GOING CHIRAL: OVERLAP VERSUS TWISTED MASS FERMIONS

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for the $\chi^F$ Collaboration

OVERVIEW

- Introduction
- Overlap fermions
- Twisted mass fermions
- Scaling test for tmQCD
- Conclusions & Outlook
Introduction

Goal: Lattice QCD simulations at the physical point, i.e. $m_\pi = 140$ MeV

- light quarks (u/d) play dominant role

Obstacles: explicit breaking of chiral symmetry on the lattice

- very low lying eigenvalues of Dirac operator
- exceptional configurations
- natural slow-down of algorithms

Challenge: better formulations of lattice QCD

- overlap fermions [Neuberger]
- twisted mass fermions [Frezzotti, Grassi, Sint, Weisz / Frezzotti, Rossi]
**Overlap Fermions**

- **Ginsparg-Wilson relation:** \( D\gamma_5 + \gamma_5 D = 2aD\gamma_5 D \)

- **Neuberger Dirac-Operator:** \( D_{ov} = (1 - \frac{m_{ov}\bar{a}}{2})D_{ov}^{(0)} + m_{ov}, \quad \bar{a} \equiv a/\rho \)

\[
D_{ov}^{(0)} = \frac{1}{a}\left\{ 1 + \frac{A}{\sqrt{A^\dagger A}} \right\}, \quad A = aD_W - \rho
\]

- Inverse square root approximated by Chebyshev polynomials

- **Standard Wilson Dirac-Operator:**

\[
D_W = \sum_{\mu=1}^{4} \frac{1}{2}[\gamma_{\mu}(\nabla^*_\mu + \nabla_{\mu}) - a\nabla^*_\mu\nabla_{\mu}]\]
Wilson twisted mass fermions

Wilson fermions with twisted mass term:

\[ D_{tm} = D_W + m_0 + i\mu\gamma_5\tau^3 \quad m_0 - m_c = m_q \cos(\omega) \quad \mu = m_q \sin(\omega) \]

Field rotations:

\[ \psi' = \exp(i \frac{\omega}{2}\gamma_5\tau^3)\psi \quad \bar{\psi}' = \bar{\psi} \exp(i \frac{\omega}{2}\gamma_5\tau^3) \]

\[ \omega = 0: \text{Wilson action}, \quad \omega = \pm \frac{\pi}{2}: \text{maximal twist} \]

Example: axial and vector current transformation

\[ A'^a_\mu = \cos(\omega)A^a_\mu + \epsilon^{3ab} \sin(\omega)V^b_\mu \quad \text{for } a=1,2 \quad A'^a_\mu = A^3_\mu \quad \text{for } a=3 \]

\[ V'^a_\mu = \cos(\omega)V^a_\mu + \epsilon^{3ab} \sin(\omega)A^b_\mu \quad \text{for } a=1,2 \quad V'^a_\mu = V^3_\mu \quad \text{for } a=3 \]
**Wilson twisted mass fermions (2)**

\( \mathcal{O}(a) \) improvement can be achieved by

- Wilson average \( \pm r \)
- mass average \( \pm m_q \)

**special case:** maximal twist \( \omega = \pm \frac{\pi}{2} \)

\[ r \leftrightarrow \omega \rightarrow \omega + \pi \]

automatic averaging for all quantities even under \( \omega \rightarrow -\omega \)

e.g. hadron masses, matrix elements, decay constants, form factors
## Comparison

<table>
<thead>
<tr>
<th>Overlap Fermions</th>
<th>Twisted Mass Fermions</th>
</tr>
</thead>
<tbody>
<tr>
<td>+ chiral symmetry</td>
<td>0</td>
</tr>
<tr>
<td>+ topological charge</td>
<td>0</td>
</tr>
<tr>
<td>+ IR safety</td>
<td>+</td>
</tr>
<tr>
<td>+ $O(a)$ improvement</td>
<td>( + )</td>
</tr>
<tr>
<td>- computational cost</td>
<td>+ +</td>
</tr>
</tbody>
</table>
Simulation Parameters

\begin{itemize}
  \item $\beta = 5.85 \iff a = 0.123 \text{ fm}$
  \item lattice sizes:
    \begin{align*}
      & 12^3 \times 24 \text{ (overlap)} \\
      & 12^3 \times 24, 14^3 \times 32, 16^3 \times 32 \text{ (twisted mass)}
    \end{align*}
  \item quark masses (MMS):
    \begin{align*}
      m_{ov}a &= 0.01, 0.02, 0.04, 0.06, 0.08, 0.10 \\
      \mu a &= 0.005, 0.01, 0.02, 0.04, 0.06, 0.08, 0.10
    \end{align*}
  \item $m_0 = m_c \iff \kappa = \kappa_c = 0.16166(2)$
    from vanishing pion mass for Wilson fermions ($\mu = 0$)
\end{itemize}
Pseudoscalar mass

\[ C_{P,\text{tm}}(x_0) = a^3 \sum_{\vec{x}} \langle P^b(\vec{x}, x_0) P^b(0) \rangle_{\text{tm}} \]

\[ b=1,2 \]

\[ C_{P,\text{ov}}(x_0) = a^3 \sum_{\vec{x}} \langle P^\dagger(\vec{x}, x_0) P(0) \rangle_{\text{ov}} \]

\[ C_{P-S,\text{ov}}(x_0) = a^3 \sum_{\vec{x}} \langle P^\dagger(\vec{x}, x_0) P(0) - S^\dagger(\vec{x}, x_0) S(0) \rangle_{\text{ov}} \]
PSEUDOSCALAR MASS SPLITTING

- neutral pion:

\[ S^3(x) = \bar{\psi}(x) \frac{\tau^3}{2} \psi(x) \]

\[ C_S^a(x_0) = a^3 \sum_x \langle S^3(x, x_0) S^3(0) \rangle \quad a=1,2 \]

- connected diagram gives sensible definition
  (proof by mapping to OS action)

- disconnected diagram is \( \mathcal{O}(a^2) \)
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Renormalization Constants (1)

- overlap

\[ Z_P = Z_S = \frac{1}{Z_m} \]

- twisted mass

\[ Z_P = \frac{1}{Z_\mu} \]

- RGI quark mass renormalization constants by matching procedure
  Hernandez, Jansen, Lellouch, Wittig; JHEP 0107 (2001) 018

- matching with Clover data (ALPHA Coll.)
  @ 3 reference points

- 2 matching conditions:
  RGI quark mass, pseudo-scalar density

<table>
<thead>
<tr>
<th>( x_{ref} )</th>
<th>( Z_{m}^{RG1,ov} )</th>
<th>( Z_{P}^{RG1,tm} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5736</td>
<td>1.02(6) 0.98(5)</td>
<td>2.27(7) 2.22(8)</td>
</tr>
<tr>
<td>3.0</td>
<td>0.98(7) 1.01(5)</td>
<td>2.32(6) 2.36(6)</td>
</tr>
<tr>
<td>5.0</td>
<td>— —</td>
<td>2.39(5) 2.55(20)</td>
</tr>
</tbody>
</table>
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PCAC QUARK MASS

\[ m_{\text{PCAC}}^{\text{ov}} = \frac{\Sigma_{\vec{x}}\langle \partial_0 A_0^\dagger(\vec{x}, x_0) P(0) \rangle}{2 \Sigma_{\vec{x}}\langle P^\dagger(\vec{x}, x_0) P(0) \rangle} \]

\[ m_{\text{PCVC}}^{\text{tm}} = \frac{\epsilon^{3bc} \Sigma_{\vec{x}}\langle \partial_0 V_0^b(\vec{x}, x_0) P^c(0) \rangle}{2 \Sigma_{\vec{x}}\langle P^c(\vec{x}, x_0) P^c(0) \rangle} \]

- apparently perfect linear behavior...
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Renormalization Constants (2)

\[ Z_A^{\text{ov}} = \lim_{m_{\text{ov}} \to 0} \frac{m_{\text{ov}}}{m_{\text{PCAC}}} = 1.448(4) \]

\[ Z_V^{\text{tm}} = \lim_{\mu \to 0} \frac{\mu}{m_{\text{PCVC}}} = ? \]

QUANTUM FIELDS IN THE ERA OF TERAFLOP-COMPUTING - ZiF, Bielefeld - November 22 to 25, 2004
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**VECTOR MESON MASS**

\[ C_{V,ov}(x_0) = \frac{a^3}{3} \sum_{k=1}^{3} \sum_{\vec{x}} \langle V_k^{\dagger}(\vec{x}, x_0)V_k(0) \rangle_{ov} \]

\[ C_{A,tm}(x_0) = \frac{a^3}{3} \sum_{k=1}^{3} \sum_{\vec{x}} \langle A_k^{b}(\vec{x}, x_0)A_k^{b}(0) \rangle_{tm} \quad b=1,2 \]
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**PSEUDO-SCALAR DECAY CONSTANT**

![Graph](image)

- using the PCAC and PCVC relation:

\[
 f_{\pi}^{ov} = \frac{2m_{ov}}{(M_{\pi}^{ov})^2} | \langle 0 | P | \pi \rangle_{ov} |
\]

\[
 f_{\pi}^{tm} = \frac{2\mu}{(M_{\pi}^{tm})^2} | \langle 0 | P^b | \pi \rangle_{tm} | 
\]

\[ b = 1, 2 \]

- competition between twisted mass term and Wilson term

\[
 m_q a \gg (\Lambda_{QCD} a)^2
\]
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**CHOICE OF CRITICAL MASS**

\[ \kappa_c \text{ from } m_{PCAC} a \to 0, \mu = 0.014 \]
\[ \kappa_c \text{ from } (m_\pi a)^2 \to 0, \mu = 0.0 \]

\[ m_V a \]
\[ f_\pi a \]

\[ \kappa_c \text{ from } m_{PCAC} a \to 0, \mu = 0.014 \]
\[ \kappa_c \text{ from } (m_\pi a)^2 \to 0, \mu = 0.0 \]

→ **\( \kappa_c \) from \( m_{PCAC} a \to 0 \) at small \( \mu \) reduces bending**

→ **work in progress:** \( \kappa_c \) from \( m_{PCAC} a \to 0 \) for \( \mu \to 0 \)
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**Scaling Test High Mass**

\[(r_0 m_{PS})^2 = 3.3 \implies m_{PS} = 720 \text{ MeV}\]

Jansen, Shindler, Urbach, IW; PLB 586 (2004) 432

- \[\beta = 5.7 - 6.2 (6.45) \iff a = 0.17 - 0.07 (0.05) \text{ fm}\]
- lattice sizes \(12^3 \times 32\) to \(24^3 \times 48 (32^3 \times 64)\)
- pseudo-scalar masses down to 250 MeV \[m_{PS}/m_V \simeq 0.29\]
Extended scaling test

pseudo-scalar decay constant

average momentum of a parton in a pion

$O(a^2)$ scaling confirmed at low masses ($\beta \geq 6.0$)
Overlap versus twisted mass

Blue: $\text{tm } \beta = 5.85$
Red: $\text{ov } \beta = 5.85$
Black: $\text{tm cont or } \beta = 6.2$

Vector meson mass versus $(r_0m_{PS})^2$

Pseudo-scalar decay constant versus $(r_0m_{PS})^2$

Scaling test for overlap fermions required for final comparison
investigated solvers: CG, CGS, MR, GMRES, SUMR

algorithmic improvements: adaptive precision, chiral projection, MMS

<table>
<thead>
<tr>
<th>Volume</th>
<th>$m_{PS}$</th>
<th>overlap</th>
<th>tm</th>
<th>rel. factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$12^4$</td>
<td>720 MeV</td>
<td>48.8(6)</td>
<td>2.6(1)</td>
<td>18.8</td>
</tr>
<tr>
<td></td>
<td>390 MeV</td>
<td>142(2)</td>
<td>4.0(1)</td>
<td>35.4</td>
</tr>
<tr>
<td></td>
<td>250 MeV</td>
<td>225(9)</td>
<td>4.9(2)</td>
<td>45.9</td>
</tr>
<tr>
<td></td>
<td></td>
<td>399(11)</td>
<td>4.9(2)</td>
<td>81.4</td>
</tr>
<tr>
<td>$16^4$</td>
<td>720 MeV</td>
<td>225(2)</td>
<td>9.0(2)</td>
<td>25.0</td>
</tr>
<tr>
<td></td>
<td>390 MeV</td>
<td>653(6)</td>
<td>17.5(6)</td>
<td>37.3</td>
</tr>
<tr>
<td></td>
<td>250 MeV</td>
<td>1949(22)</td>
<td>22.1(8)</td>
<td>88.6</td>
</tr>
</tbody>
</table>

Wilson twisted mass fermions are 20-80 times cheaper than overlap fermions

fastest solver: CGS (twisted mass), GMRES$_{ap}$ [CG$_{Xap}$] (overlap)
Conclusions

- pseudo-scalar masses of $O(250 \text{ MeV})$ are reachable with overlap and Wilson twisted mass fermions

- twisted mass fermions show $O(a^2)$ scaling for low masses ($\beta \geq 6.0$)

- pion splitting is under control

- renormalization factors through suitable matching conditions

- dependence of scaling on the choice of $\kappa_c$ needs further investigation

- conceptual versus practical advantages

- next step: dynamical simulations
Going chiral: overlap versus twisted mass fermions

**Outlook: dynamical tm fermions**

- complicated phase structure

Ilgenfritz, Kerler, Müller-Preussker, Sternbeck, Stüben; PRD69 (2004) 074511

Farchioni, Jansen, Montvay, Scholz, Scorzato, Shindler, Ukita, Urbach, IW
hep-lat/0410031