)\varphi(x)\varphi(y)\right)}$$

3. In contrast to the equilibrium case the density matrix, $\mathcal{D}(0) \not\sim e^{-\beta H}$, cannot be interpreted as an evolution operator in imaginary time!

Example: the most general *Gaussian* density matrix can be written as

$$\langle\varphi^{(1)}|\mathcal{D}(0)|\varphi^{(2)}\rangle =$$

$$\frac{1}{\sqrt{2\pi\xi^2}}\exp\left\{i\dot{\phi}_0(\varphi^{(1)}-\varphi^{(2)})-\frac{\sigma^2+1}{8\xi^2}\left[(\varphi^{(1)}-\phi_0)^2+(\varphi^{(2)}-\phi_0)^2\right]\right.$$

$$\left.+i\,\frac{\eta}{2\xi}\left[(\varphi^{(1)}-\phi_0)^2-(\varphi^{(2)}-\phi_0)^2\right]+\frac{\sigma^2-1}{4\xi^2}(\varphi^{(1)}-\phi_0)(\varphi^{(2)}-\phi_0)\right\}$$

... where we neglect the spatial dependencies for a moment. Proof:

a) The density matrix is equivalent to the set of *initial conditions:*

$$\phi_0 \equiv \phi(t)_{|t=0} = \mathsf{Tr}\left\{\mathcal{D}(0)\Phi(t)\right\}_{|t=0}$$
$$\dot{\phi}_0 \equiv \partial_t \phi(t)_{|t=0}$$

$$F_0 \equiv G(t,t')_{|t=t'=0} = \left[\mathsf{Tr}\left\{\mathcal{D}(0)\Phi(t)\Phi(t')\right\} - \phi(t)\phi(t')\right]_{|t=t'=0}$$
$$Q_0 \equiv \frac{1}{2}\left[\partial_t G(t,t') + \partial_{t'} G(t,t')\right]_{|t=t'=0}$$
$$K_0 \equiv \partial_t \partial_{t'} G(t,t')_{|t=t'=0}$$

with $F_0 = \xi^2$, $Q_0 = \xi\eta$, $K_0 = \eta^2 + \sigma^2/(4\xi^2)$

b) Higher initial time derivatives are *not* independent since we obtain from the field equation of motion:

$$\langle \partial_t^2 \Phi \rangle = -m^2 \langle \Phi \rangle - \frac{\lambda}{6N} \langle \Phi^3 \rangle$$

with zero connected initial three-point function for Gaussian $\mathcal{D}(0)$

c) The equivalence between initial density matrix and initial conditions can be checked explicity:

$$\begin{aligned}
\mathrm{Tr}\mathcal{D}(0) &= \int_{-\infty}^{\infty} \mathrm{d}\varphi \, \langle\varphi|\mathcal{D}(0)|\varphi\rangle \\
&= \frac{1}{\sqrt{2\pi\xi^2}} \int_{-\infty}^{\infty} \mathrm{d}\varphi \exp\left\{ -\frac{1}{2\xi^2}(\varphi-\phi_0)^2 \right\} = 1 \\
\mathrm{Tr}\left\{\mathcal{D}(0)\Phi(0)\right\} &= \frac{1}{\sqrt{2\pi\xi^2}} \int_{-\infty}^{\infty} \mathrm{d}\varphi \, \varphi \exp\left\{ -\frac{1}{2\xi^2}(\varphi-\phi_0)^2 \right\} \stackrel{\varphi\to\varphi+\phi_0}{=} \phi_0
\end{aligned}$$

etc. (q.e.d.)

Similarly one finds

$$\mathrm{Tr}\,\mathcal{D}^2(0) = \int_{-\infty}^{\infty} \mathrm{d}\varphi \int_{-\infty}^{\infty} \mathrm{d}\varphi' \langle\varphi|\mathcal{D}(0)|\varphi'\rangle\langle\varphi'|\mathcal{D}(0)|\varphi\rangle = \frac{1}{\sigma}$$

$\rightsquigarrow$ for $\sigma > 1$ the initial conditions with $\eta^2 + \frac{\sigma^2}{4\xi^2} \equiv \partial_t\partial_{t'}G(t,t')_{|t=t'=0}$ describe a *mixed state*. For $\sigma = 1$ the "mixing term" in $\mathcal{D}(0)$ is absent and one obtains a pure-state density matrix.

Generalizing the example, parametrize the most general density matrix:

$$\langle\varphi^{(1)}|\mathcal{D}(0)|\varphi^{(2)}\rangle = \mathcal{N}\, e^{ifc[\varphi]}$$

with
$$\begin{aligned} fc[\varphi] &= \alpha_0 + \int_x \alpha_1(x)\varphi(x) + \frac{1}{2}\int_{xy} \alpha_2(x,y)\varphi(x)\varphi(y) \\ &\quad + \frac{1}{3!}\int_{xyz} \alpha_3(x,y,z)\varphi(x)\varphi(y)\varphi(z) \\ &\quad + \frac{1}{4!}\int_{xyzw} \alpha_4(x,y,z,w)\varphi(x)\varphi(y)\varphi(z)\varphi(w) + \ldots \end{aligned}$$

Here $\varphi(0^+,\mathbf{x}) = \varphi^{(1)}(\mathbf{x})$, $\varphi(0^-,\mathbf{x}) = \varphi^{(2)}(\mathbf{x})$ and, since $\mathcal{D}(0)$ is specified at time zero, $\alpha_i(x_1,\ldots x_n) \equiv 0$ for $x_i \neq 0\,,\; i = 1,\ldots n$

$\leadsto$ The α_i represent initial-time *sources*

- α_0 is irrelevant constant, α_1 can be absorbed in J, and α_2 in R
- for Gaussian $\mathcal{D}(0)$: initial-time sources $\alpha_i \equiv 0$ for $i \geq 3$

$\leadsto$ For *Gaussian* initial density matrix: $Z[J,R;\mathcal{D}] \to Z[J,R]$ (!)

Summary: Nonequilibrium generating functional

- Gaussian initial density matrix $\mathcal{D}(0)$:

$$\boxed{\begin{array}{l}\text{Nonequilibrium 2PI effective action} = \\ \Gamma[\phi, G] \text{ with closed time path } \mathcal{C}\end{array}}$$

$\rightsquigarrow$ Previous loop or $1/N$ expansions remain unchanged except that for nonequilibrium the time integrals involve a closed time path!

- Most general initial Gaussian $\mathcal{D}(0)$: Only one- (ϕ) and two-point function (G) and derivatives need to be specified at initial time

  Note: represents *no approximation* for the dynamics, i.e. it constrains only the 'experimental' setup and higher irreducible correlations can build up corresponding to a non-Gaussian density matrix for $t > 0$

- Non-Gaussian initial density matrices pose no problems in principle but require taking into account additional initial-time sources

Nonequilibrium evolution equations

Consider first $\Gamma[\phi = 0, G] \equiv \Gamma[G]$. Equation of motion for G:

$$\frac{\delta \Gamma[G]}{\delta G(x,y)} = 0 \quad \Rightarrow \quad G^{-1}(x,y) = G_0^{-1}(x,y) - \Sigma(x,y;G)$$

Rewrite as partial differential equation suitable for initial value problems by convolution with G: $\quad (\delta(x-y) \equiv \delta_{\mathcal{C}}(x^0 - y^0)\delta(\mathbf{x} - \mathbf{y}))$

$$\int_z G_0^{-1}(x,z)G(z,y) = \int_z \Sigma(x,z)G(z,y) + \delta(x-y)$$

For $G_0^{-1}(x-y) = i\left[\Box_x + m^2\right]\delta(x-y)$:

$$\boxed{\left(\Box_x + m^2\right)G(x,y) + i\int_z \Sigma(x,z;G)G(z,y) = -i\delta(x-y)}$$

Very useful further rewriting: ($\rightsquigarrow$ simple physical interpretation)

1. Separate Σ in 'local' and 'nonlocal' part:

$$\Sigma(x,y;G) = -i\Sigma^{(0)}(x;G)\delta(x-y) + \overline{\Sigma}(x,y;G)$$

Time-dependent mass shift $\Sigma^{(0)}$: $M^2(x;G) = m^2 + \Sigma^{(0)}(x;G)$

2. Decompose into 'statistical' and 'spectral' components:

$$\text{anti-commutator:} \qquad F(x,y) = \frac{1}{2}\langle\{\Phi(x),\Phi(y)\}\rangle$$

$$\text{commutator:} \qquad \rho(x,y) = i\langle[\Phi(x),\Phi(y)]\rangle$$

Spectral function ρ encodes equal-time commutation relations:

$$\rho(x,y)|_{x^0=y^0} = 0 \quad , \quad \partial_{x^0}\rho(x,y)|_{x^0=y^0} = \delta(\mathbf{x}-\mathbf{y})$$

$$\rightsquigarrow \qquad \boxed{G(x,y) = F(x,y) - \frac{i}{2}\rho(x,y)\,\mathrm{sign}_{\mathcal{C}}(x^0-y^0)}$$

with $\mathrm{sign}_{\mathcal{C}}(x^0-y^0) \equiv \Theta_{\mathcal{C}}(x^0-y^0) - \Theta_{\mathcal{C}}(y^0-x^0)$

Check:

$$\begin{aligned} G(x,y) &= \langle \Phi(x)\Phi(y)\rangle \Theta_{\mathcal{C}}(x^0-y^0) + \langle \Phi(y)\Phi(x)\rangle \Theta_{\mathcal{C}}(y^0-x^0) \\ &= \frac{1}{2}\langle\{\Phi(x),\Phi(y)\}\rangle \left(\Theta_{\mathcal{C}}(x^0-y^0)+\Theta_{\mathcal{C}}(y^0-x^0)\right) \\ &+ \frac{1}{2}\langle[\Phi(x),\Phi(y)]\rangle \underbrace{\left(\Theta_{\mathcal{C}}(x^0-y^0)-\Theta_{\mathcal{C}}(y^0-x^0)\right)}_{\mathrm{sign}_{\mathcal{C}}(x^0-y^0)} \end{aligned}$$

- Equivalent decomposition for the self-energy:

$$\overline{\Sigma}(x,y) = \Sigma^F(x,y) - \frac{i}{2}\Sigma^\rho(x,y)\,\mathrm{sign}_{\mathcal{C}}(x^0-y^0)$$

- $\rho(x,y)\Theta(x^0-y^0)$ is the retarded propagator, $-\rho(x,y)\Theta(y^0-x^0)$ the advanced one, etc. but *out of equilibrium*

  there are only two independent two-point functions! (e.g. F and ρ)

  For real, scalar fields F and ρ are real functions with symmetry properties $F(x,y) = F(y,x)$ and $\rho(x,y) = -\rho(y,x)$

<u>Detour</u>: Consider for a moment **thermal equilibrium** (**not** employed)

$\rightsquigarrow$ two-point functions depend only on relative coordinates:

$$\Rightarrow \; F^{(\mathrm{eq})}(x,y) = \int \frac{\mathrm{d}\omega \mathrm{d}^d p}{(2\pi)^{d+1}} e^{-i\omega(x^0-y^0)+i\mathbf{p}(\mathbf{x}-\mathbf{y})} F^{(\mathrm{eq})}(\omega,\mathbf{p})$$

$\rightsquigarrow$ periodicity ("KMS") condition in imaginary time, $\mathcal{C} = [0, -i\beta]$:
$G(x-y)|_{x^0=0} = G(x-y)|_{x^0=-i\beta}$

$$\Rightarrow \; F^{(\mathrm{eq})}(\omega,\mathbf{p}) + \frac{i}{2}\rho^{(\mathrm{eq})}(\omega,\mathbf{p}) = e^{-\beta\omega}\left(F^{(\mathrm{eq})}(\omega,\mathbf{p}) - \frac{i}{2}\rho^{(\mathrm{eq})}(\omega,\mathbf{p})\right)$$

Fluctuation-dissipation relation:

$$\Leftrightarrow \qquad \boxed{F^{(\mathrm{eq})}(\omega,\mathbf{p}) = -i\left(n_{\mathrm{BE}}(\omega) + \frac{1}{2}\right)\rho^{(\mathrm{eq})}(\omega,\mathbf{p})}$$

with $n_{\mathrm{BE}}(\omega) = (e^{\beta\omega} - 1)^{-1}$ (Bose-Einstein)

In contrast: Out of equilibrium F and ρ are independent functions (**no** fluctuation-dissipation relation!)

$\rightarrow$ back to nonequilibrium

3. Evaluation along the time contour C:

$$i\int_{\mathcal{C}} \mathrm{d}z^0\, \overline{\Sigma}(x,z;G)G(z,y) = i\int_{\mathcal{C}} \mathrm{d}z^0 \Big\{\Sigma^F(x,z)F(z,y)$$
$$-\frac{i}{2}\Sigma^F(x,z)\rho(z,y)\,\mathrm{sign}_{\mathcal{C}}(z^0-y^0) - \frac{i}{2}\Sigma^\rho(x,z)F(z,y)\,\mathrm{sign}_{\mathcal{C}}(x^0-z^0)$$
$$-\frac{1}{4}\Sigma^\rho(x,z)\rho(z,y)\,\mathrm{sign}_{\mathcal{C}}(x^0-z^0)\mathrm{sign}_{\mathcal{C}}(z^0-y^0)\Big\}$$

- The first term vanishes because of the *closed* time contour
- For the second term split the contour integral:

  $\int_{\mathcal{C}} \mathrm{d}z^0\, \mathrm{sign}_{\mathcal{C}}(z^0-y^0) = \int_0^{y^0} \mathrm{d}z^0(-1) + \int_{y^0}^0 \mathrm{d}z^0 = -2\int_0^{y^0} \mathrm{d}z^0$, etc.
- For the last term consider:

  (a) <u>$x^0 > y^0$:</u>

  $\int_{\mathcal{C}} \mathrm{d}z^0\, \mathrm{sign}_{\mathcal{C}}(x^0-z^0)\mathrm{sign}_{\mathcal{C}}(z^0-y^0) = \int_0^{y^0} \mathrm{d}z^0(-1) + \int_{y^0}^{x^0} \mathrm{d}z^0 + \int_{x^0}^0 \mathrm{d}z^0(-1)$

  (b) <u>$y^0 > x^0$:</u>

  $\int_{\mathcal{C}} \mathrm{d}z^0\, \mathrm{sign}_{\mathcal{C}}(x^0-z^0)\mathrm{sign}_{\mathcal{C}}(z^0-y^0) = \int_0^{x^0} \mathrm{d}z^0(-1) + \int_{x^0}^{y^0} \mathrm{d}z^0 + \int_{y^0}^0 \mathrm{d}z^0(-1)$

  Note that (a) und (b) differ by a sign $\rightsquigarrow$ $\mathrm{sign}_{\mathcal{C}}(x^0-y^0)$

Combining the integrals gives:

$$i\int_{\mathcal{C}} \mathrm{d}z^0\, \overline{\Sigma}(x,z;G)G(z,y) = -\frac{i}{2}\mathrm{sign}_{\mathcal{C}}(x^0-y^0)\int_{y^0}^{x^0} \mathrm{d}z^0\, \Sigma^{\rho}(x,z)\rho(z,y)$$
$$+\int_0^{x^0} \mathrm{d}z^0\, \Sigma^{\rho}(x,z)F(z,y) - \int_0^{y^0} \mathrm{d}z^0\, \Sigma^{F}(x,z)\rho(z,y)$$

4. Finally, use that

$$\Box_x G(x,y) = \Box_x F(x,y) - \frac{i}{2}\mathrm{sign}_{\mathcal{C}}(x^0-y^0)\Box_x\rho(x,y) - \underbrace{i\delta(x-y)}_{\text{cancels with } \delta \text{ in EOM}}$$

Check:

$$-\frac{i}{2}\partial_{x^0}^2\left[\rho(x,y)\,\mathrm{sign}_{\mathcal{C}}(x^0-y^0)\right] = -\frac{i}{2}\partial_{x^0}\Big[\mathrm{sign}_{\mathcal{C}}(x^0-y^0)\partial_{x^0}\rho(x,y) \hookleftarrow$$
$$\hookrightarrow + 2\underbrace{\rho(x,y)\delta_{\mathcal{C}}(x^0-y^0)}_{\text{zero since } \rho(x,y)|_{x^0=y^0}=0}\Big] = -\frac{i}{2}\mathrm{sign}_{\mathcal{C}}(x^0-y^0)\partial_{x^0}^2\rho(x,y)$$
$$-\underbrace{i\delta_{\mathcal{C}}(x^0-y^0)\partial_{x^0}\rho(x,y)}_{i\delta(x-y)}$$

since $\partial_{x^0}\rho(x,y)|_{x^0=y^0} = \delta(\mathbf{x}-\mathbf{y})$

$\rightsquigarrow$ **Coupled evolution equations for spectral and statistical components:** (*exact* for known self-energies)

$$\left[\Box_x \delta_{ac} + M^2_{ac}(x)\right] \rho_{cb}(x,y) \;=\; -\int_{y^0}^{x^0} \mathrm{d}z\, \Sigma^{\rho}_{ac}(x,z) \rho_{cb}(z,y)$$

$$\left[\Box_x \delta_{ac} + M^2_{ac}(x)\right] F_{cb}(x,y) \;=\; \int_{0}^{y^0} \mathrm{d}z\, \Sigma^{F}_{ac}(x,z) \rho_{cb}(z,y) \;-\; \int_{0}^{x^0} \mathrm{d}z\, \Sigma^{\rho}_{ab}(x,z) F_{cb}(z,y)$$

- Causal equations with "**memory**" integrals $\left(\int_{y^0}^{x^0} \mathrm{d}z \equiv \int_{y^0}^{x^0} \mathrm{d}z^0 \int \mathrm{d}^3 z\right)$
- For $\phi = 0$: $M^2 = M^2(F)$, $\Sigma^F = \Sigma^F(\rho, F)$, $\Sigma^\rho = \Sigma^\rho(\rho, F)$

  For $\phi \neq 0$: $M^2 = M^2(\phi, F)$, $\Sigma^F = \Sigma^F(\phi, \rho, F)$, $\Sigma^\rho = \Sigma^\rho(\phi, \rho, F)$

Similarly, for $\phi \neq 0$ one finds for the **field equation**: $\delta\Gamma[\phi, G]/\delta\phi = 0$

$$\Rightarrow \left(\left[\Box_x + \frac{\lambda}{6N}\phi^2(x) \right] \delta_{ab} + M^2_{ab}(x; \phi = 0, F) \right) \phi_b(x)$$

$$= -\int_0^{x^0} \mathrm{d}y \, \Sigma^\rho_{ab}(x, y; \phi = 0, F, \rho) \; \phi_b(y)$$

($\phi^2 \equiv \phi_a \phi_a$) at NLO in the $1/N$ expansion:

$$M^2_{ab}(x; \phi, F) \;=\; \left(m^2 + \frac{\lambda}{6N} \left[F_{cc}(x, x) + \phi^2(x) \right] \right) \delta_{ab} \quad \sim \mathcal{O}(N^0)$$

$$+ \;\; \frac{\lambda}{3N} \left[F_{ab}(x, x) + \phi_a(x)\phi_b(x) \right] \quad \sim \mathcal{O}(1/N)$$

$$\Sigma^{F,\rho}(\phi, F, \rho) \;\sim\; \mathcal{O}(1/N)$$

$\rightsquigarrow$ $\mathrm{RHS} \equiv 0$ at LO $(N \to \infty)$: **No memory terms at LO !**

Nonequilibrium evolution from the 2PI $1/N$ expansion

Consider spacially homogeneous ensembles: $\phi_a(x^0) = \langle \Phi_a(x^0, \mathbf{x}) \rangle$,
$F_{ab}(x,y) = F_{ab}(x^0, y^0; \mathbf{x} - \mathbf{y}) = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} F_{ab}(x^0, y^0; \mathbf{p})$

(Note that a *homogeneous ensemble* average contains all fluctuations from an *inhomogeneous* $\Phi(x)$ as well)

LO approximation: (Long history!)

$$
\begin{aligned}
\left(\left[\partial_t^2 + \mathbf{p}^2\right] \delta_{ac} + M^2_{ac}(t;\phi,F)\right) F_{cb}(t,t';\mathbf{p}) &= 0 \\
\left(\left[\partial_t^2 + \mathbf{p}^2\right] \delta_{ac} + M^2_{ac}(t;\phi,F)\right) \rho_{cb}(t,t';\mathbf{p}) &= 0 \\
\left([\partial_t^2 + \frac{\lambda}{6N}\phi^2(t)]\delta_{ab} + M^2_{ab}(t;\phi=0,F)\right)\phi_b(t) &= 0
\end{aligned}
$$

- No memory integrals (neglects 'direct scattering')
- Evolution of ϕ, F completely decoupled from spectral function ρ

Important consequence of the LO approximation:

Infinite number of **additional conserved quantities** $n_0(\mathbf{p})$ which are not present in the fully interacting theory:

$$n_0(\mathbf{p}) + \frac{1}{2} = \left(F(t,t';\mathbf{p})\, \partial_t \partial_{t'} F(t,t';\mathbf{p}) - (\partial_t F(t,t';\mathbf{p}))^2 \right)^{\frac{1}{2}} \Big|_{t=t'}$$

$\rightsquigarrow$ Details of nonthermal, initial $n_0(\mathbf{p})$ determine late-time behavior!

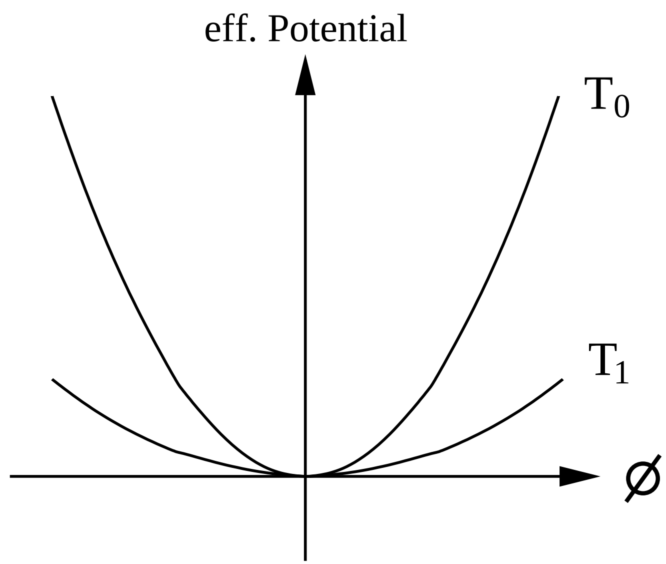
1. Example: 'Quench' from high temperature (T_0) with sudden drop in the effective mass:

$$M^2(0) = 2M_0^2 \to M^2(0^+) = M_0^2$$

Initial particle number distribution:

$$n_0(p) = 1/(\exp[\sqrt{p^2 + 2M_0^2}/T_0] - 1)$$

$$T_0 = 2M_0, \ \phi_0 = \dot{\phi}_0 = 0$$

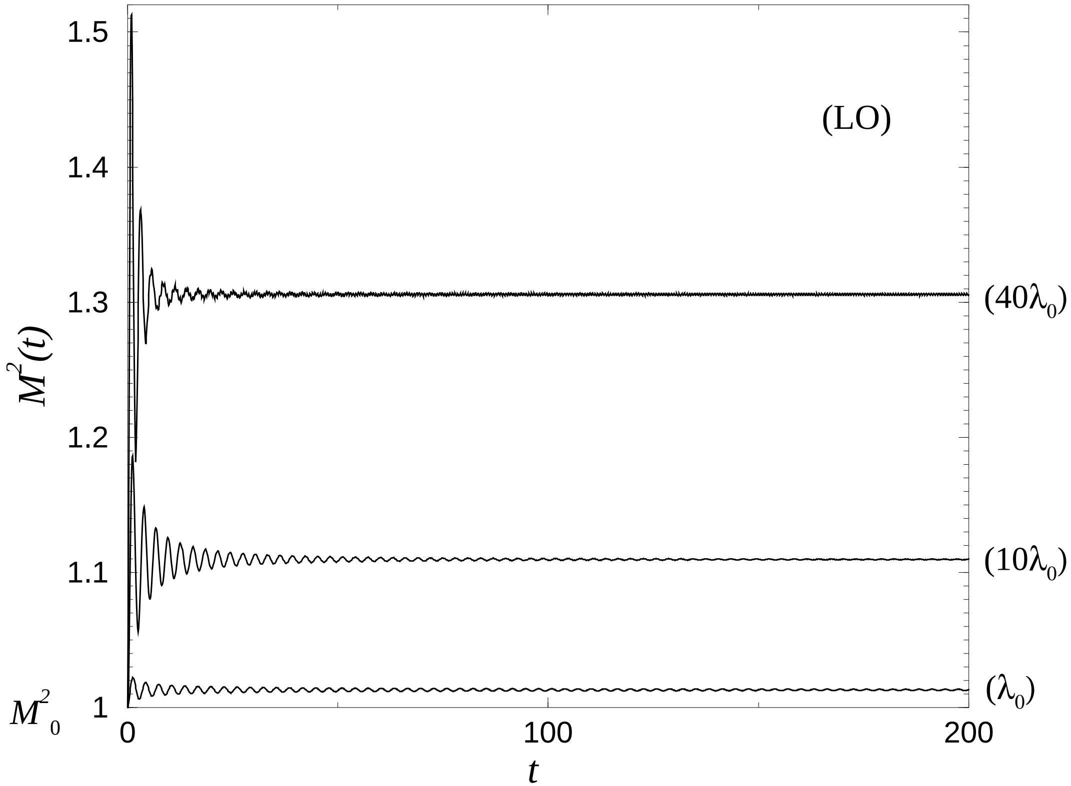


Should at late times approach thermal $n_1(p)$ with $T_1 < T_0$ $\rightsquigarrow$ Check!

Numerical solution of time evolution at LO in $1+1$ dimensions:

Wide range of couplings: $\lambda/\lambda_0 = \{1, 10, 40\}$, $\lambda_0 = 0.5 M_0^2$

Renormalization condition: $M^2(0^+) = M_0^2 \quad (\equiv 1)$ to absorb log divergence into m^2



$\rightsquigarrow$ $M^2(t)$ quickly approaches solution of static LO gap equation with initial *(conserved)* particle number $n_0(p)$ $\rightsquigarrow$ *not* thermal equilibrium:

$$\lim_{t\to`\infty`} M^2(t) = \{1.01, 1.11, 1.31\} M_0^2 \simeq M^2_{\rm GAP}(n_0)$$

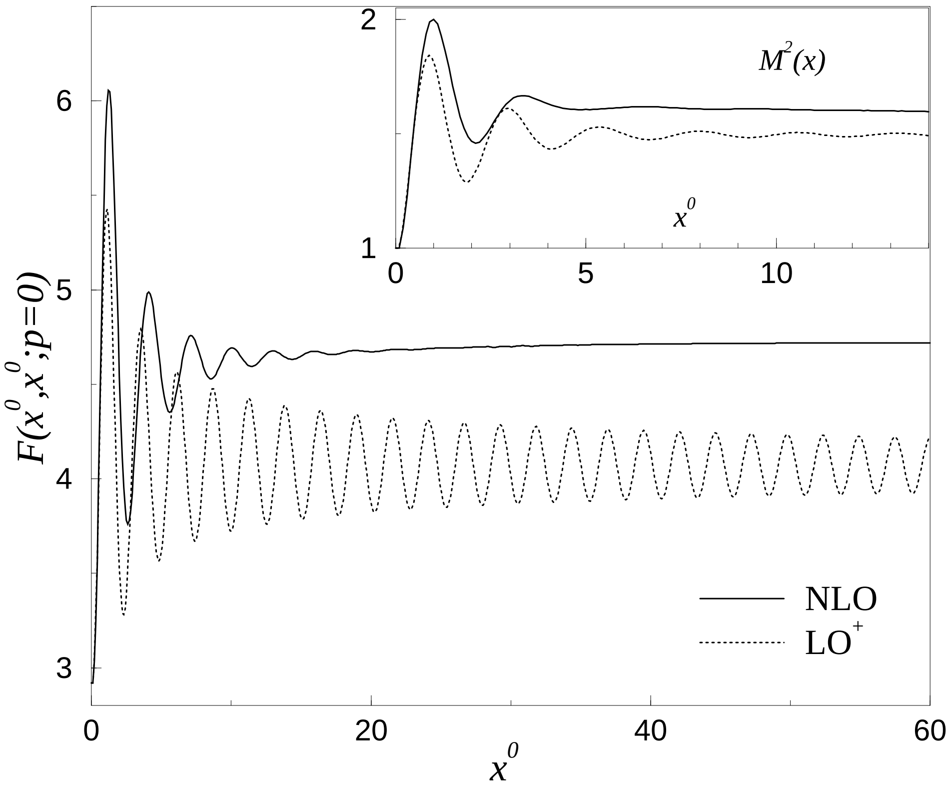
$$M^2_{\rm GAP}(n_0) = m^2 + \frac{\lambda}{6}\int \frac{dp}{2\pi}\left(n_0(p) + \frac{1}{2}\right)\frac{1}{\sqrt{p^2 + M^2_{\rm GAP}(n_0)}}$$

$$= \{1.01, 1.10, 1.29\} M_0^2$$

Comparison LO/NLO evolution

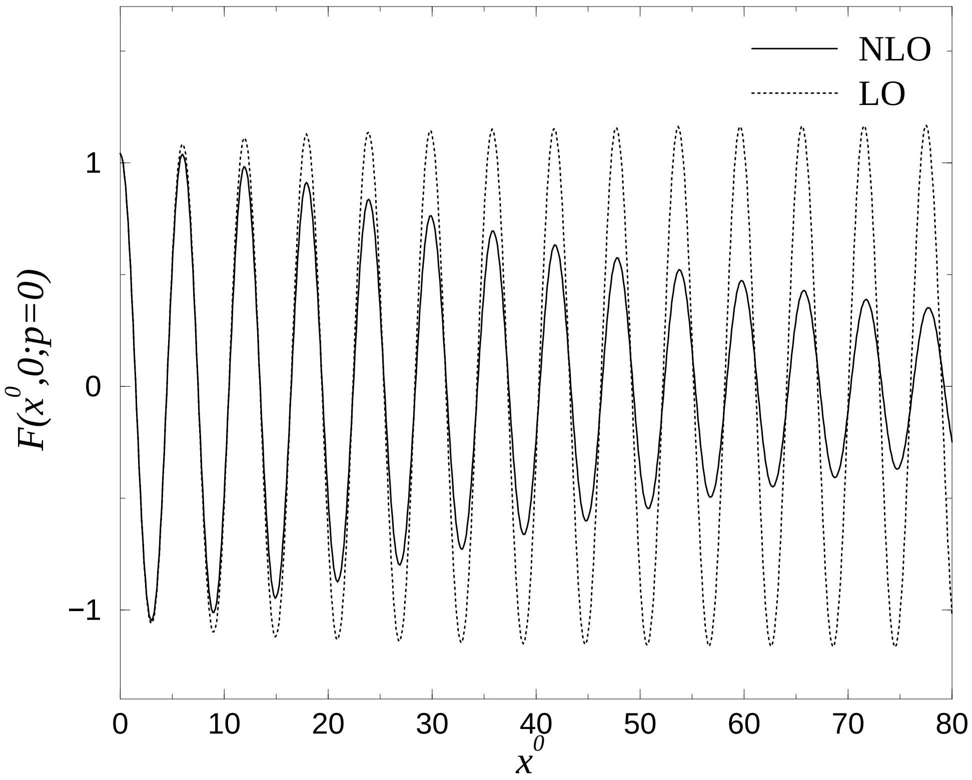
Equal-time:

$(\lambda/6N = 0.17 M_0^2, N = 4)$



Unequal-time:

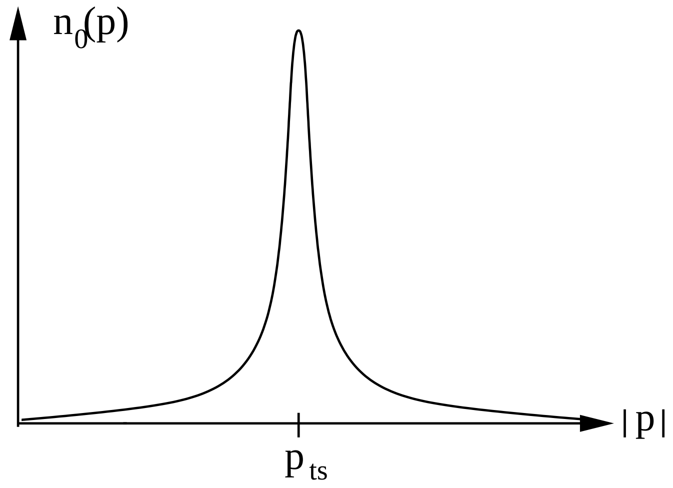
$(\lambda/6N = 0.083 M_0^2, N = 10)$



- At NLO modes quickly approach exponentially damped behavior
- NLO correlations with initial time $F(t,0;p)$ are suppressed $\rightsquigarrow$ *effective loss of details of initial conditions* (prerequisite for thermalization!)

2. Example: 'Tsunami'-type initial condition, i.e.

high particle number density at a characteristic momentum $p = \pm p_{\rm ts}$



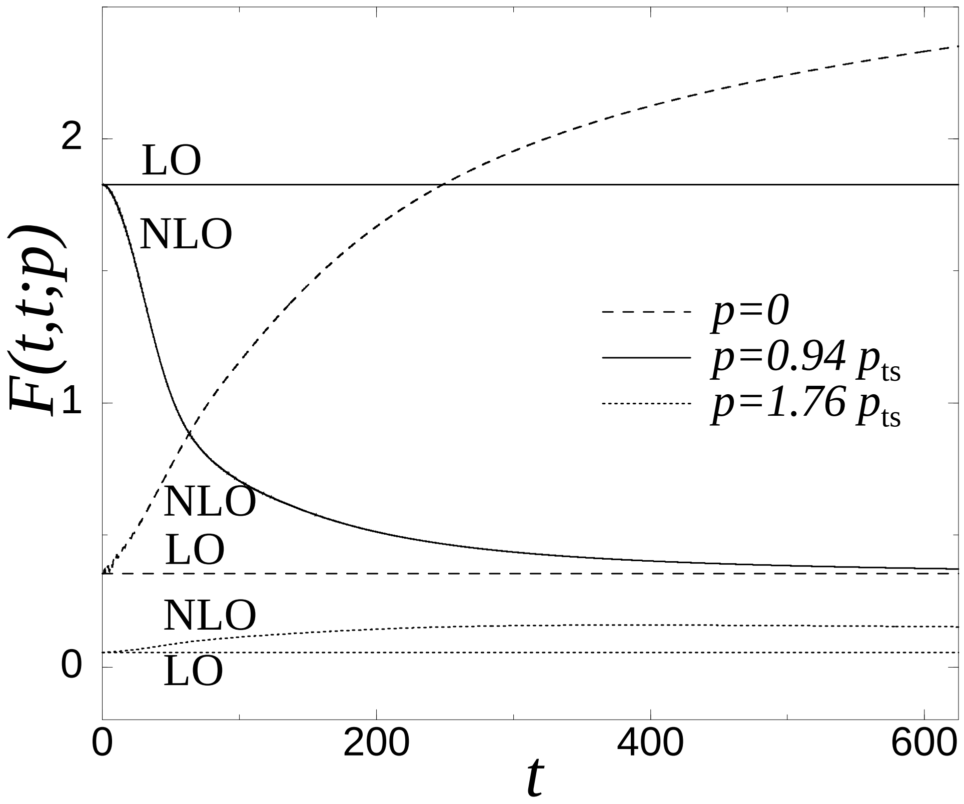
Comparison LO/NLO evolution:

$N = 10$, $\lambda/6N = 0.1 M_0^2$

$n_0(p) = \mathcal{A} e^{-\frac{1}{2\sigma^2}(|p|-p_{\rm ts})^2}$

$p_{\rm ts} = 5M_0$, $\sigma = 0.5 M_0$, $\mathcal{A} = 10$

$\phi_0 = \dot{\phi}_0 = 0$



- 'Tsunami' decays at NLO

Strong qualitative difference between LO and NLO because spurious conserved quantities are removed once scattering is taken into account!

Check: Derive evolution equation for (effective) particle number $n_{\mathbf{p}}(t)$ and show that $\partial_t n_{\mathbf{p}}(t) \equiv 0$ not true beyond LO !

Consider: $$(n_{\mathbf{p}}(t) + 1/2)^2 = F(t,t;\mathbf{p})K(t,t;\mathbf{p}) - Q^2(t,t;\mathbf{p})$$

$$\begin{aligned} K(t,t';\mathbf{p}) &= \partial_t \partial_{t'} F(t,t';\mathbf{p}) \, , \\ Q(t,t';\mathbf{p}) &= \frac{1}{2} \left[\partial_t F(t,t';\mathbf{p}) + \partial_{t'} F(t,t';\mathbf{p})\right] \end{aligned}$$

$\rightsquigarrow$ $n_{\mathbf{p}}(t=0) \equiv n_0(\mathbf{p})$ is the initial particle number introduced above

Since $F(t,t';\mathbf{p}) = F(t',t;\mathbf{p})$: $\partial_t F(t,t;\mathbf{p}) = 2\partial_t F(t,t';\mathbf{p})|_{t=t'}$

$$\Rightarrow \quad (n_{\mathbf{p}} + 1/2)\, \partial_t n_p = \partial_t \left[F(t,t';\mathbf{p})K(t,t';\mathbf{p}) - Q^2(t,t';\mathbf{p})\right] |_{t=t'}$$

RHS can be obtained from the exact evolution equation for $F(t,t';\mathbf{p})$ (act with $\partial_{t'}$, multiply with $F(t,t';\mathbf{p})$, subtract/add $\partial_t Q^2(t,t';\mathbf{p})$)

$\rightsquigarrow$ Exact ("quantum Boltzmann" :-) equation for $n_{\mathbf{p}}(t)$:

$$\left(n_{\mathbf{p}}(t) + \frac{1}{2}\right) \partial_t n_{\mathbf{p}}(t) =$$

$$\int_{t_0}^{t} dt'' \Big\{ \left[\Sigma^{\rho}(t,t'';\mathbf{p}) F(t'',t;\mathbf{p}) - \Sigma^{F}(t,t'';\mathbf{p}) \rho(t'',t;\mathbf{p})\right] \left[\partial_t F(t,t';\mathbf{p})\right] |_{t=t}$$

$$- \left[\Sigma^{\rho}(t,t'';\mathbf{p}) \partial_t F(t'',t;\mathbf{p}) - \Sigma^{F}(t,t'';\mathbf{p}) \partial_t \rho(t'',t;\mathbf{p})\right] F(t,t;\mathbf{p}) \Big\} |_{t=t'}$$

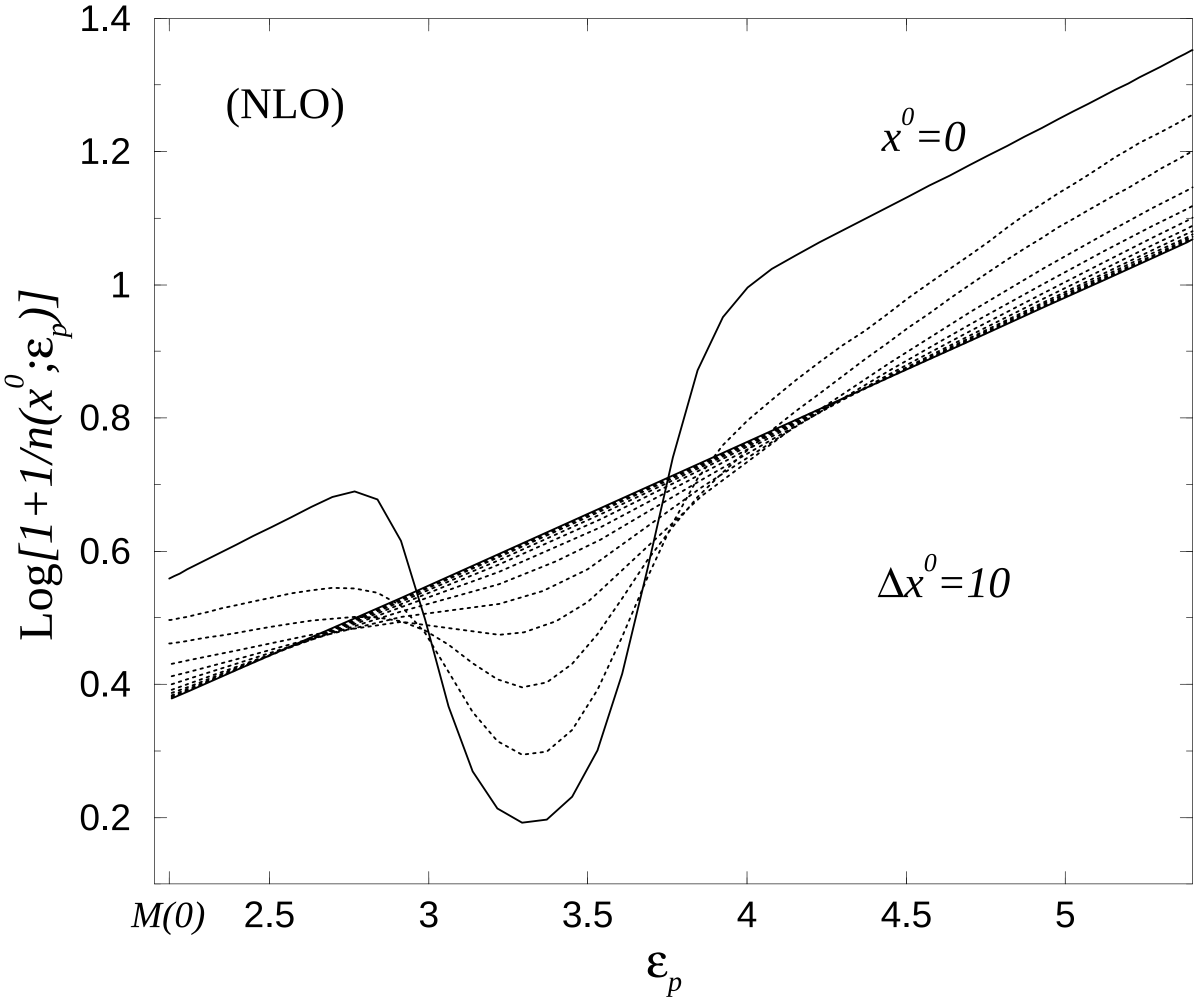
with initial time t_0. Since $\Sigma^{F,\rho} \sim \mathcal{O}(1/N)$:

$\rightsquigarrow$ at LO ($N \to \infty$): $\Sigma^{F,\rho} \equiv 0 \;\Rightarrow\; \partial_t n_{\mathbf{p}}(t) \equiv 0$ (strictly conserved!)

$\rightsquigarrow$ first non-zero contribution to $\Sigma^{F,\rho}$ at NLO: $\partial_t n_{\mathbf{p}}(t) \neq 0$ in general

Evolution of the effective particle number at NLO:

'Tsunami' at $p_{\rm ts} = 2.5 M_0$ with initial 'thermal background' $T_0 = 4 M_0$ $(\lambda/6N = 0.5 M_0^2\ , N = 4)$



For Bose-Einstein $n(\epsilon_p) \to 1/(e^{\epsilon_p/T_{\rm eq}} - 1)$: $\ln[1 + 1/n(\epsilon_p)] \to \epsilon_p/T_{\rm eq}$

$\leadsto$ inverse slope parameter at late time: $T_{\rm eq} \simeq 4.7 M_0 > T_0$

$\rightsquigarrow$ late-time evolution shows a *Bose-Einstein distributed effective particle number* $n_{\mathbf{p}}(t)$

$\rightsquigarrow$ characteristic for a system of weakly interacting 'quasiparticles'

Details: Average energy per mode for 'quasiparticles' (free-field form):

$$[n_{\mathbf{p}}(t) + 1/2]\, \epsilon_{\mathbf{p}}(t) \stackrel{(!)}{=} \frac{1}{2} \left[K(t,t;\mathbf{p}) + \epsilon_{\mathbf{p}}^2(t) F(t,t;\mathbf{p}) \right]$$

and $Q(t,t;\mathbf{p}) \equiv 0$, i.e. $n_{\mathbf{p}}(t) + 1/2 = \sqrt{K(t,t;\mathbf{p}) F(t,t;\mathbf{p})}$

$$\Rightarrow \qquad \epsilon_{\mathbf{p}}(t) \;=\; \sqrt{K(t,t;\mathbf{p})/F(t,t;\mathbf{p})}$$

Plotted above is $n_{\mathbf{p}}(t)$ as a function of the 'quasiparticle' energy $\epsilon_{\mathbf{p}}(t)$

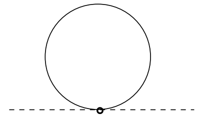
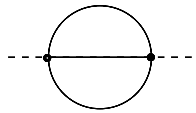
Characteristic time scales

Question: good *quasiparticle* description of nonequilibrium dynamics?

In particular: reliable description in terms of a Boltzmann equation?

To obtain a Boltzmann equation from the exact equation for $\partial_t n$:

1. 2PI loop expansion of $\Gamma[G]$ to three-loop order

   $\rightsquigarrow$ self-energy corrections up to two loops:  , 

2. quasiparticle ansatz (free-field form):

$$\begin{aligned} F(t'', t; \mathbf{p}) &\to (n_{\mathbf{p}} + 1/2) \cos\left[\omega_{\mathbf{p}}(t'' - t)\right] / \omega_{\mathbf{p}} \\ \rho(t'', t; \mathbf{p}) &\to \sin\left[\omega_{\mathbf{p}}(t'' - t)\right] / \omega_{\mathbf{p}} \end{aligned}$$

$$\begin{aligned} \partial_t F(t'', t; \mathbf{p}) &\to (n_{\mathbf{p}} + 1/2) \sin\left[\omega_{\mathbf{p}}(t'' - t)\right] \\ \partial_t \rho(t'', t; \mathbf{p}) &\to -\cos\left[\omega_{\mathbf{p}}(t'' - t)\right] \end{aligned}$$

3. consider time dependent $n_{\mathbf{p}} \to n_{\mathbf{p}}(t)$ with t the latest time, equivalently $\omega_{\mathbf{p}} \to \omega_{\mathbf{p}}(t) = \sqrt{\mathbf{p}^2 + M^2(t)}$

4. send the initial time t_0 to the remote past: $t_0 \to -\infty$

Alternative derivation (standard): employs first-order gradient expansion in Wigner coordinates and quasiparticle ansatz (δ-form for spectral function ρ)

Assumptions 1. – 3. straightforwardly lead to "gain & loss" structure:

$$\partial_t n_{\mathbf{p}}(t) \;\to\; \frac{\lambda^2}{3}\int_{\mathbf{sqk}} (2\pi)^d \delta(\mathbf{p}-\mathbf{q}-\mathbf{k}-\mathbf{s})\frac{1}{2\omega_{\mathbf{p}}2\omega_{\mathbf{q}}2\omega_{\mathbf{k}}2\omega_{\mathbf{s}}}$$

$$\Big\{[(1+n_{\mathbf{p}})(1+n_{\mathbf{q}})(1+n_{\mathbf{k}})(1+n_{\mathbf{s}}) - n_{\mathbf{p}}n_{\mathbf{q}}n_{\mathbf{k}}n_{\mathbf{s}}] \qquad \textbf{(I)}$$

$$\int_{t_0}^{t} dt'' \cos\left[(\omega_{\mathbf{p}}+\omega_{\mathbf{q}}+\omega_{\mathbf{k}}+\omega_{\mathbf{s}})(t-t'')\right]$$

$$+\;\; 3\left[(1+n_{\mathbf{p}})(1+n_{\mathbf{q}})(1+n_{\mathbf{k}})n_{\mathbf{s}} - n_{\mathbf{p}}n_{\mathbf{q}}n_{\mathbf{k}}(1+n_{\mathbf{s}})\right] \qquad \textbf{(II)}$$

$$\int_{t_0}^{t} dt'' \cos\left[(\omega_{\mathbf{p}}+\omega_{\mathbf{q}}+\omega_{\mathbf{k}}-\omega_{\mathbf{s}})(t-t'')\right]$$

$$+\;\; 3\left[(1+n_{\mathbf{p}})(1+n_{\mathbf{q}})n_{\mathbf{k}}n_{\mathbf{s}} - n_{\mathbf{p}}n_{\mathbf{q}}(1+n_{\mathbf{k}})(1+n_{\mathbf{s}})\right] \qquad \textbf{(III)}$$

$$\int_{t_0}^{t} dt'' \cos\left[(\omega_{\mathbf{p}}+\omega_{\mathbf{q}}-\omega_{\mathbf{k}}-\omega_{\mathbf{s}})(t-t'')\right]$$

$$+\;\; \left[(1+n_{\mathbf{p}})n_{\mathbf{q}}n_{\mathbf{k}}n_{\mathbf{s}} - n_{\mathbf{p}}(1+n_{\mathbf{q}})(1+n_{\mathbf{k}})(1+n_{\mathbf{s}})\right] \qquad \textbf{(IV)}$$

$$\int_{t_0}^{t} dt'' \cos\left[(\omega_{\mathbf{p}}-\omega_{\mathbf{q}}-\omega_{\mathbf{k}}-\omega_{\mathbf{s}})(t-t'')\right]\Big\}$$

Simple interpretation in the 'quasiparticle' limit:

(I) production and annihilation of four 'quasiparticles' ($0 \to 4$, $4 \to 0$)

(II) and (IV) describe $1 \to 3$ and $3 \to 1$ processes

(III) describes $2 \leftrightarrow 2$ scattering processes ('Boltzmann equation')

- Only (III) leads to a non-zero contribution for the limit

$$\lim_{(t-t_0)\to\infty} \int_{t_0}^{t} \mathrm{d}t'' \cos\left[\omega(t-t'')\right] = \lim_{(t-t_0)\to\infty} \frac{\sin\left[\omega(t-t_0)\right]}{\omega} = \pi\delta(\omega)$$

$\rightsquigarrow$ *energy δ-functions for (I), (II), (IV) cannot be fulfilled for on-shell 'quasiparticles'*

- *For one spatial dimension the contribution (III) vanishes as well:*

$$\begin{aligned} & p + q - k - s = 0 \\ \wedge \quad & \sqrt{p^2 + M^2} + \sqrt{q^2 + M^2} - \sqrt{k^2 + M^2} - \sqrt{s^2 + M^2} = 0 \end{aligned}$$

$\rightsquigarrow$ 'ineffective' two-to-two scattering with $p = k$, $q = s$

Conclude: The effective loss of nonequilibrium initial $n_0(\mathbf{p})$ leading to *late-time thermalization must be due to 'off-shell' processes* in $1+1$ d

$\rightsquigarrow$ $\boxed{\text{requires nontrivial (i.e. not '}\delta\text{'-) form of spectral function } \rho}$

Consider the Fourier transform in Wigner coordinates $X^0 = (t+t')/2$, $s^0 = t - t'$: (bounded integral by $\pm 2X^0$ because of finite initial time)

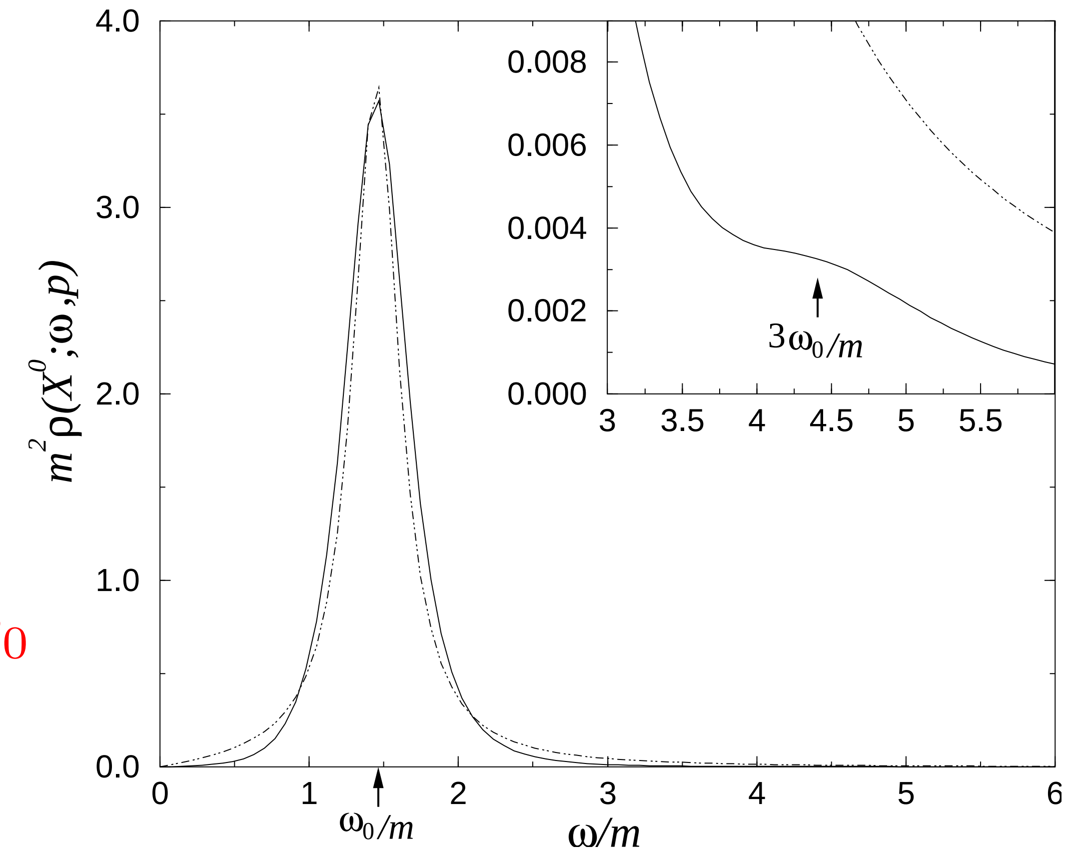
$$\rho(X^0;\omega,p) = -i \int_{-2X^0}^{2X^0} ds^0 \, e^{i\omega s^0} \rho(X^0 + s^0/2, X^0 - s^0/2; p)$$

Spectral function:

(full 2PI three-loop,
'tsunami' with $\lambda/m^2 = 4$
$mX^0 = 35.1$ for $p = 0$)

- **nonzero 'width'**
- **small second peak at $3\omega_0$**

(Dotted line: Breit-Wigner fit)



$\rightsquigarrow$ **Characteristic time scales** associated to

1. rapid oscillations of correlation functions with period $\sim 1/\epsilon_{\mathbf{p}}$
2. damping of oscillations with inverse rate $1/\gamma_{\mathbf{p}}^{(\mathrm{damp})}$

   $\rightsquigarrow$ *described by nonzero 'width'* $\Gamma_{\mathbf{p}} = 2\gamma_{\mathbf{p}}^{(\mathrm{damp})}$ *of* $\rho(\omega;\mathbf{p})$:

- $\Gamma_{\mathbf{p}}$ can be estimated by equilibrium 'on-shell' width $\Gamma^{(\mathrm{eq})}(\omega = \epsilon_{\mathbf{p}}, \mathbf{p})$ with 'on-shell' energy $\epsilon_{\mathbf{p}}$ where

$$2\omega\Gamma^{(\mathrm{eq})}(\omega,\mathbf{p}) \equiv -\Sigma_{\rho}^{(\mathrm{eq})}(\omega,\mathbf{p}) \;\sim\; \mathcal{O}(\lambda^2/N)$$

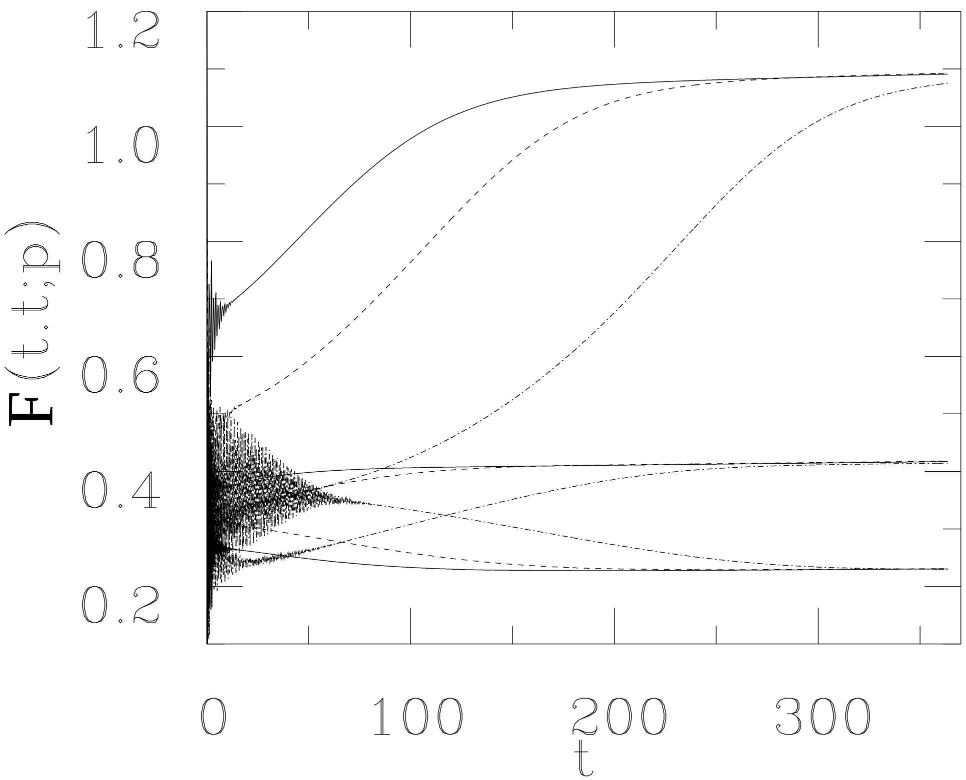
3. 'late-time' thermalization with inverse rate $1/\gamma_{\mathbf{p}}^{(\mathrm{therm})}$

   $\rightsquigarrow$ *because of 'off-shell' number changing processes*:

   Number changing processes require nonzero 'width' $\sim \lambda^2/N$ effect in $\mathcal{O}(\lambda^2/N)$ evolution equation for $n_{\mathbf{p}}(t)$:

   $\rightsquigarrow$ expected suppression by additional power of λ^2/N ("slow!")

Overview:



Propagator for (very) different initial conditions with *same* $\langle E \rangle$

$p = 0, 3, 5$; all in initial mass units

(here: 3-loop, $1+1$ dimensions, $\phi \equiv 0$)

Initial conditions

'Damping'

exponential suppression of correlations with initial time

rate: $\gamma^{(\mathrm{damp})}$

$F(t,0)=\langle \phi(t)\, \phi(0) \rangle \longrightarrow 0^+$

'Drifting'

smooth, parametrically slow change of modes

weak dependence of $F(t,t')$ on $t + t'$

'Thermalization'

approach to quantum thermal equilibrium

rate: $\gamma^{(\mathrm{therm})} \neq \gamma^{(\mathrm{damp})}$

$F(t,t') \longrightarrow F^{(\mathrm{eq})}(t-t')$

Effective loss of initial conditions

early **intermediate** **late** **time**

Summary: Far-from-equilibrium dynamics beyond LO

- Three characteristic regimes:
  1. Rapid loss of details of initial conditions $\sim 1/\gamma^{(\mathrm{damp})}$ (system typically still far from equilibrium)
  2. Smooth change of correlations wrt $t+t'$ ($\rightsquigarrow$ gradient expansion relevant for kinetic descriptions!) at intermediate times
  3. Late-time approach to thermal equilibrium
- 1.–3. not described by LO (or mean-field/Hartree) approximation
- Boltzmann equation may only be used for times $t > 1/\gamma^{(\mathrm{damp})}$. In $1+1$ dimensions ($2 \leftrightarrow 2$ processes) fail to describe 2. and 3.

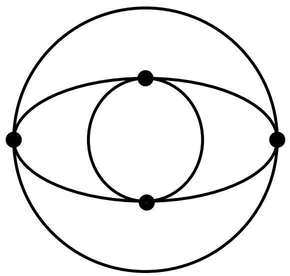
$\rightsquigarrow$ Small 'off-shell' contributions crucial for effective loss of initial conditions, i.e. *to reach* thermal equilibrium

$\rightsquigarrow$ Small 'off-shell' contributions not crucial for certain aspects *in* thermal equilibrium (good quasiparticle description for ϕ^4!) ($\rightsquigarrow$ 'initial'-value problem/nonequilibrium very different from 'boundary'-value/equilibrium problem)

Implications for 'realistic' $3+1$ dimensions:

- One finds qualitatively same *three characteristic ranges* as in $1+1\ d$ (Cf. also chiral 'quark-meson' model below)
- Important quant. difference: *'on-shell' $2 \leftrightarrow 2$ processes contribute* However: they do *not change integrated 'quasiparticle' number*
- Global 'quasiparticle' number changing 'off-shell'/'on-shell' processes required in general to reach thermal equilibrium

$\rightsquigarrow$ Question whether *'off-shell' at NLO 2PI* dominante over *'on-shell' number changing processes appearing at NNLO.* E.g. ϕ^4: lowest order 'on-shell' $(2 \to 4)$ contribution at 2PI five-loop/NNLO:



How important are NNLO and higher corrections?
$\rightsquigarrow$ Precision test!

Classical statistical field theory limit: Precision test

Classical equation of motion for N-component field $\phi_a(x)$:

$$\left[\Box_x + m^2 + \lambda\phi_b(x)\phi_b(x)/6N\right]\phi_a(x) = 0$$

Define classical 'statistical' two-point function:

$$F_{ab,\mathrm{cl}}(x,y) = \langle\phi_a(x)\phi_b(y)\rangle_{\mathrm{cl}} \equiv \int D\pi D\phi\, W[\pi,\phi]\phi_a(x)\phi_b(y)$$

$W[\pi,\phi]$: normalized probability functional at initial time with canonical momentum $\pi_a(x) = \partial_t\phi_a(x)$, $\phi_a(0,\mathbf{x}) \equiv \phi_a(\mathbf{x})$, $\pi_a(0,\mathbf{x}) \equiv \pi_a(\mathbf{x})$

Integration over classical phase-space: $\int D\pi D\phi = \int \prod_{a=1}^{N} \prod_{\mathbf{x}} \mathrm{d}\pi_a(\mathbf{x})\mathrm{d}\phi_a(\mathbf{x})$

Similarly, define classical spectral function:

$$\begin{aligned}\rho_{ab,\mathrm{cl}}(x,y) &= -\langle\, [\phi_a(x),\phi_b(y)]_{\mathrm{PoissonBracket}}\, \rangle_{\mathrm{cl}} \\ \Rightarrow\ \rho_{ab,\mathrm{cl}}(x,y)|_{x^0=y^0} &= 0, \quad \partial_{x^0}\rho_{ab,\mathrm{cl}}(x,y)|_{x^0=y^0} = \delta_{ab}\delta(\mathbf{x}-\mathbf{y})\end{aligned}$$

A) The classical statistical field theory can be simulated "exactly"

- up to controlled statistical errors
- by numerical integration of the field equation and MC methods

$\rightsquigarrow$ $\boxed{\text{Numerical computation of } F_{\rm cl},\ \rho_{\rm cl} \text{ including ``all orders in } 1/N\text{''}}$

B) Analytical form of the exact classical evolution equations for $F_{\rm cl}$ and $\rho_{\rm cl}$ is *identical* to the respective quantum ones with replacements

$$\Sigma^F \;\to\; \Sigma^F_{\rm cl} \;=\; \Sigma^F(F^2 \gg \rho^2)$$
$$\Sigma^\rho \;\to\; \Sigma^\rho_{\rm cl} \;=\; \Sigma^\rho(F^2 \gg \rho^2)$$

$\rightsquigarrow$ $\boxed{\text{2PI } 1/N \text{ expansion gives } F_{\rm cl},\ \rho_{\rm cl} \text{ up to given order in } 1/N}$

C) For *same initial conditions* direct comparison of nonequilibrium classical and quantum evolution possible

Note: classical limit suffers from Rayleigh-Jeans divergences and has to be regulated

To B): Proof either using resummed classical perturbation theory or classical functional methods (cf. ref.)

Consider self-energies at three-loop order (equivalently for NLO $1/N$):

$$
\begin{aligned}
\Sigma^F(t,t';\mathbf{p}) &= -\frac{\lambda^2}{6}\int_{\mathbf{q},\mathbf{k}} F(t,t';\mathbf{p}-\mathbf{q}-\mathbf{k}) \\
&\quad \left[F(t,t';\mathbf{q})F(t,t';\mathbf{k}) - \frac{3}{4}\rho(t,t';\mathbf{q})\rho(t,t';\mathbf{k})\right] \\
\Sigma^\rho(t,t';\mathbf{p}) &= -\frac{\lambda^2}{2}\int_{\mathbf{q},\mathbf{k}} \rho(t,t';\mathbf{p}-\mathbf{q}-\mathbf{k}) \\
&\quad \left[F(t,t';\mathbf{q})F(t,t';\mathbf{k}) - \frac{1}{12}\rho(t,t';\mathbf{q})\rho(t,t';\mathbf{k})\right]
\end{aligned}
$$

$$
\begin{aligned}
\Sigma^F_{\rm cl}(t,t';\mathbf{p}) &= -\frac{\lambda^2}{6}\int_{\mathbf{q},\mathbf{k}} F(t,t';\mathbf{p}-\mathbf{q}-\mathbf{k})F(t,t';\mathbf{q})F(t,t';\mathbf{k}) \\
\Sigma^\rho_{\rm cl}(t,t';\mathbf{p}) &= -\frac{\lambda^2}{2}\int_{\mathbf{q},\mathbf{k}} \rho(t,t';\mathbf{p}-\mathbf{q}-\mathbf{k})F(t,t';\mathbf{q})F(t,t';\mathbf{k})
\end{aligned}
$$

$\Rightarrow$ *sufficient* condition for classical evolution:

$$\boxed{|F(t,t';\mathbf{q})F(t,t';\mathbf{k})| \gg \frac{3}{4}\,|\rho(t,t';\mathbf{q})\rho(t,t';\mathbf{k})|}$$

i.e. for *all* times and momenta

1. First estimate on required lower bound for effective particle number:

For 'quasiparticles' (in the sense used above to derive Boltzmann eq.)

$$\overline{F^2}(t,t';\mathbf{p}) \equiv \frac{\omega_\mathbf{p}}{2\pi}\int_{t-2\pi/\omega_\mathbf{p}}^{t}\mathrm{d}t'\,F^2(t,t';\mathbf{p}) \;\rightarrow\; \frac{(n_\mathbf{p}(t)+1/2)^2}{2\omega_\mathbf{p}^2(t)}$$

and for time-averaged spectral function: $\overline{\rho^2}(t,t';\mathbf{p}) \rightarrow 1/2\omega_\mathbf{p}^2(t)$

$$\rightsquigarrow \qquad \left[n_\mathbf{p}(t)+\frac{1}{2}\right]^2 \gg \frac{3}{4} \qquad \text{or} \qquad n_\mathbf{p}(t) \gg 0.37$$

Compare with Bose-Einstein: below $n_{\mathrm{BE}}(\omega_\mathbf{p}=T) \simeq 0.58$ deviations from classical thermal distribution become sizeable

2. Thermal equilibrium: *fluctuation-dissipation relation* ensures $F^{(\mathrm{eq})2}(\omega,\mathbf{p}) \gg \rho^{(\mathrm{eq})2}(\omega,\mathbf{p})$ for high temperature modes $T \gg \omega$:

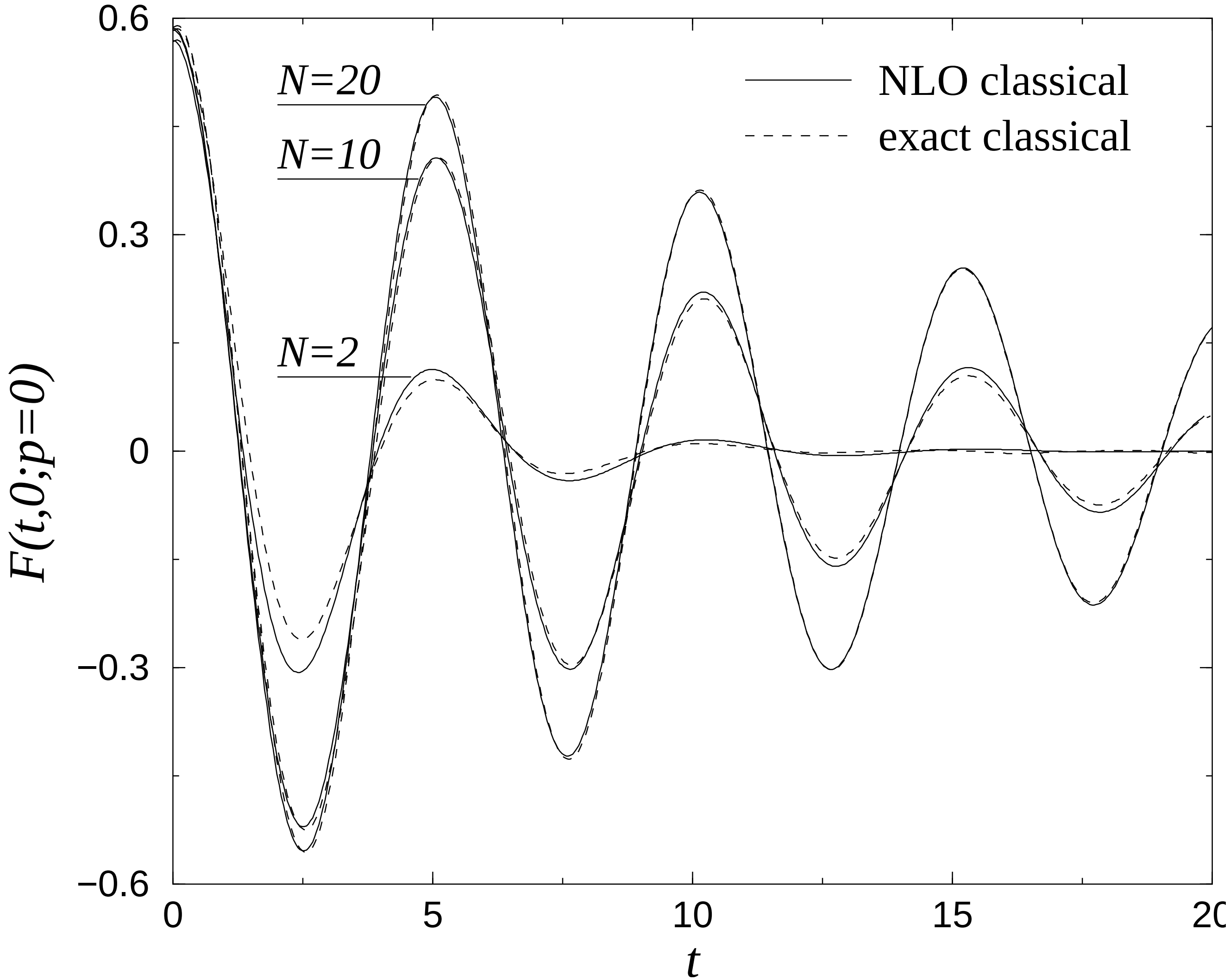
$$F^{(\mathrm{eq})}(\omega,\mathbf{p}) = -i\Big(n_{\mathrm{BE}}(\omega) + \frac{1}{2}\Big)\,\rho^{(\mathrm{eq})}(\omega,\mathbf{p}) \overset{T\gg\omega}{\simeq} -i\,\frac{T}{\omega}\,\rho^{(\mathrm{eq})}(\omega,\mathbf{p})$$

$\overset{1.,2.}{\rightsquigarrow}$ Expectation: F, ρ well approximated by F_{cl}, ρ_{cl} for large initial $n_0(\mathbf{p}) = n_{\mathbf{p}}(t=0)$ and not too late times ($\sim 1/\gamma^{(\mathrm{therm})}$), before the approach to quantum thermal equilibrium sets in

1. Damping (effective loss of details of initial conditions)

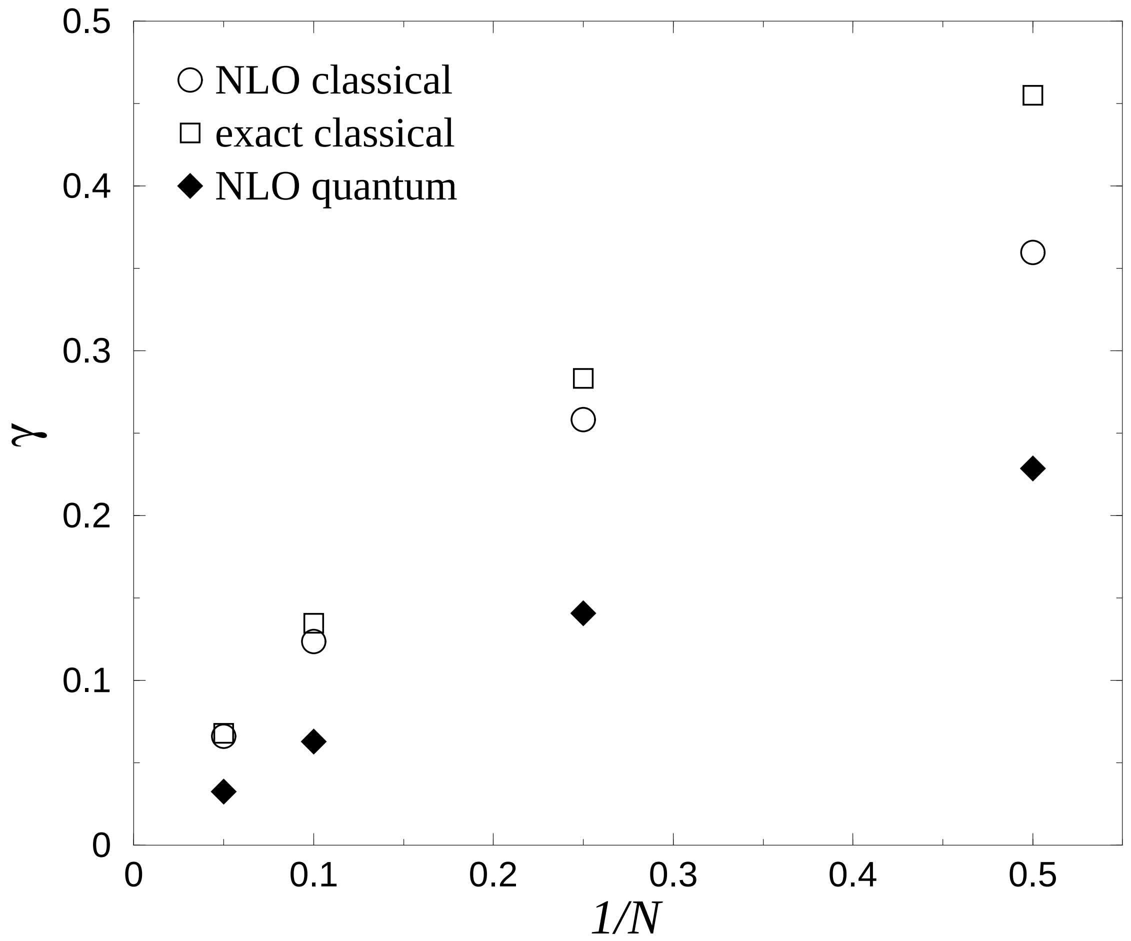
$(1+1)\ d$

$$\frac{\lambda}{M_0^2} = 30$$



$\Rightarrow$ Convergence of classical *NLO* and *exact* (MC) results already for moderate values of N (!)

Parametric behavior/comparison with quantum theory:



- Inverse damping rate $1/\gamma$ scales $\sim N$ for large N

$\rightsquigarrow$ zero damping at LO ($N \to \infty$): LO breakdown for times $t \sim 1/\gamma$

- Damping enhanced if quantum corrections are neglected

$$\boxed{\text{Here: } \quad n_0(p) \ll \frac{1}{2} \quad \rightsquigarrow \quad \text{quantum corrections important!}}$$

2. "Late-time" (i.e. for quantum theory) behavior:

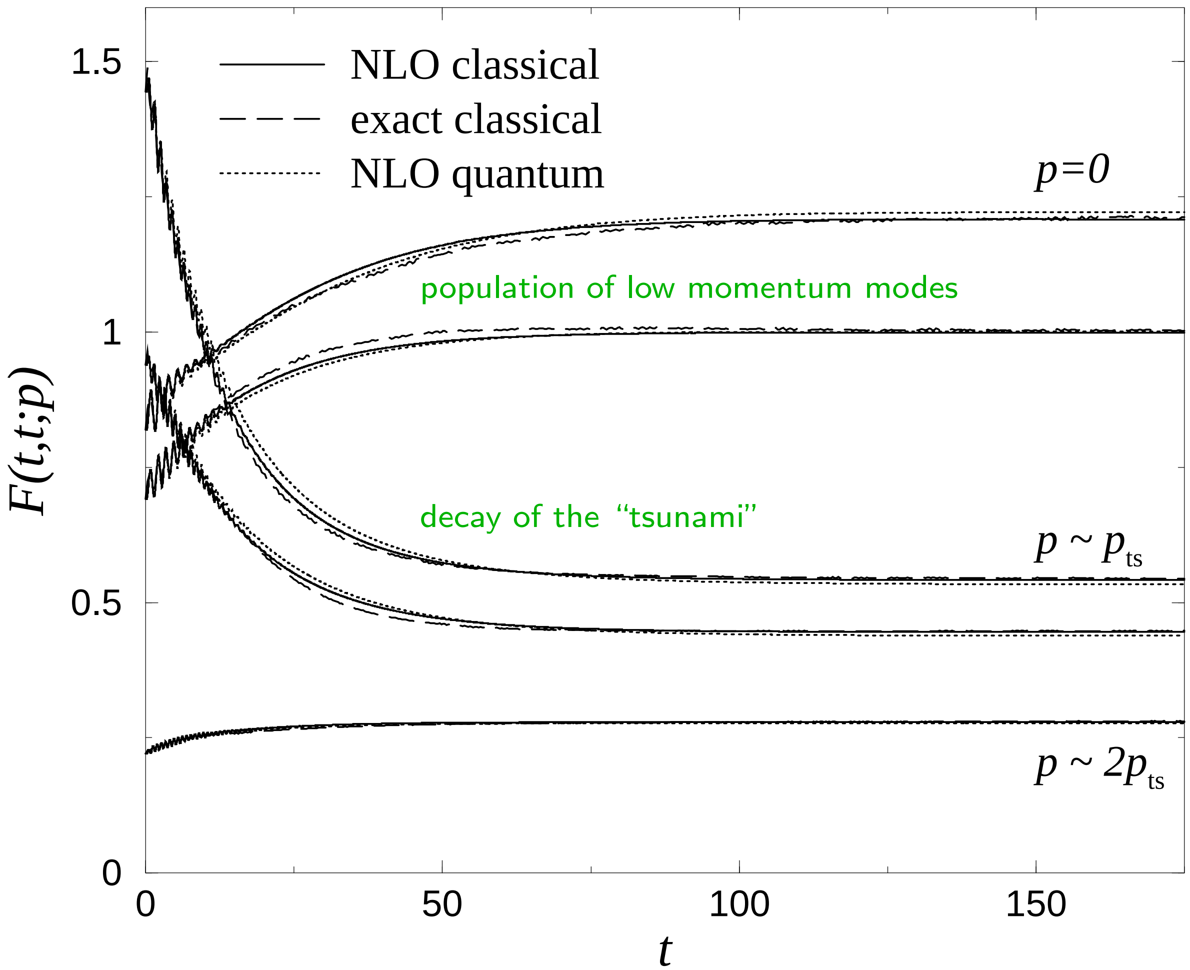
High initial particle number n_0 peaked around $p = |p_{\rm ts}| \simeq 4M_0$

"Tsunami" :

$(1+1)\, d$

$\boxed{\text{N=4 (!)}}$

$\lambda = 12\, M_0^{-2}$

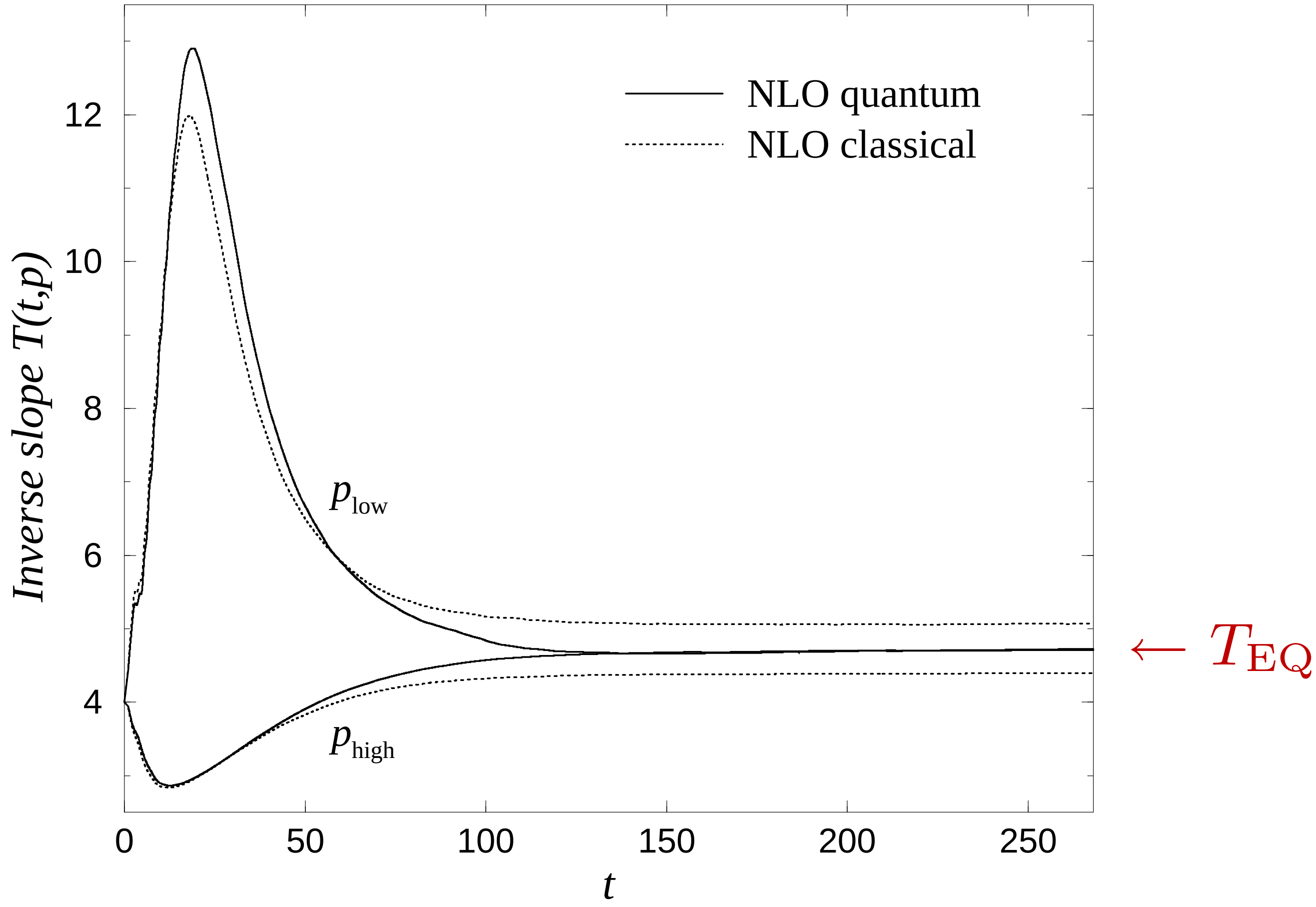


- *NLO* quantum evolution well approximated by *exact* classical result!

Here: $n_0(p_{\rm ts}) \gg \dfrac{1}{2}$ and $n_0(p) \gtrsim 0.35$ for $p \lesssim 2p_{\rm ts}$

$\leadsto$ small quantum corrections for displayed modes

Quantum vs classical thermalization:



Inverse slope parameter*: $T(t,p) \equiv -n(t,\epsilon_p)[n(t,\epsilon_p)+1](dn/d\epsilon_p)^{-1}$

$\Rightarrow$ constant for $n(t,\epsilon_p) = 1/[e^{\epsilon_p/T_{\rm EQ}} - 1]$ (Bose-Einstein) *cf. above

- Classical theory does (of course) not reach Bose-Einstein distribution

Summary: Classical statistical limit/precision test

- 2PI $1/N$ expansion in *classical statistical field theory:*
  - $\rightsquigarrow$ Strong qualitative difference between LO and NLO
  - $\rightsquigarrow$ Quantitatively good agreement between NLO and *exact* results already for moderate values of N
  - $\rightsquigarrow$ No indications for relevance of NNLO 'on-shell' number changing processes *though NLO 'off-shell' processes crucial in $1+1$ d!*
- *NLO quantum theory* well approximated by (NLO/exact) classical
  - $\rightsquigarrow$ for sufficiently high effective particle number per mode
  - $\rightsquigarrow$ not too late times (quantum vs 'classical' thermal equilibrium)

2PI $1/N$ expansion beyond leading order:

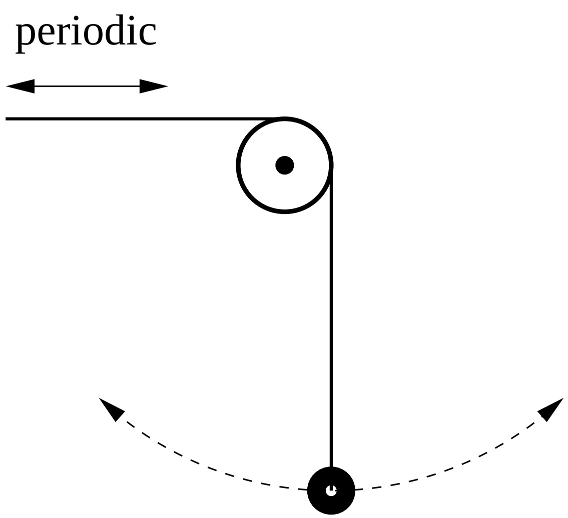
- provides powerful quantitative tool for nonequilibrium dynamics from 'first principles' with a *nonperturbative* expansion parameter

$\rightsquigarrow$ can be used in particular to study far-from-equilibrium dynamics with large fluctuations!

Application I: Parametric resonance in QFT

$\rightsquigarrow$ 'paradigm' for far-from-equilibrium dynamics with large fluctuations

- Remember classical mechanics: resonant amplitude amplification of a *'driven'* oscillator with time-dependent periodic frequency



$\ddot{x} + \omega^2(t)x = 0\,, \quad \omega(t+T) = \omega(t)$

invariant under $t \to t + T$:

$\rightsquigarrow \quad x(t+T) = c\,x(t)$, i.e.

$x(t) = c^{t/T}\,\Pi(t)\,, \quad \Pi(t+T) = \Pi(t)$

for real $c > 1$:

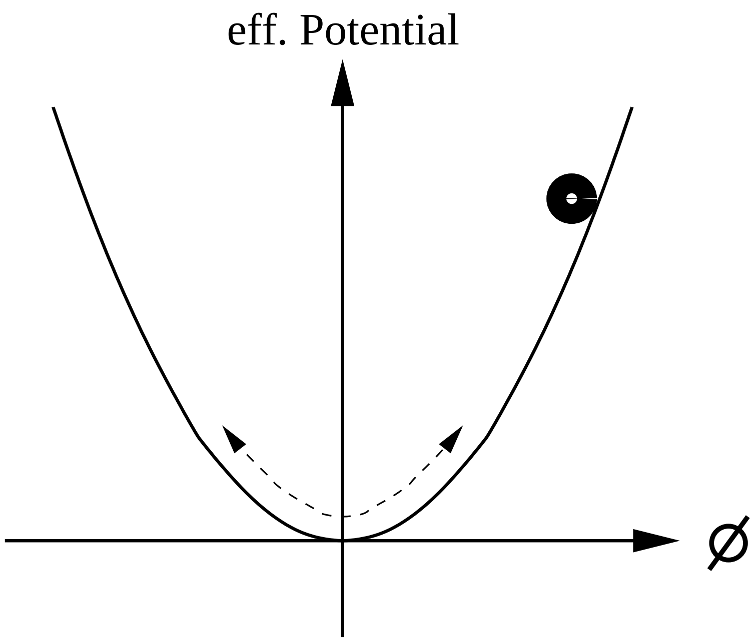
instability with *exponential growth!*

- Cosmology: Preheating after inflation in the early universe

  large coherent field amplitude $\rightsquigarrow$ large quantum fluctuations ("particle production")

- Operative in heavy–ion collisions?

In quantum field theory an oscillating macroscopic field $\phi = \langle\Phi\rangle$ can trigger the resonant amplification of fluctuations $G = \langle\Phi\Phi\rangle - \langle\Phi\rangle^2$ (*no external source!*)



'<u>End of inflation</u>':

Large field: $\phi \sim \mathcal{O}(1/\sqrt{\lambda})\,,\ \lambda \ll 1$

Small fluctuations: $G \sim \mathcal{O}(1) \ll \phi^2$

<u>Parametric resonance</u>:

$G \sim e^{\gamma t}$ until $G \sim \mathcal{O}(1/\lambda)$

Nonperturbatively large $G \sim \mathcal{O}(1/\lambda)$:

$\rightsquigarrow$ *2PI loop or coupling expansion not applicable!*
(infinite number 2PI diagrams of $\mathcal{O}(\lambda^0)$)

$\rightsquigarrow$ *2PI $1/N$ expansion beyond LO can provide controlled description!*

Parametric resonance in the $O(N)$ model:

Initial conditions: Pure–state density matrix (Gaussian), i.e. zero initial particle number, $n_0 \equiv 0$

- Large homogeneous field amplitude ($\lambda \ll 1$)

$$\phi_a(t) = \sigma(t) M_0 \sqrt{6N/\lambda}\, \delta_{a1}$$

with $\sigma(0) = \sigma_0$ and $\partial_t \sigma(t)|_{t=0} = 0$

- Initial propagator $F_{ab} = \mathrm{diag}\{F_\parallel, F_\perp, \ldots, F_\perp\} \sim \mathcal{O}(1)$

with $\partial_t F(t, 0; \mathbf{p})|_{t=0} = 0$, (Pure-state density matrix:

$$\rightsquigarrow \;\; \partial_t \partial_{t'} F(t, t'; \mathbf{p})|_{t=t'=0} \equiv F^{-1}(0, 0; \mathbf{p})/4 \;)$$

To set the scale we use the initial longitudinal mass^2: $M_0^2 \equiv M^2(t=0)$

$$\begin{aligned} M^2(t) &= m^2 + \frac{\lambda}{6N}\Big[3T_\parallel(t) + (N-1)T_\perp(t)\Big] \\ T_{\parallel,\perp}(t) &= \int^\Lambda \frac{\mathrm{d}^3 p}{(2\pi)^3} F_{\parallel,\perp}(t, t; \mathbf{p}) \end{aligned}$$

NLO self-energies: ('minimal' approximation for an analytical description!)

$$\begin{aligned} \Sigma^F_{ab}(x,y) &= -\frac{\lambda}{3N}\Big\{\mathbf{I}_F(x,y)\left[\phi_a(x)\phi_b(y)+F_{ab}(x,y)\right]-\frac{1}{4}\mathbf{I}_\rho(x,y)\rho_{ab}(x,y) \\ &\quad +\mathbf{P}_F(x,y)F_{ab}(x,y)-\frac{1}{4}\mathbf{P}_\rho(x,y)\rho_{ab}(x,y)\Big\}, \end{aligned} \tag{1}$$

$$\begin{aligned} \Sigma^\rho_{ab}(x,y) &= -\frac{\lambda}{3N}\Big\{\mathbf{I}_\rho(x,y)\left[\phi_a(x)\phi_b(y)+F_{ab}(x,y)\right]+\mathbf{I}_F(x,y)\rho_{ab}(x,y) \\ &\quad +\mathbf{P}_\rho(x,y)F_{ab}(x,y)+\mathbf{P}_F(x,y)\rho_{ab}(x,y)\Big\}, \end{aligned} \tag{2}$$

with 'chain' summation functions:

$$\begin{aligned} \mathbf{I}_F(x,y) &= \frac{\lambda}{6N}\left(F^2_{ab}(x,y)-\frac{1}{4}\rho^2_{ab}(x,y)\right) \\ &- \frac{\lambda}{6N}\int d\mathbf{z}\left\{\int_0^{x^0} dz^0\,\mathbf{I}_\rho(x,z)\left(F^2_{ab}(z,y)-\frac{1}{4}\,\rho^2_{ab}(z,y)\right)\right. \\ &\qquad \left. -\,2\int_0^{y^0} dz^0\,\mathbf{I}_F(x,z)F_{ab}(z,y)\rho_{ab}(z,y)\right\}\;, \end{aligned} \tag{3}$$

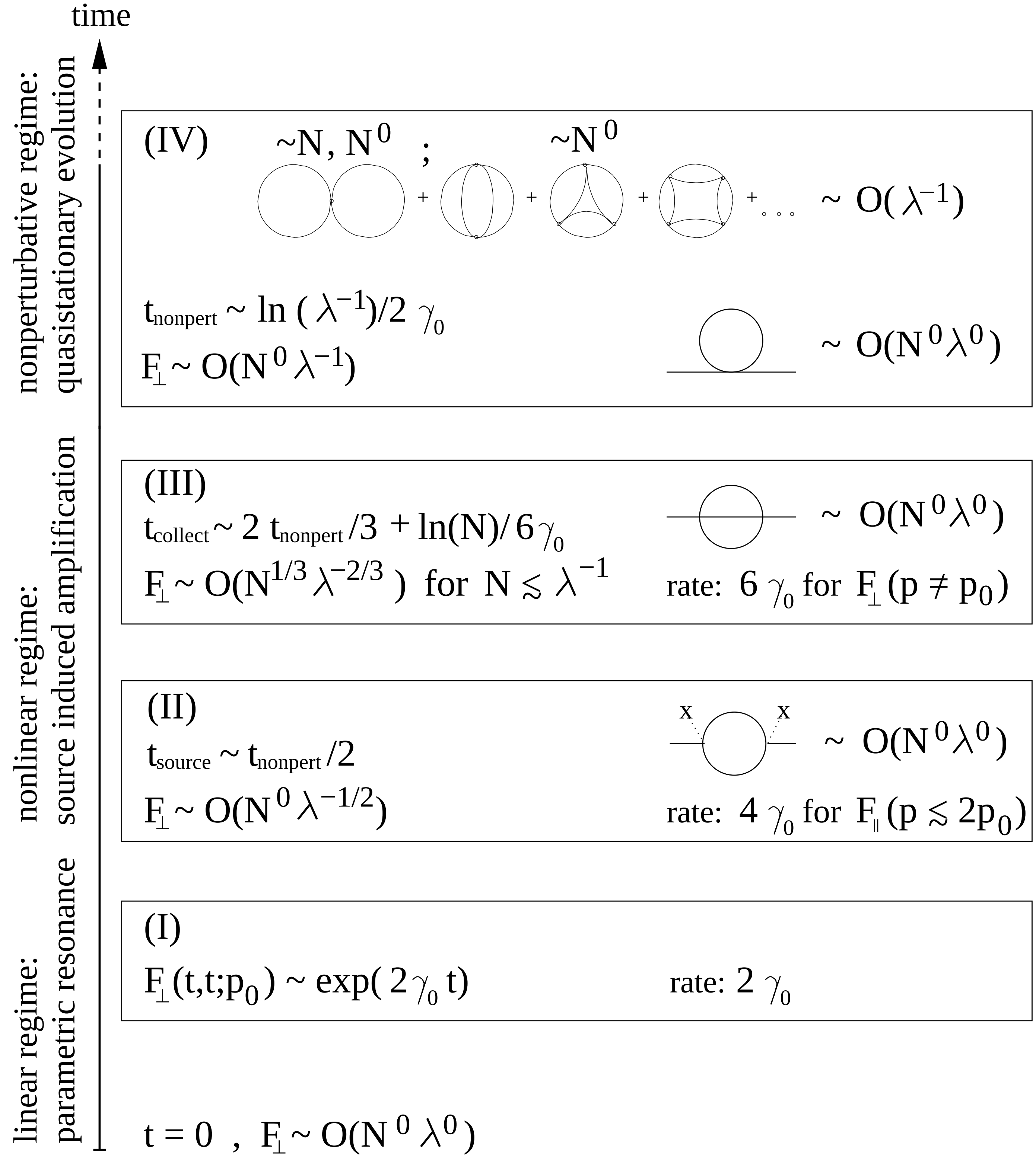
$$\mathbf{I}_\rho(x,y) = \frac{\lambda}{3N}F_{ab}(x,y)\rho_{ab}(x,y)-\frac{\lambda}{3N}\int d\mathbf{z}\int_{y^0}^{x^0} dz^0\,\mathbf{I}_\rho(x,z)F_{ab}(z,y)\rho_{ab}(z,y)\;, \tag{4}$$

$$
\begin{aligned}
\mathbf{P}_F(x,y) \;=\; & -\frac{\lambda}{3N}\Big\{\Delta_F(x,y) \\
& -\int_0^{x^0} dz\;[\Delta_\rho(x,z)\mathbf{I}_F(z,y)+\mathbf{I}_\rho(x,z)\Delta_F(z,y)] \\
& +\int_0^{y^0} dz\;[\Delta_F(x,z)\mathbf{I}_\rho(z,y)+\mathbf{I}_F(x,z)\Delta_\rho(z,y)] \\
& -\int_0^{x^0} dz\int_0^{y^0} dv\,\mathbf{I}_\rho(x,z)\Delta_F(z,v)\mathbf{I}_\rho(v,y) \\
& +\int_0^{x^0} dz\int_0^{z^0} dv\,\mathbf{I}_\rho(x,z)\Delta_\rho(z,v)\mathbf{I}_F(v,y) \\
& +\int_0^{y^0} dz\int_{z^0}^{y^0} dv\,\mathbf{I}_F(x,z)\Delta_\rho(z,v)\mathbf{I}_\rho(v,y)\Big\},
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\mathbf{P}_\rho(x,y) \;=\; & -\frac{\lambda}{3N}\Big\{\Delta_\rho(x,y) \\
& -\int_{y^0}^{x^0} dz\;[\Delta_\rho(x,z)\mathbf{I}_\rho(z,y)+\mathbf{I}_\rho(x,z)\Delta_\rho(z,y)] \\
& +\int_{y^0}^{x^0} dz\int_{y^0}^{z^0} dv\,\mathbf{I}_\rho(x,z)\Delta_\rho(z,v)\mathbf{I}_\rho(v,y)\Big\},
\end{aligned} \tag{6}
$$

with $\Delta_F(x,y) = -\phi_a(x)F_{ab}(x,y)\phi_b(y)$ and $\Delta_\rho(x,y) = -\phi_a(x)\rho_{ab}(x,y)\phi_b(y)$.

Overview:



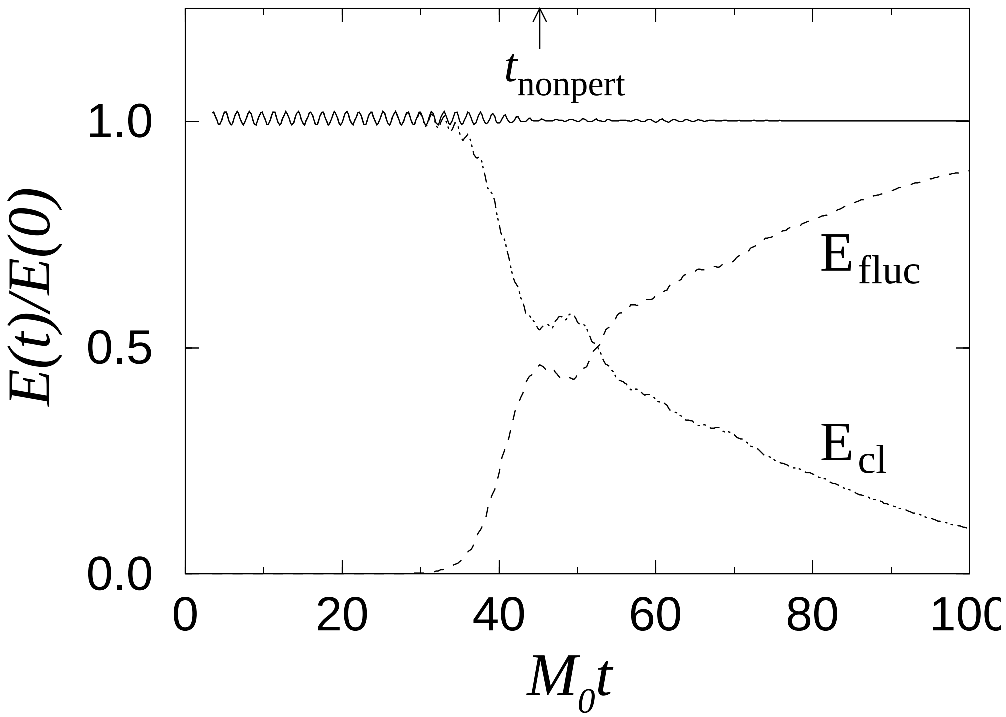
time
nonperturbative regime: quasistationary evolution
nonlinear regime: source induced amplification
linear regime: parametric resonance
(IV)
$\sim N, N^0$; $\sim N^0$
$\sim O(\lambda^{-1})$
$t_{nonpert} \sim \ln(\lambda^{-1})/2\gamma_0$
$F_\perp \sim O(N^0 \lambda^{-1})$
$\sim O(N^0 \lambda^0)$
(III)
$t_{collect} \sim 2\, t_{nonpert}/3 + \ln(N)/6\gamma_0$
$F_\perp \sim O(N^{1/3} \lambda^{-2/3})$ for $N \lesssim \lambda^{-1}$
$\sim O(N^0 \lambda^0)$
rate: $6\gamma_0$ for $F_\perp (p \neq p_0)$
(II)
$t_{source} \sim t_{nonpert}/2$
$F_\perp \sim O(N^0 \lambda^{-1/2})$
x
x
$\sim O(N^0 \lambda^0)$
rate: $4\gamma_0$ for $F_\parallel (p \lesssim 2p_0)$
(I)
$F_\perp(t,t;p_0) \sim \exp(2\gamma_0 t)$
rate: $2\gamma_0$
$t = 0$, $F_\perp \sim O(N^0 \lambda^0)$

Total energy $E_{\rm tot} = \rm const$ initially dominated by classical $E_{\rm cl}(t=0)$:

$$\begin{aligned} E_{\rm cl}(t)/V &= \frac{1}{2}\partial_t\phi_a(t)\partial_t\phi_a(t) + \frac{m^2}{2}\phi_a(t)\phi_a(t) + \frac{\lambda}{4!N}[\phi_a(t)\phi_a(t)]^2 \\ &= \frac{3N}{\lambda}M_0^2\left[\partial_t\partial_{t'} + m^2 + \frac{1}{2}M_0^2\sigma^2(t)\right]\sigma(t)\sigma(t')\Big|_{t=t'} \\ &\sim \mathcal{O}(N\lambda^{-1}) \gg E_{\rm fluc}/V \end{aligned}$$

with $E_{\rm fluc} = E_{\rm tot} - E_{\rm cl} \sim \mathcal{O}(N\lambda^0)$

Classical field regime $\xrightarrow{\text{‘particle production‘}}$ fluctuation dominated regime:



$\boxed{t_{\rm nonpert}:\ E_{\rm fluc} \simeq E_{\rm cl}}$

$(N=4,\ \lambda=10^{-6},\ 3+1\ d)$

(I) Early-time (linear) regime: *Parametric resonance*

$\mathcal{O}(\lambda^0)$ evolution equations: $(M_0 \equiv 1)$

1. Classical equation of motion for the field:

$$\partial_t^2 \sigma(t) + \sigma(t) + \sigma^3(t) = 0$$

Periodic solution with $\sigma(t + \pi/\omega_0) = -\sigma(t)$,

(Jacobian cosine: $\sigma(t) = \sigma_0 \, \mathrm{cn}[t\sqrt{1+\sigma_0^2}, \sigma_0/\sqrt{2(1+\sigma_0^2)}]$,
period average $\overline{\sigma^2(t)} \simeq \sigma_0^2/2$)

$\rightsquigarrow$ $\boxed{\text{rapid oscillations with characteristic frequency } \omega_0 \simeq 2\sqrt{1+\sigma_0^2}}$

2. Free-field like equations with time–dependent mass terms for $F_{\parallel,\perp}$:

$$\begin{aligned} \left[\partial_t^2 + \mathbf{p}^2 + 1 + \sigma^2(t)\right] F_\perp(t,t';\mathbf{p}) &= 0 \\ \left[\partial_t^2 + \mathbf{p}^2 + 1 + 3\sigma^2(t)\right] F_\parallel(t,t';\mathbf{p}) &= 0 \end{aligned}$$

$\rightsquigarrow$ at $\mathcal{O}(\lambda^0)$ $F_{\parallel,\perp}$ can be factorized using 'mode functions' $f_{\parallel,\perp}$, e.g.:

$$F_\perp(t,t';\mathbf{p}) = \frac{1}{2}\left[f_\perp(t;\mathbf{p})f_\perp^*(t';\mathbf{p}) + f_\perp^*(t;\mathbf{p})f_\perp(t';\mathbf{p})\right]$$
$$\Rightarrow \quad \left[\partial_t^2 + \mathbf{p}^2 + 1 + \sigma^2(t)\right] f_\perp(t;\mathbf{p}) = 0$$

$\rightsquigarrow$ problem equivalent to the one known from classical mechanics!

<u>Analytical solution of Lamé–type equations:</u>

- Exponential amplification of $F_\perp(t,t';\mathbf{p})$ for *bounded* momentum range $\boxed{0 \le \mathbf{p}^2 \le \sigma_0^2/2}$ $\rightsquigarrow$ $\mathbf{p}^2_{\max} + 1 + \overline{\sigma^2(t)} \simeq 1 + \sigma_0^2 = (\omega_0/2)^2$
- Separation of scales: $\omega_0 \gg \gamma_0$, with maximum amplification rate: $\gamma_0 \simeq 2\delta\omega_0$ for $\mathbf{p}^2 = \mathbf{p}_0^2 \simeq \sigma_0^2/4$, $(\delta \le e^{-\pi} = 0.043\ldots)$
- Much smaller growth in narrow momentum range for $F_\parallel$

Time-averaged over $\sim \omega_0^{-1}$, for $t,t' \gg \gamma_0^{-1}$: $\boxed{F_\perp(t,t';\mathbf{p}_0) \sim e^{\gamma_0(t+t')}}$

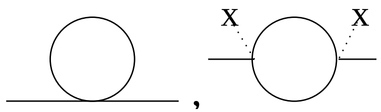
(II) Source-induced (nonlinear) amplification regime:

Strongly enhanced particle production for longitudinal modes

$\mathcal{O}(\lambda^0)$ approximation for longitudinal modes breaks down at

$$t \simeq t' = t_{\rm source}: \qquad F_\perp(t,t';\mathbf{p}_0) \sim \mathcal{O}(N^0\lambda^{-1/2})$$

Can be derived from $\mathcal{O}(\lambda)$ evolution equations:



$$\left(\partial_t^2 + \mathbf{p}^2 + M^2(t) + 3\sigma^2(t)\right) F_\parallel(t,t';\mathbf{p}) \simeq$$

$$\frac{2\lambda(N-1)}{3N}\sigma(t)\left\{\int_0^t {\rm d}t''\sigma(t'')\Pi_\perp^\rho(t,t'';\mathbf{p})F_\parallel(t'',t';\mathbf{p})\right.$$

$$\left.-\frac{1}{2}\int_0^{t'} {\rm d}t''\sigma(t'')\Pi_\perp^F(t,t'';\mathbf{p})\rho_\parallel(t'',t';\mathbf{p})\right\}$$

$\Pi_\perp^A(t,t'';\mathbf{p}) = \int {\rm d}\mathbf{q}/(2\pi)^3\, F_\perp(t,t'';\mathbf{p}-\mathbf{q})\, A_\perp(t,t'';\mathbf{q})$, $A = \{F,\rho\}$;

$\Pi_\perp \gg \Pi_\parallel$; $F_\perp^2 \gg \rho_\perp^2$ (classical dom.)

For $F_\perp \sim \mathcal{O}(N^0\lambda^{-1/2})$ $\rightsquigarrow$ RHS $\sim \mathcal{O}(1)$ cannot be neglected!

Evaluating the 'memory integrals':

Effective time locality $\quad (t \lesssim t_{\mathrm{nonpert}})$

Exponential growth $\leadsto$ latest-time contributions dominate

1. Consider 'time-average' over $\omega_0^{-1} \ll \gamma_0^{-1}$:

$$\int_0^t \mathrm{d}t'' \;\longrightarrow\; \int_{t-c/\omega_0}^t \mathrm{d}t'' \qquad (c \sim 1)$$

2. Taylor expand memory kernel(s) around latest time t (t'):

$$\rho_{\parallel,\perp}(t,t'';\mathbf{p}) \simeq \partial_{t''}\rho_{\parallel,\perp}(t,t'';\mathbf{p})|_{t=t''}(t''-t) \equiv (t-t'')$$

$$F_{\parallel,\perp}(t,t'';\mathbf{p}) \simeq F_{\parallel,\perp}(t,t;\mathbf{p})$$

$$\Rightarrow \quad \mathrm{RHS} \simeq \lambda\,\sigma^2(t)\,F_{\parallel}(t,t';\mathbf{p})\,\frac{c^2}{\omega_0^2}\,\frac{(N-1)}{3N}\,T_{\perp}(t) \qquad \text{(mass term)}$$

$$+ \; \lambda\,\sigma(t)\sigma(t')\,\frac{c^2}{\omega_0^2}\,\frac{(N-1)}{6N}\,\Pi_{\perp}^{F}(t,t';\mathbf{p}) \qquad \text{(source term)}$$

with "tadpole" $\quad T_{\perp}(t) = \int^{\Lambda} \frac{\mathrm{d}^3p}{(2\pi)^3}\,F_{\perp}(t,t;\mathbf{p})$

Momentum integrals dominated by $\mathbf{p} \simeq \mathbf{p}_0 \rightsquigarrow$ use saddle point app.:

$$F_\perp(t,t',\mathbf{p}) \simeq F_\perp(t,t',\mathbf{p}_0)\exp[-|\gamma_0''|(t+t')(p-p_0)^2/2] \qquad (*)$$

$\overset{(*)}{\rightsquigarrow}$ mass term: $\quad \sim \lambda T_\perp(t) \;\simeq\; \lambda \dfrac{p_0^2\, F_\perp(t,t;\mathbf{p}_0)}{2(\pi^3|\gamma_0''|t)^{1/2}}$

To this order in λ it is correct to use $F_\perp(t,t';\mathbf{p}_0) \sim e^{\gamma_0(t+t')}$

$\rightsquigarrow \; \lambda T_\perp \sim \mathcal{O}(1) \quad$ for $\quad \boxed{t_{\rm nonpert} \simeq (\ln\lambda^{-1})/(2\gamma_0)} \quad (\lambda \ll 1)$

$\overset{(*)}{\rightsquigarrow}$ source term: $\quad \lambda\,\Pi_\perp^F(t,t';0) \;\simeq\; \lambda \dfrac{p_0^2\, F_\perp^2(t,t';\mathbf{p}_0)}{4(\pi^3|\gamma_0''|(t+t'))^{1/2}}$

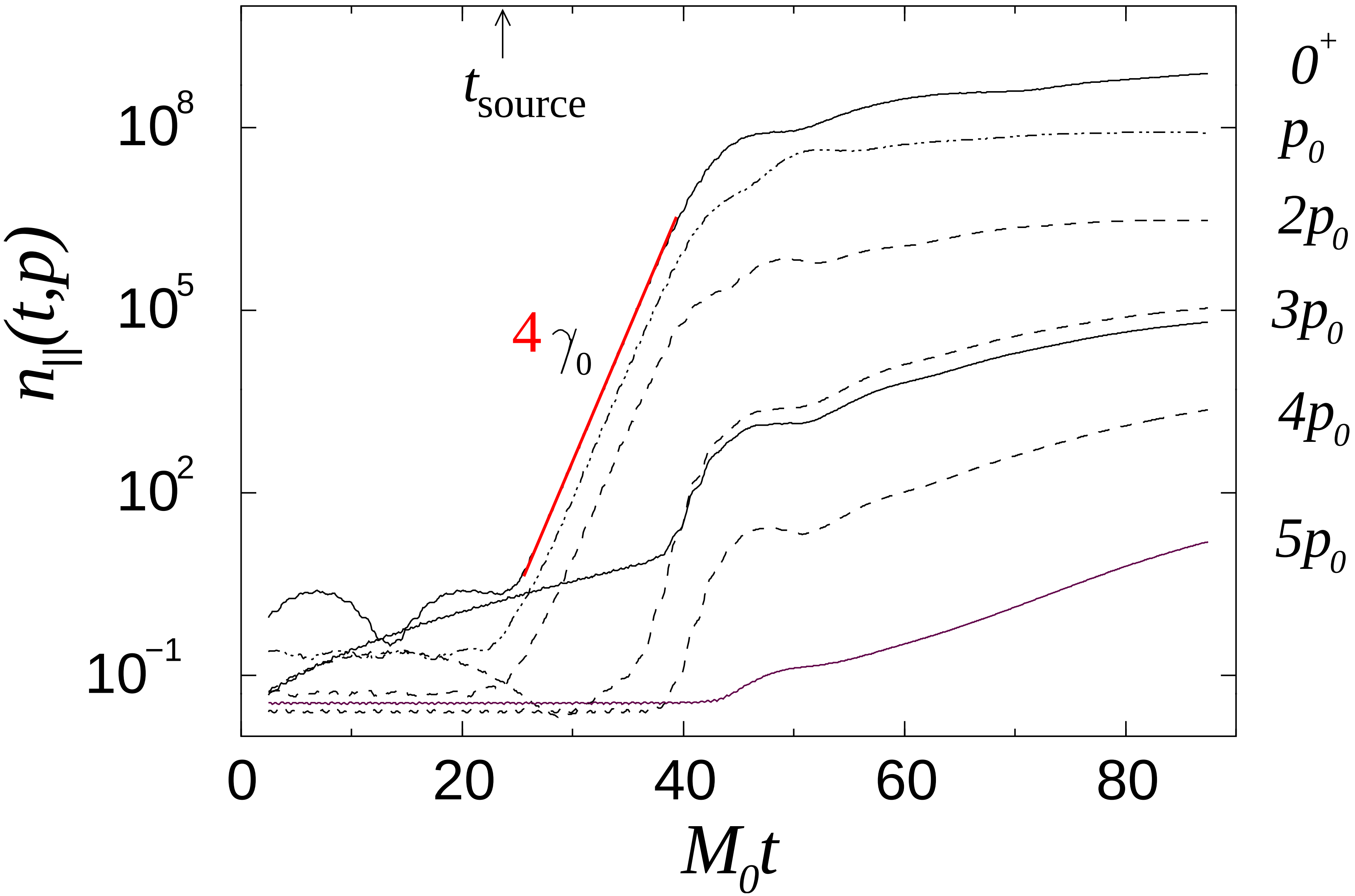
$\rightsquigarrow \; \lambda\,\Pi_\perp^F \sim \mathcal{O}(1) \quad$ for $\quad \boxed{t_{\rm source} \simeq t_{\rm nonpert}/2} \quad$ *(earlier!)*

For $t_{\rm source} \lesssim t \lesssim t_{\rm nonpert}$, $0 \lesssim p \lesssim 2p_0$:

$$F_{||}(t,t;\mathbf{p}) \sim \lambda\, F_\perp^2(t,t;\mathbf{p}_0) \sim \lambda\, e^{4\gamma_0 t} \quad \textit{(twice the rate } 2\gamma_0 \textit{ !)}$$

Comparison with numerical NLO solution ($N = 4, \lambda = 10^{-6}$)

Longitudinal effective particle number:



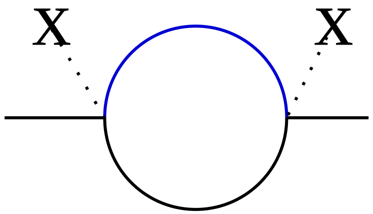
$\rightsquigarrow$ very good agreement with analytical estimates for t_{source} and rates

(III) Collective amplification regime: *Explosive particle production in a broad momentum range for transverse modes*

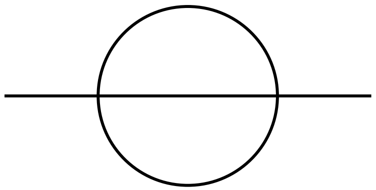
$\mathcal{O}(\lambda)$ approximation breaks down at

$$t \simeq t' = t_{\rm collect} : \qquad F_\perp(t,t';\mathbf{p}_0) \sim \mathcal{O}(N^{1/3}\lambda^{-2/3})$$

Similar analysis for the $\mathcal{O}(\lambda^2)$ evolution equations yields source terms:



$$\sim \frac{\lambda}{N} F_\| F_\perp \overset{\text{cf. (II)}}{\sim} \frac{\lambda^2}{N} F_\perp^3$$



$$\sim \frac{\lambda^2}{N} F_\perp^3$$

$$\Bigg\} \sim \frac{\lambda^2}{N} e^{6\gamma_0 t}$$

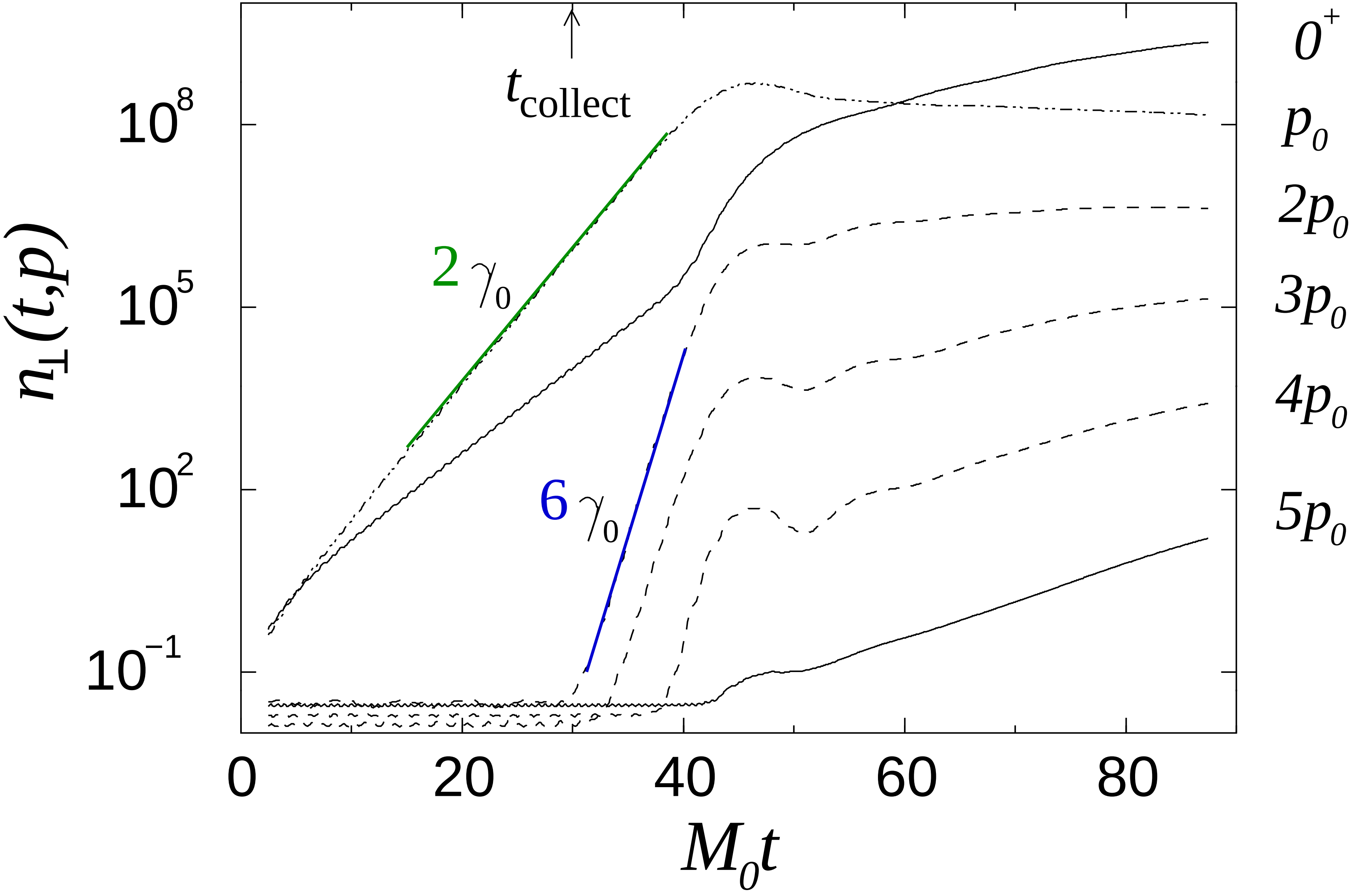
Sources of $\mathcal{O}(1)$ $\overset{t_{\rm nonpert} \simeq \frac{\ln(1/\lambda)}{2\gamma_0}}{\rightsquigarrow}$ $\boxed{t_{\rm collect} \simeq 2\, t_{\rm nonpert}/3 + (\ln N)/(6\gamma_0)}$

$\rightsquigarrow$ For $t_{\rm collect} \le t_{\rm nonpert} \;\Leftrightarrow\; N \le 1/\lambda$:

Enhanced rate $6\gamma_0$ *in a momentum range* $0 \lesssim p \lesssim 3p_0$*, etc.*

Comparison with numerical NLO solution ($N = 4, \lambda = 10^{-6}$)

Transverse effective particle number:



$\rightsquigarrow$ very good agreement with analytical estimates for t_{collect} and rates

(IV) Nonperturbative, fluctuation dominated regime:

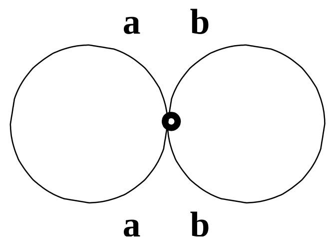
'Saturated' densities $n_{\|,\perp}(\mathbf{p}) \sim 1/\lambda$, *'quasistationary' evolution*

$\mathcal{O}(\lambda^2)$ evolution equations break down at

$t \simeq t' = t_{\rm nonpert}$: $\qquad F_\perp(t,t') \sim F_\|(t,t') \sim \mathcal{O}(N^0 \lambda^{-1})$

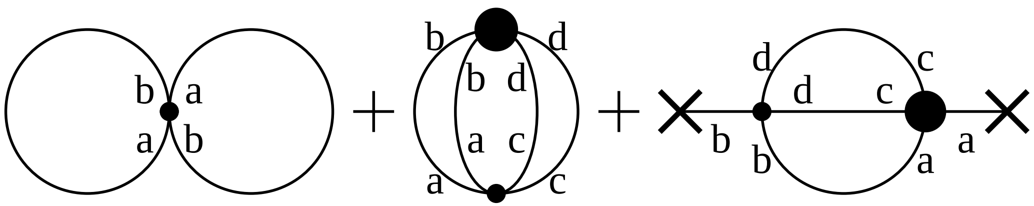
$\rightsquigarrow$ *all loop orders contribute at the same order in* λ *!*

<u>LO</u>

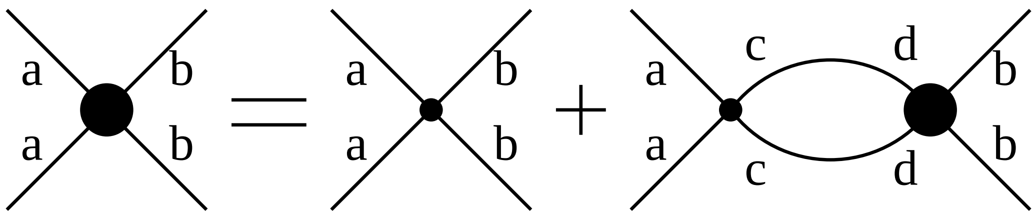


$\sim \mathcal{O}(N\lambda^{-1})$

<u>NLO</u>



$\sim \mathcal{O}(N^0\lambda^{-1})$



Vertex corrections!

$\rightsquigarrow$ 2PI $1/N$ expansion crucial for a quantitative description

Check: $$\Sigma^F_\perp(t,t';\mathbf{p}) = -\frac{\lambda}{3N}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^d}\Big\{\mathbf{I}_F(t,t';\mathbf{q})F_\perp(t,t';\mathbf{p}-\mathbf{q}) - \frac{1}{4}\mathbf{I}_\rho(t,t';\mathbf{q})\rho_\perp(t,t';\mathbf{p}-\mathbf{q}) + (\text{terms with } \phi\neq 0)\Big\}$$

$\mathbf{I}_{F,\rho}$ resum "chain" graphs to infinite loop order:

$$\mathbf{I}_F(t,t') = -\frac{\lambda}{3N}\Pi_F(t,t') + \int_0^t dt''\,\mathbf{I}_\rho(t,t'')\,\frac{\lambda}{3N}\Pi_F(t'',t') - \int_0^{t'} dt''\,\mathbf{I}_F(t,t'')\,\frac{\lambda}{3N}\Pi_\rho(t'',t'),$$

$$\mathbf{I}_\rho(t,t') = -\frac{\lambda}{3N}\Pi_\rho(t,t') + \int_{t'}^t dt''\,\mathbf{I}_\rho(t,t'')\,\frac{\lambda}{3N}\Pi_\rho(t'',t')$$

With $F_\perp(t,t') \sim \mathcal{O}(N^0\lambda^{-1})$, $\rho_\perp(t,t') \sim \mathcal{O}(N^0\lambda^0)$:

$$\frac{\lambda}{N}\Pi_F \sim \frac{\lambda}{N}(NF_\perp F_\perp) \sim \mathcal{O}(N^0\lambda^{-1}) \;,\; \frac{\lambda}{N}\Pi_\rho \sim \lambda F_\perp\rho_\perp \sim \mathcal{O}(N^0\lambda^0)$$

$\rightsquigarrow$ Higher loops not suppressed by additional powers of λ, $1/N$ counting remains unchanged

Behavior of the distribution functions:

- Densities reach 'maximum': $n_{\parallel,\perp}(\mathbf{p}) \sim 1/\lambda$ monotonous in p
- Shortly after $t_{\rm nonpert}$ comparably slow, *'quasistationary'* evolution sets in (all terms of the same order in λ)

$\rightsquigarrow$ Approach to true thermal equilibrium exceedingly slow for small coupling $\lambda = 10^{-6}$ (for $N = 4$ negligible parametric resonance regime for $\lambda \gtrsim 10$)

Behavior of the field:

- Around $t \lesssim t_{\rm nonpert}$ earliest time when sizeable *corrections to the classical field equation* appear:

$$\left[\partial_t^2 + 1 + \delta M^2(t) + \sigma^2(t)\right] \sigma(t) = - \int_0^t \mathrm{d}t' \Sigma_{\parallel}^{\rho}(t,t';\mathbf{p}=0)|_{\sigma=0}\, \sigma(t')$$

For $t \to t_{\rm nonpert}$: $\delta M^2 \simeq \frac{\lambda(N-1)}{6N} T_\perp \sim \mathcal{O}(N^0 \lambda^0)$, $\Sigma_{\parallel}^{\rho} \sim \mathcal{O}(N^{-1}\lambda^0)$

Before $t_{\rm nonpert}$ where memory expansion was valid we had:

LO: Perturbatively for $\sigma = \sigma^{(0)} + \delta\sigma$, slowly varying small correction $\delta\sigma$ and small δM^2:

$$\sigma(t) \simeq \left(1 - \frac{1}{1+3\sigma_0^2}\frac{\lambda(N-1)}{6N}T_\perp(t)\right)\sigma^{(0)}(t)$$

$\rightsquigarrow$ exponential decrease of field amplitude at LO since $T_\perp(t) \sim e^{2\gamma_0 t}$

Dominant NLO correction:

$$\int_{t-c/\omega_0}^{t} dt'\, \Sigma_\parallel^\rho(t,t';\mathbf{p}=0)|_{\sigma=0}\,\sigma(t') \simeq \frac{c^2}{2\omega_0^2}\left(\partial_t \Sigma_\parallel^\rho(t,t';\mathbf{p}=0)|_{\sigma=0}\right)|_{t=t'}\,\sigma(t)$$

$$\overset{\mathcal{O}(\lambda^2)}{\simeq} -\frac{c^2}{2\omega_0^2}\frac{\lambda^2}{18N}\int^\Lambda \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3} F_\perp(t,t;-\mathbf{q}-\mathbf{k})F_\perp(t,t;\mathbf{k})\,\sigma(t)$$

$$\simeq -\frac{c^2}{2\omega_0^2}\frac{\lambda^2}{18N}T_\perp^2(t)\,\sigma(t)$$

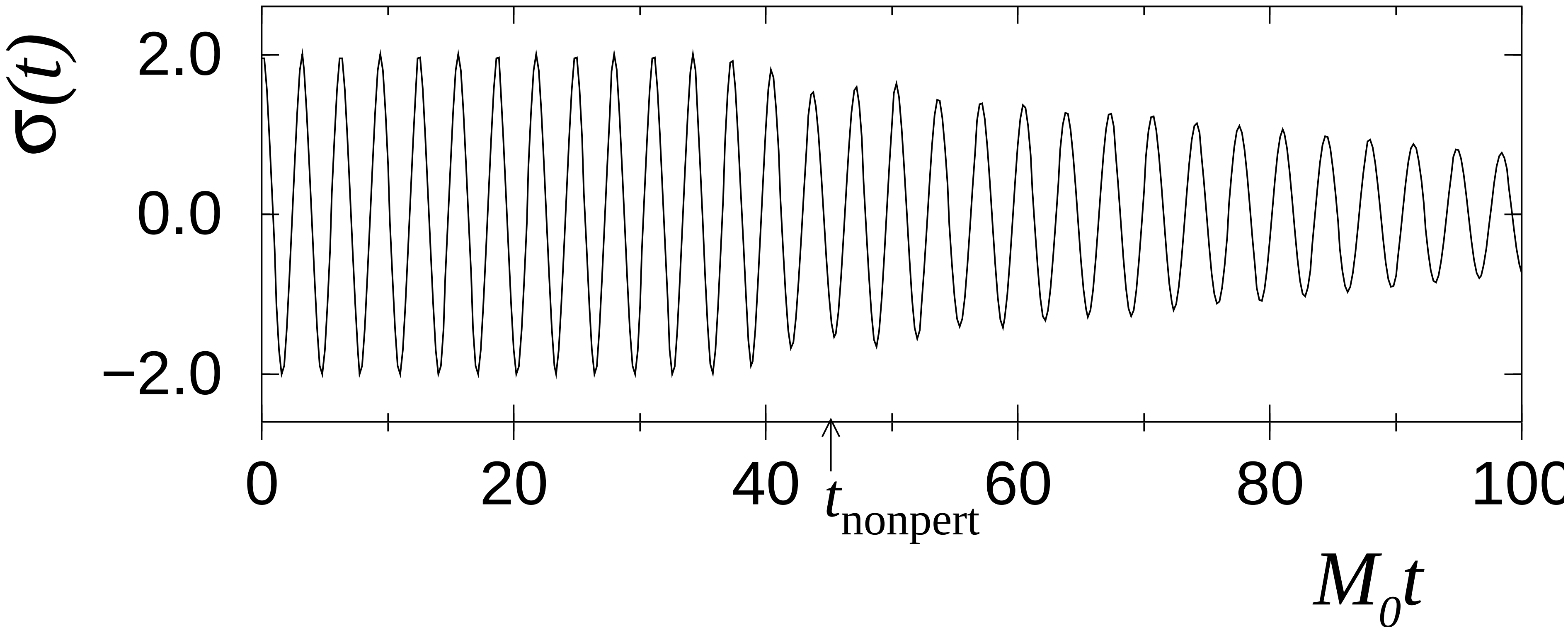
$\rightsquigarrow$ NLO correction of effective mass term with opposite sign than LO!

For $t \to t_{\rm nonpert}$ all terms become of the same order in λ:

$\leadsto$ cancellations may lead (temporarily) to reverse field decay

Numerical NLO solution: $(N = 4, \lambda = 10^{-6})$

After $t_{\rm nonpert}$ strong nonlinearities appear: The field decay 'overshoots' and is shortly reversed by feedback from modes, overshoots again etc.



Summary: Parametric resonance

Far-from equilibrium phenomenon with nonperturbatively large fluctuations. 2PI $1/N$ expansion at NLO:

- $\rightsquigarrow$ Provides controlled, *practicable* approach in QFT including "rescattering" effects
- $\rightsquigarrow$ Description equivalent to 'two-loop' self-energy with strong effective vertex corrections relevant for large densities n
- $\rightsquigarrow$ Four characteristic regimes:
  - (I) Early-time (linear) regime: *Parametric resonance*
  - (II) Source-induced (nonlinear) amplification regime: *Strongly enhanced particle production for longitudinal modes*
  - (III) Collective amplification regime: *Explosive particle production in a broad momentum range for transverse modes* $(N \leq 1/\lambda)$
  - (IV) Nonperturbative, fluctuation dominated regime: *'Saturated' densities* $n \sim \mathcal{O}(\lambda^{-1})$, *'quasistationary' evolution*
- $\rightsquigarrow$ (II) – (IV) not accessible by LO or 'Hartree'–type approximations

Application II: Chiral 'Quark-Meson' Model

$SU(2)_L \times SU(2)_R \sim O(4)$ symmetric model involving

- two fermion flavors ("quarks") coupled to
- a scalar σ–field and a triplet of pseudoscalar "pions" π^a $(a = 1, 2, 3)$

Classical action:

$$S = \int \mathrm{d}^4 x \Big\{ \bar{\psi}\, i\partial\!\!\!/\, \psi + \frac{1}{2} \left[\partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \pi^a \partial^\mu \pi^a \right] + g\, \bar{\psi}(\sigma + i\gamma_5 \tau^a \pi^a)\psi - V(\sigma^2 + \pi^2) \Big\}$$

$V(\sigma^2 + \pi^2) = \frac{1}{2} m_0^2 (\sigma^2 + \pi^2) + V_{\text{int}}$; ($\pi^2 \equiv \pi^a \pi^a$; Pauli matrices τ^a)

$\rightsquigarrow$ well-known "linear σ–model" incorporating chiral symmetry of massless two-flavor QCD

$\rightsquigarrow$ valuable toy-model: describes equilibrium phase diagram of the type shown in the introduction (chiral phase transition, (tri-)critical point at $T, \mu \neq 0$, 'color' superconductivity with add. global $SU_C(3)$)

2PI effective action for fermions:

Consider first purely fermionic theory with classical action

$$\begin{aligned} S^{(f)} &= S_0^{(f)} + S_{\rm int}^{(f)} \\ S_0^{(f)} &= \int \mathrm{d}^4x\mathrm{d}^4y\, \bar{\psi}_i(x) i\Delta_{0,ij}^{-1}(x,y)\psi_j(y) \end{aligned}$$

with $i = 1, \ldots, N_f$, interaction term $S_{\rm int}^{(f)}(\bar{\psi}\psi)$, free inverse propagator:

$$i\Delta_{0,ij}^{-1}(x,y) = (i\partial\!\!\!/ - m_f)\,\delta(x-y)\,\delta_{ij}$$

Construction of the 2PI effective action for fermions proceeds along the same lines as for bosons!

2PI effective action: (for $\langle\psi\rangle = \langle\bar{\psi}\rangle = 0$)

$$\Gamma[\Delta] = -i\mathrm{Tr}\ln\Delta^{-1} - i\mathrm{Tr}\,\Delta_0^{-1}\Delta + \Gamma_2[\Delta] + \mathrm{const}$$

Only important difference compared to bosonic case:

- Anticommuting (Grassmann) behavior of the fermionic fields

$\rightsquigarrow$ different factor for 'one-loop' ($\Gamma_2 = 0$) part: $-\frac{1}{2}$ replaced by 1

$$\text{Fermions:}\quad -i\ln\int \mathscr{D}\bar{\psi}\mathscr{D}\psi\, e^{iS_0^{(f)}} = -i\ln\left(\det\Delta_0^{-1}\right) \;=\; -i\,\mathrm{Tr}\ln\Delta_0^{-1}$$

$$\text{Bosons:}\quad -i\ln\int \mathscr{D}\varphi\, e^{iS_0} = -i\ln\left(\det G_0^{-1}\right)^{-\frac{1}{2}} \;=\; \frac{i}{2}\mathrm{Tr}\ln G_0^{-1}$$

- $\Gamma_2[\Delta]$ contains all 2PI diagrams with propagator lines associated to $\Delta_{ij}(x,y) = \langle T_{\mathcal{C}}\psi_i(x)\bar{\psi}_j(y)\rangle$

  (diagrams get additional minus sign from each closed fermion loop)

- 'Tr' includes integration over closed time path $\mathcal{C}$, integration over spatial coordinates, summation over flavor and Dirac indices

Equation of motion: $\delta\Gamma[\Delta]/\delta\Delta = 0$

$$\Rightarrow \qquad \Delta^{-1}_{ij}(x,y) = \Delta^{-1}_{0,ij}(x,y) - \Sigma^{(f)}_{ij}(x,y;\Delta)$$

$$\Sigma^{(f)}_{ij}(x,y;\Delta) \equiv -i\frac{\delta\Gamma_2[\Delta]}{\delta\Delta_{ji}(y,x)}$$

with proper self-energy $\Sigma^{(f)}$

Convoluting with Δ gives time evolution equation:

$$(i\partial\!\!\!/_x - m_f)\Delta_{ij}(x,y) - i\int_z \Sigma_{ik}(x,z;\Delta)\Delta_{kj}(z,y) = i\delta(x-y)\delta_{ij}$$

Separation into spectral and statistical components: (cf. bosons)

$$\Delta_{ij}(x,y) = F^{(f)}_{ij}(x,y) - \frac{i}{2}\rho^{(f)}_{ij}(x,y)\,\mathrm{sign}_{\mathcal{C}}(x^0-y^0)$$

$$\Sigma^{(f)}_{ij}(x,y) = \mathbf{A}_{ij}(x,y) - \frac{i}{2}\mathbf{C}_{ij}(x,y)\,\mathrm{sign}_{\mathcal{C}}(x^0-y^0)$$

$$\left(\stackrel{(!)}{\rightsquigarrow}\ \rho^{(f)}_{ij}(x,y) = i\langle\{\psi_i(x),\bar\psi_j(y)\}\rangle\,,\ F^{(f)}_{ij}(x,y) = \frac{1}{2}\langle[\psi_i(x),\bar\psi_j(y)]\rangle\right)$$

Gives along very same lines as for bosons the *exact* evolution equations:

$$
\begin{aligned}
(i\partial\!\!\!/_x - m_f)\rho^{(f)}_{ij}(x,y) &= \int_{y^0}^{x^0} \mathrm{d}z\, \mathbf{A}_{ik}(x,z)\rho^{(f)}_{kj}(z,y)\,, \\
(i\partial\!\!\!/_x - m_f)F^{(f)}_{ij}(x,y) &= \int_{0}^{x^0} \mathrm{d}z\, \mathbf{A}_{ik}(x,z)F^{(f)}_{kj}(z,y) \\
&- \int_{0}^{y^0} \mathrm{d}z\, \mathbf{C}_{ik}(x,z)\rho^{(f)}_{kj}(z,y)
\end{aligned}
$$

- Form of RHS completely equivalent to the one for scalar fields
- First order differential equations:

$\rightsquigarrow$ fermion anticommutation relation

$$
\gamma^0 \rho^{(f)}_{ij}(x,y)|_{x^0=y^0} = i\delta(\mathbf{x}-\mathbf{y})\,\delta_{ij}
$$

uniquely specifies the initial condition for the spectral function!

$\rightsquigarrow$ specify nonequilibrium initial conditions for $F^{(f)}_{ij}(x,y)$

Initial conditions:

For simplicity consider *spatially homogeneous, isotropic* initial conditions *(P, CP invariant)*. For the *chirally symmetric* case the most general (Gaussian) initial conditions are then given by : $\quad (p = |\mathbf{p}|)$

$$
\begin{aligned}
F_V(t,t';p)|_{t=t'=0} &\equiv \frac{1}{4}\,\mathrm{tr}\,\frac{p^i\gamma^i}{p}\,F(t,t';\mathbf{p})|_{t=t'=0} = \frac{1}{2} - n_0^f(p) \\
F_V^0(t,t';p)|_{t=t'=0} &\equiv \frac{1}{4}\,\mathrm{tr}\,\gamma^0 F(t,t';\mathbf{p})|_{t=t'=0} = 0
\end{aligned}
$$

Note that:

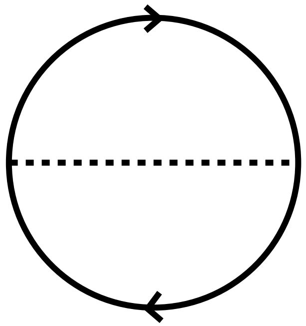
- $F_V^0 \sim \mathrm{tr}\langle\bar{\psi}\gamma^0\psi\rangle$ counts the net charge density $\sim n - \bar{n} \overset{CP}{\equiv} 0$, while
- $F_V \sim \vec{v}\,\mathrm{tr}\langle\bar{\psi}\vec{\gamma}\psi\rangle$ counts particle number $(n+\bar{n})/2 = n_0^f$ at initial time
- In particular a mass term m_f is absent because of chiral symmetry

$\rightsquigarrow$ If S_{int} respects the symmetries of the initial conditions all other components of $F(t,t';\mathbf{p})$ ($\rho(t,t';\mathbf{p})$) vanish in the symmetric case

Effective action for fermions and bosons: 'quark-meson' model

$$\Gamma[G,\Delta] = \frac{i}{2}\mathrm{Tr}\ln G^{-1} + \frac{i}{2}\mathrm{Tr}G_0^{-1}G - i\mathrm{Tr}\ln\Delta^{-1} - i\mathrm{Tr}\,\Delta_0^{-1}\Delta$$
$$+ \quad \Gamma_2[G,\Delta] + \mathrm{const}$$

Consider the following 2-loop approximation for Γ_2: (cf. NLO $1/N_f$)



solid: fermion propagator Δ
dashed: scalar propagator G

$\rightsquigarrow$ lowest order which contributes to scalar/fermion self-energies: Σ_F, Σ_ρ, $\mathbf{A}$, $\mathbf{C}$ of $\mathcal{O}(g^2)$

$\rightsquigarrow$ without this contribution the 'free-field' evolution for fermions:

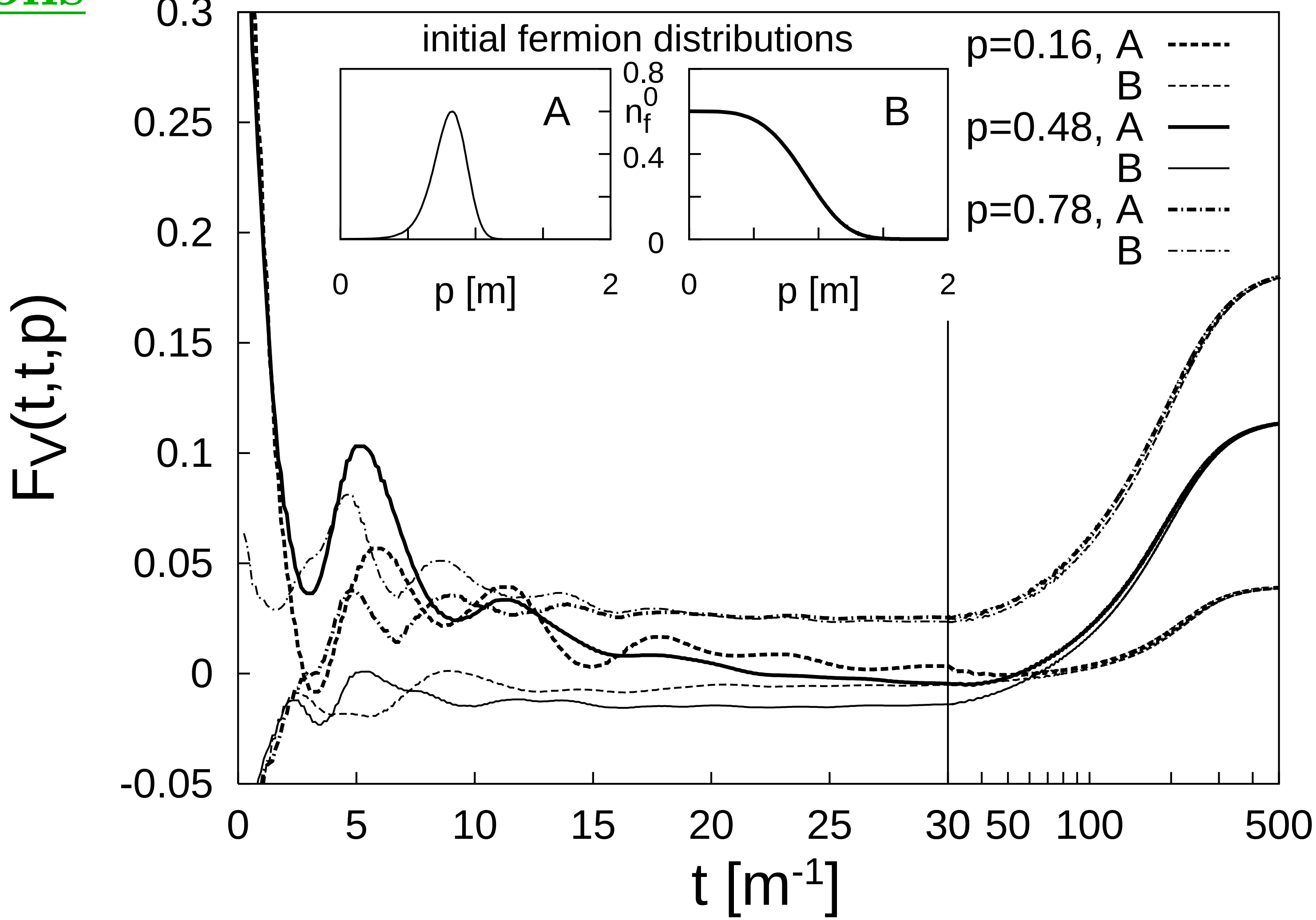
$$F_V(t,t';p) = \left(\frac{1}{2} - n_0^f(p)\right)\cos[p(t-t')]$$

strictly conserves particle number $\sim F_V(t,t;p)$

Effective loss of details of initial conditions:

- Two different initial conditions (A), (B) with *same* energy density

Fermions

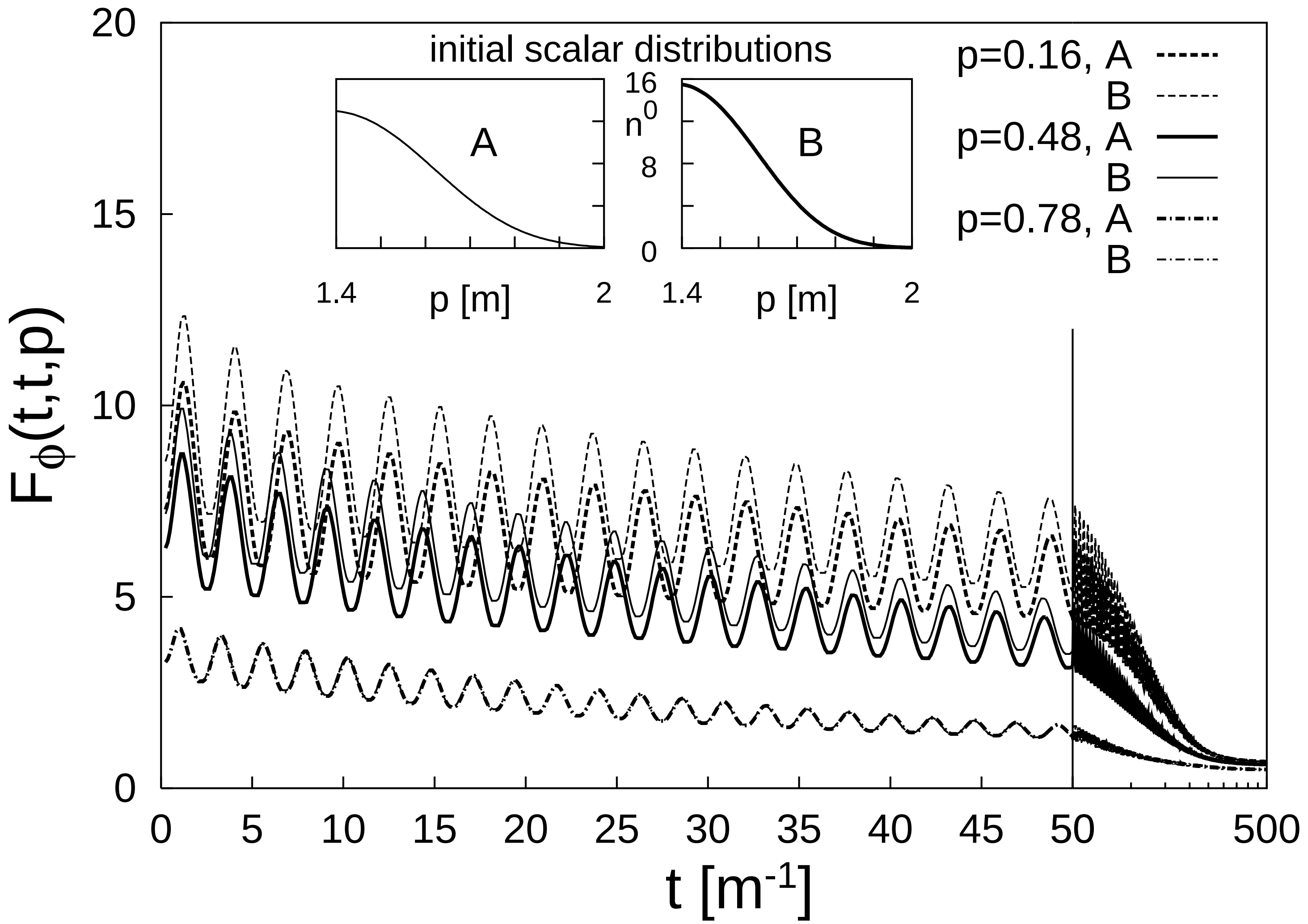


(g = 1)

Here: $1/\gamma_f^{(\mathrm{damp})}(p=0) = 15(1)m^{-1}$, $1/\gamma_f^{(\mathrm{therm})} = 95(5)m^{-1}$

in units of the renormalized scalar *thermal* mass m.

Scalars



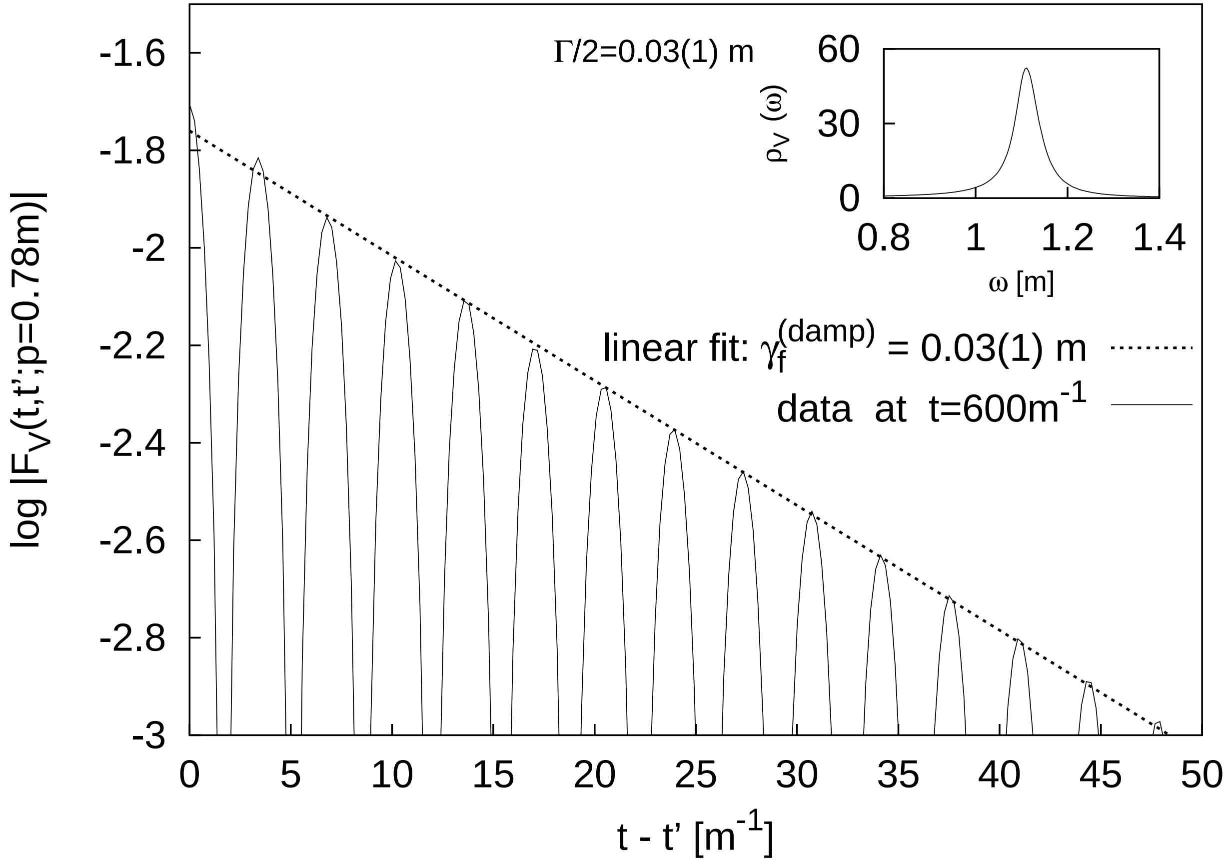
Here: $1/\gamma^{(\mathrm{damp})}(p=0) \;=\; 50(5)m^{-1}$, $1/\gamma^{(\mathrm{therm})} \;=\; 90(5)m^{-1}$

$\rightsquigarrow$ Characteristic 'two-stage' loss of initial conditions: (cf. $O(N)$-model)

- $t \gtrsim 1/\gamma_{(f)}^{(\mathrm{damp})}(p)$ details about initial (A) vs. (B) effectively lost (still far from equilibrium)
- $t \gtrsim 1/\gamma_{(f)}^{(\mathrm{therm})}$ approach to thermal equilibrium

$\rightsquigarrow$ $\gamma_f^{(\mathrm{therm})} \simeq \gamma^{(\mathrm{therm})}$ $(\gamma_f^{(\mathrm{damp})} \neq \gamma^{(\mathrm{damp})})$

$\rightsquigarrow$ Damping rate well described by equilibrium width $\Gamma^{(\mathrm{eq})}(\omega = \epsilon_{\mathbf{p}}, \mathbf{p})$:



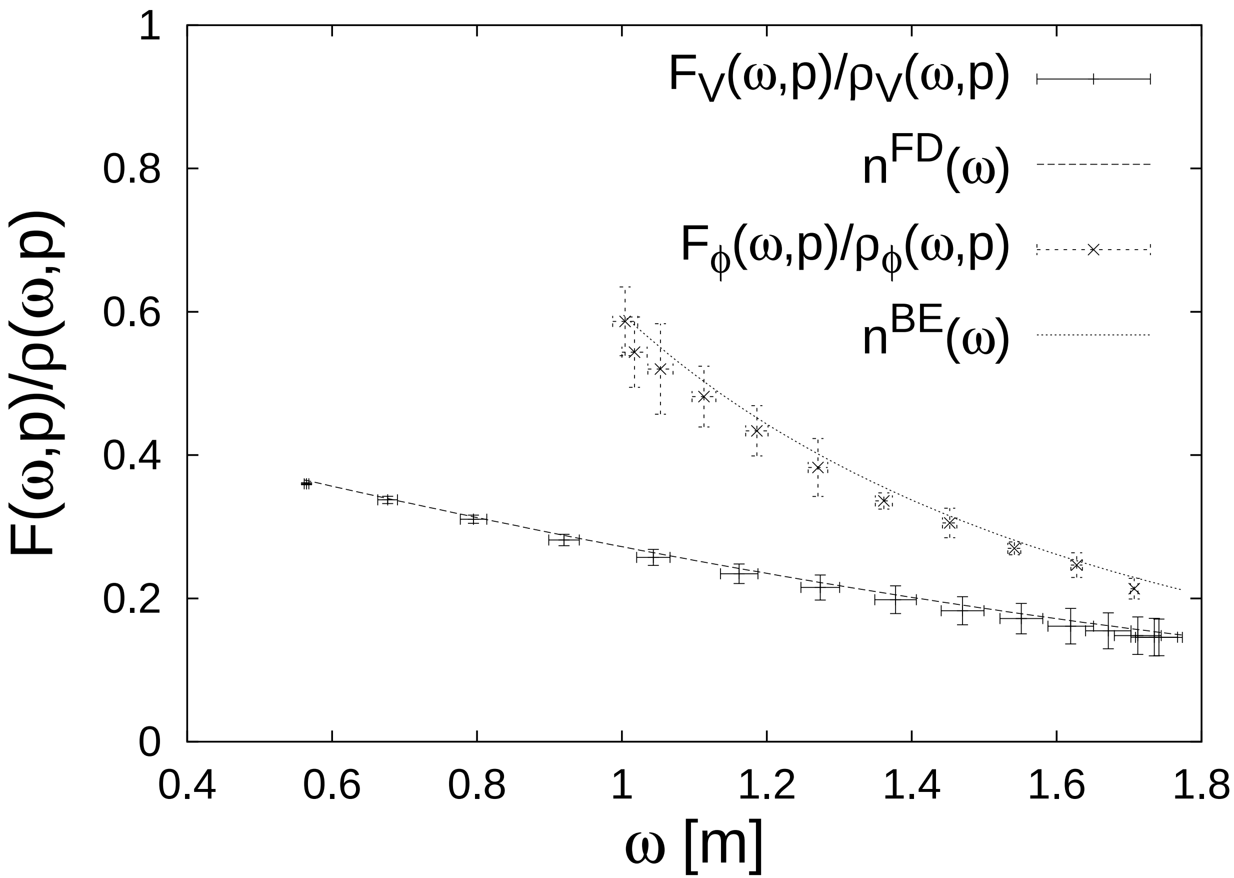
$(g = 1)$

Quantum thermal equilibrium: Bose-Einstein vs. Fermi-Dirac

- At late times $F(t,t';\mathbf{p}) \simeq F(t-t';\mathbf{p})$, $\rho(t,t';\mathbf{p}) \simeq \rho(t-t';\mathbf{p})$

$\rightsquigarrow$ *direct measurement* of Bose-Einstein/Fermi-Dirac distribution possible in frequency space: (no effective particle number involved)

fermion: $\dfrac{F_V(\omega,p)}{\rho_V(\omega,p)} \to n^{FD}(\omega)$, scalar: $\dfrac{F_\phi(\omega,p)}{\rho_\phi(\omega,p)} \to n^{BE}(\omega)$



Note: $T \stackrel{!}{=} T_f = T_\phi = 0.94\,m$ not fitted here (inverse slope parameter)

Conclusions

2PI loop / 2PI $1/N$ expansion of the effective action:

$\rightsquigarrow$ Powerful quantitative tool for nonequilibrium dynamics from 'first principles'

Practical means to go beyond leading order:

$\rightsquigarrow$ Includes scattering, memory and 'off-shell' effects (nonzero width of the spectral function!)

$\rightsquigarrow$ Quantitative description of far-from-equilibrium dynamics and subsequent approach to quantum thermal equilibrium (Bose–Einstein / Fermi–Dirac)

- Controlled dynamics for nonperturbatively large fluctuations (large densities/critical phenomena from 2PI $1/N$)

Most pressing: Extension to gauge theories (gauge-fixing dependence!)

Selected further reading:

- A recent short review with a list of references is given in: *Progress in nonequilibrium quantum field theory*, J. Berges and J. Serreau, http://arXiv:hep-ph/0302210
- The 2PI $1/N$ expansion is discussed in: *Controlled nonperturbative dynamics of quantum fields out of equilibrium*, J. Berges, Nucl. Phys. **A699** (2002) 847; *Far-from-equilibrium dynamics with broken symmetries from the 1/N expansion of the 2PI effective action*, G. Aarts, D. Ahrensmeier, R. Baier, J. Berges and J. Serreau, Phys. Rev. **D66** (2002) 045008; *Quantum dynamics of phase transitions in broken symmetry ϕ^4 field theory*, F. Cooper, J. F. Dawson and B. Mihaila, Phys. Rev. **D67** (2003) 056003, and applied to the question of thermalization in the first and third reference
- The quantum theory is compared to the 2PI $1/N$ expansion in the classical statistical field theory limit and to exact results in: *Classical aspects of quantum fields far from equilibrium*, G. Aarts and J. Berges, Phys. Rev. Lett. **88** (2002) 0416039

- Far-from-equilibrium quantum fields with large fluctuations are discussed in: *Parametric resonance in quantum field theory*, J. Berges and J. Serreau, Phys. Rev. Lett. **91** (2003) 111601
- Nonequilibrium dynamics/thermalization of a chiral 'quark-meson' model is discussed in: *Thermalization of fermionic quantum fields*, J. Berges, Sz. Borsányi and J. Serreau, Nucl. Phys. **B660** (2003) 52
- The renormalization of 2PI approximation schemes is discussed in (see also the talk by U. Reinosa, this school): *Renormalizability of Φ-derivable approximation schemes in scalar ϕ^4-theory*, J.-P. Blaizot, E. Iancu and U. Reinosa, Phys. Lett. **B568** (2003) 160; *Renormalization in self-consistent approximations schemes at finite temperature. I: Theory*, H. van Hees and J. Knoll, Phys. Rev. D **65** (2002) 025010
- Gauge-fixing dependence of 2PI approximation schemes is discussed in: *Gauge-fixing dependence of Φ-derivable approximations*, A. Arrizabalaga and J. Smit, Phys. Rev. **D66** (2002) 065014

Appendix A

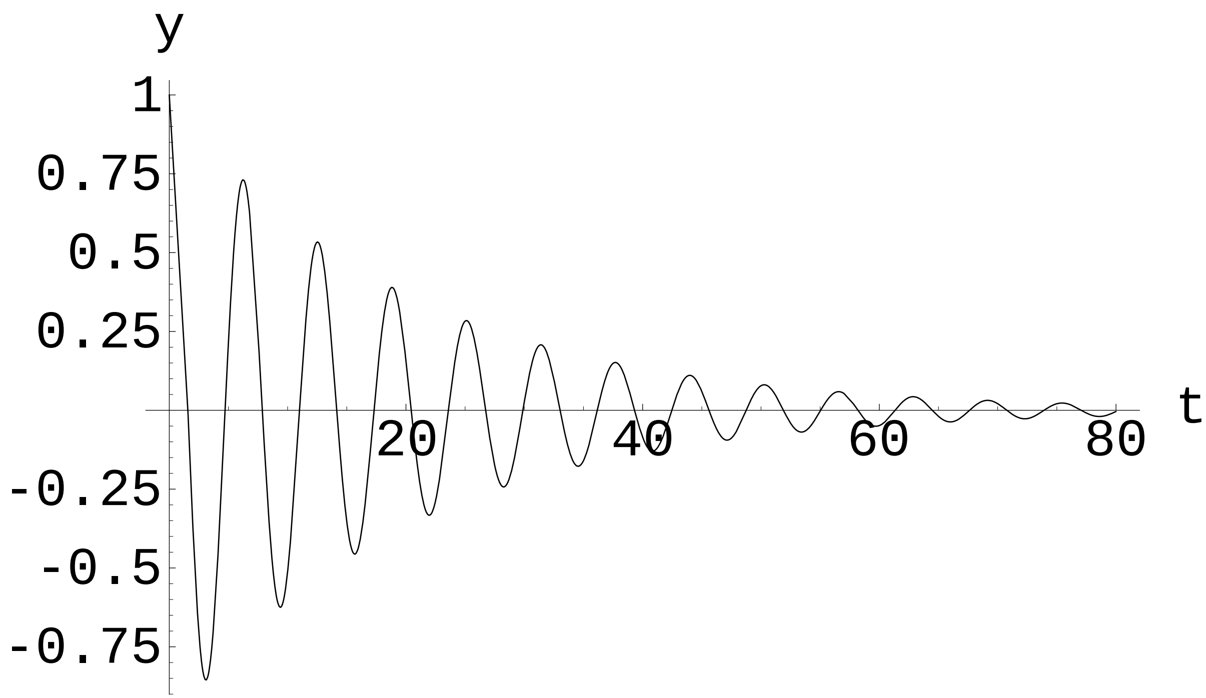
Secularity: An illustrative example

Consider a damped anharmonic oscillator described by

$$\ddot{y} + y \;=\; -\epsilon\dot{y} - (\epsilon y)^3 - (\epsilon y)^5 - (\epsilon y)^7 - \ldots \quad , \quad \epsilon \ll 1$$

for $y(0) = 1$, $\dot{y}(0) = -\epsilon/2$ $(t \geq 0)$, $\sum_{n=3}^{\infty}(\epsilon y)^n = (\epsilon y)^3/(1-\epsilon^2 y^2)$.

Numerical solution:
$(\epsilon = 0.1)$



Question: Accurate description of full y if contributions from higher powers of ϵ are neglected?

I. Consider perturbative expansion of y in ϵ:

$$y_{\rm pert}(t) = y_0(t) + \epsilon y_1(t) + \frac{\epsilon^2}{2} y_2(t) + \mathcal{O}(\epsilon^3)$$

$$\rightarrow \quad \ddot{y}_0(t) + y_0(t) = 0 \quad , \quad \ddot{y}_1(t) + y_1(t) = -\dot{y}_0 \quad , \quad \ldots$$

Iterative solution:

$$y_0(t) = \frac{1}{2} e^{it} + \text{c.c.} \quad , \quad y_1(t) = -\frac{1}{4} e^{it} t + \text{c.c.} \quad , \quad \ldots$$

Up to second order in ϵ one finds:

$$y_{\rm pert}(t) = \frac{1}{2} e^{it} \left(1 - \frac{\epsilon}{2} t + \frac{\epsilon^2}{8} \left[t^2 - it \right] \right) + \text{c.c.}$$

Contains *secular terms* that grow with powers of t (!)

$$\boxed{\Rightarrow \text{Expansion only valid for } \epsilon t \ll 1}$$

II. "2PI" type: Classify the contributions to the evolution equation according to powers of ϵ

without expanding the dynamical variable $y \sim \mathcal{O}(\epsilon^0)$

Lowest order: (same as for perturbation theory)

$$\ddot{y}^{(0)}_{\mathrm{2PI}} + y^{(0)}_{\mathrm{2PI}} = 0 \quad \Rightarrow \quad y^{(0)}_{\mathrm{2PI}}(t) = \frac{1}{2} e^{it} + \text{c.c.}$$

Up to second order in ϵ:

$$\ddot{y}^{(2)}_{\mathrm{2PI}} + y^{(2)}_{\mathrm{2PI}} = -\epsilon \dot{y}^{(2)}_{\mathrm{2PI}} \quad \Rightarrow \quad y^{(2)}_{\mathrm{2PI}}(t) = \frac{1}{2} e^{it\sqrt{1-\epsilon^2/4} - \epsilon t/2} + \text{c.c.}$$

- Both the lowest order approximation and $y^{(2)}_{\mathrm{2PI}}(t)$ do not exceed $\mathcal{O}(\epsilon^0)$ for all times, as required by the classification scheme
- Simple reason for success: "self-consistency" (note that the presence of "external" oscillating *source terms* in the perturbative hierarchy, such as $\sim y_0$ driving y_1, leads to secular behavior)

Comparison of approximation I (perturbative) and II ("2PI"):

$$
\begin{aligned}
y^{(2)}_{\rm 2PI}(t) &= \frac{1}{2} e^{it\sqrt{1-\epsilon^2/4} - \epsilon t/2} + \text{c.c.} \\
&= \frac{1}{2} e^{it} \left(1 - \frac{\epsilon}{2} t + \frac{\epsilon^2}{8} \left[t^2 - it \right] \right) + \mathcal{O}(\epsilon^3) + \text{c.c.} \\
&\overset{(!)}{=} y_{\rm pert}(t) + \mathcal{O}(\epsilon^3) + \text{c.c.}
\end{aligned}
$$

⇒ The "2PI" result corresponds to the perturbative one up to the order of approximation (here: $\sim \epsilon^2$).

⇒ They differ for higher order terms since $y_{\rm 2PI}(t)$ *sums terms up to infinite order in* ϵ

Infinite summation is required *(necessary, but not sufficient in general)* to obtain a uniform approximation to the exact solution, i.e. that the error stays of a given order at all times