# Analytic Approaches to Thermalization in Heavy Ion Collisions

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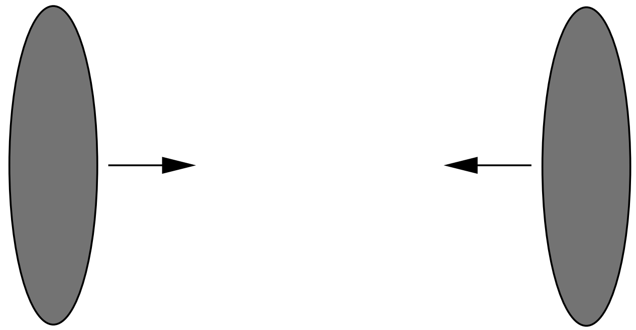
# Outline

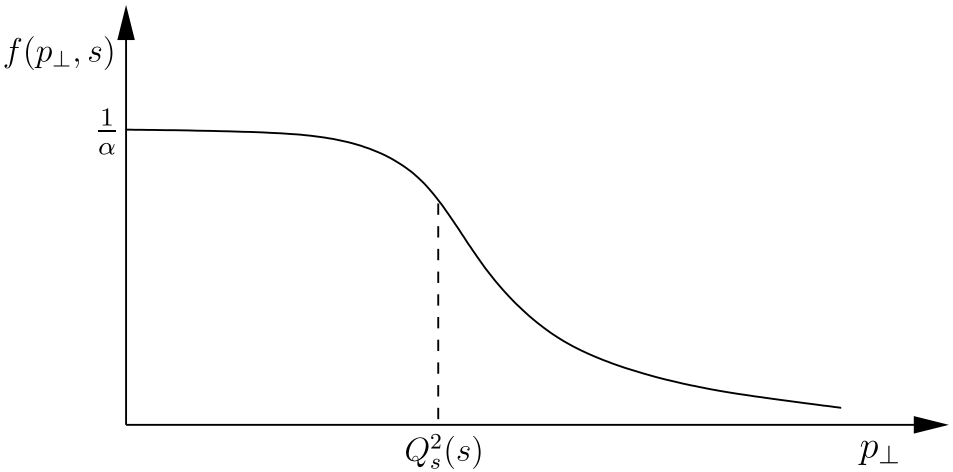
- High energy heavy ion collisions
- Bottom-Up thermalization szenario
  - early period of time
  - intermediate period of time
  - final period of time
- Plasma instabilities
  - change the bottom-up picture
  - no fast thermalization
- Wave turbulence as a possible mechanism for thermalization

# Heavy ion collisions at RHIC and LHC

- Just before the collision:



- gluon occupancy in the high energy heavy ion




◇ $Q_s(s)$ – saturation scale

$\simeq 1\text{GeV} \quad \text{at} \quad \text{RHIC}$

$\simeq 2 - 3\text{GeV} \quad \text{at} \quad \text{LHC}$

◇ $p_\perp \leq Q_s$: gluon saturation/CGC

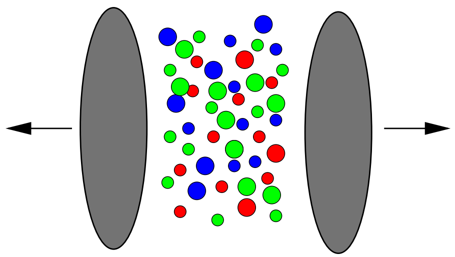
◇ $f \sim 1/\alpha$, large if $\alpha$ small

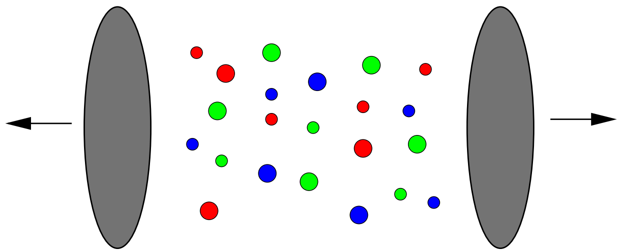
- During the collision:
  - ◇ gluons are freed, $p \sim Q_s$
  - ◇ typical gluon number/rapidity:

    $\simeq$ 600 at RHIC

    $\simeq$ 1500 at LHC

- Key issue:





CGC/initial stage
- ◇ high density, $f \sim 1/\alpha$
- ◇ in non-equilibrium

QGP/final stage
- ◇ lower density, $f \sim 1$
- ◇ in equilibrium

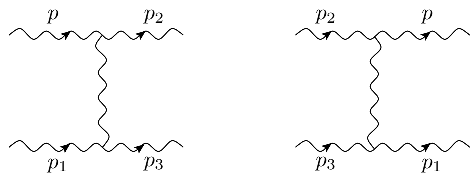
# Bottom-Up thermalization szenario

[Baier, Mueller, Schiff, Son 2001]

- Initial conditions given by saturation physics:
  - ◇ hard gluons $Q_s$, $f \sim 1/\alpha$
  - ◇ $\alpha(Q_s) \ll 1$, weak coupling methods
  - ◇ $f$ does not depend on $x, y$
- Dynamics given by Boltzmann equation

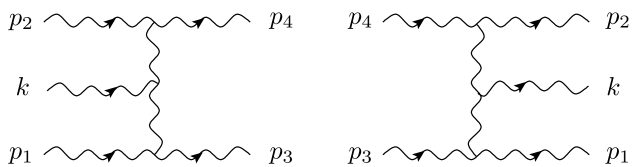
$$\left(\frac{\partial}{\partial\tau} + \frac{p_z}{\tau}\frac{\partial}{\partial p_z}\right) f(\tau, p) = C_{el}(p) + C_{inel}(p)$$

◇ Elastic interactions:



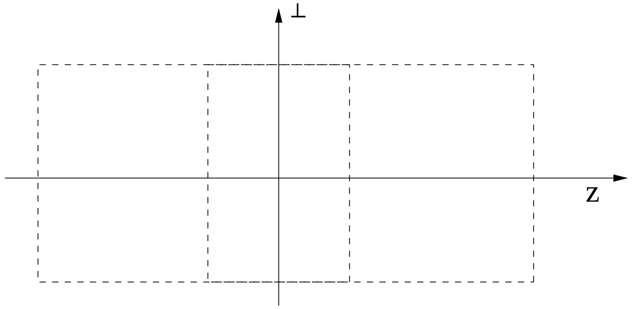

$$C_{el}(p) = \int \frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_3}{(2\pi)^3 2E_3}(2\pi)^4\delta^4(p+p_1-p_2-p_2)\cdot|M|^2$$
$$\times\left[f_p f_{p_1}(1+f_{p_2})(1+f_{p_3}) - f_{p_2} f_{p_3}(1+f_p)(1+f_{p_1})\right]$$

◇ Inelastic interactions:



# Early time period: $1 \ll Q_s\tau \ll \alpha^{-3/2}$

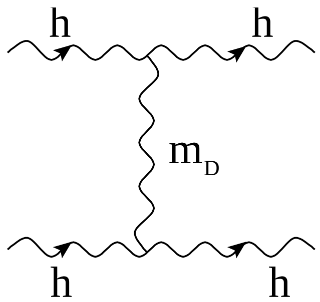
- Longitudinal expansion




$$N_h \sim \frac{Q_s^3}{\alpha(Q_s\tau)}$$

◇ density of hard gluons: $N_h = \int d^3p f_h(p)$

- Elastic interactions (small angle)




▷ $p_z$ - broadening

$$p_z^2 \sim N_{coll}\, m_D^2$$
$$\sim \sigma N_h(1+f_h)\tau m_D^2 \sim \frac{\alpha^2}{m_D^2} N_h f_h \tau m_D^2$$

$\Rightarrow p_z \sim (Q_s\tau)^{2/3}\frac{1}{\tau}$ $\quad$ [$p_z \sim \frac{1}{\tau}$, if no scatterings]

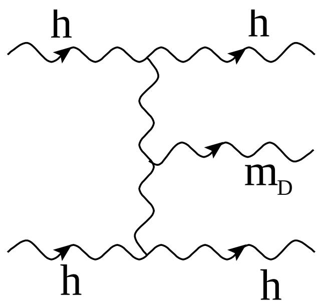
◇ Debye mass: $m_D^2 \sim \alpha \int d^3p\, \frac{f_h(p)}{p} \sim \alpha \frac{N_h}{Q_s} \sim \frac{Q_s^2}{Q_s\tau}$

▷ occupancy decreases with time

◇ $f_h \sim \frac{N_h}{Q_s^2\, p_z} \sim \frac{1}{\alpha(Q_s\tau)^{2/3}}$ $\quad$ [$f \sim \frac{1}{\alpha}$ , if no scattering]

◇ $f_h \sim 1$ at $(Q_s\tau) \sim \alpha^{-3/2}$ !!

- Inelastic interactions $\Rightarrow$ soft gluon production




$$N_s \sim \tau \, \frac{\partial N_s}{\partial \tau}$$
$$\sim \tau \frac{\alpha^3}{m_D^2} N_h^2 (1 + f_h)^2 \sim \frac{Q_s^3}{\alpha (Q_s \tau)^{4/3}}$$

- ◇ $m_D \Rightarrow k_s \sim p_z$ due to multiple scattering with hard gluons
- ◇ $N_h \gg N_s$

# Intermediate time period: $\alpha^{-3/2} \ll Q_s\tau \ll \alpha^{-5/2}$

- occupancy of hard gluons
  - $f_h < 1$ when $(Q_s\tau) > \alpha^{-3/2}$

- therefore
  - ▷ $k_s^2 \sim \sigma N_h(1+f_h)\tau m_D^2 \sim \frac{\alpha^2}{m_D^2} N_h \tau m_D^2 \sim \alpha Q_s$
  - ▷ $N_s \sim \tau \frac{\alpha^3}{m_D^2} N_h^2 (1+f_h)^2 \sim \frac{\alpha^{1/4} Q_s^3}{(Q_s\tau)^{1/2}}$

  $\Rightarrow m_D^2 \sim \frac{\alpha N_s}{k_s} \sim \frac{\alpha^{3/4} Q_s^2}{(Q_s\tau)^{1/2}}$ dominated by soft gluons

- note that at
  - $Q_s\tau \sim \alpha^{-3/2}$: $N_h \gg N_s$
  - $Q_s\tau \sim \alpha^{-5/2}$: $N_h \sim N_s \sim \alpha^{3/2} Q_s$

# Final time period; $\alpha^{-5/2} \ll Q_s\tau \ll \alpha^{-13/5}$

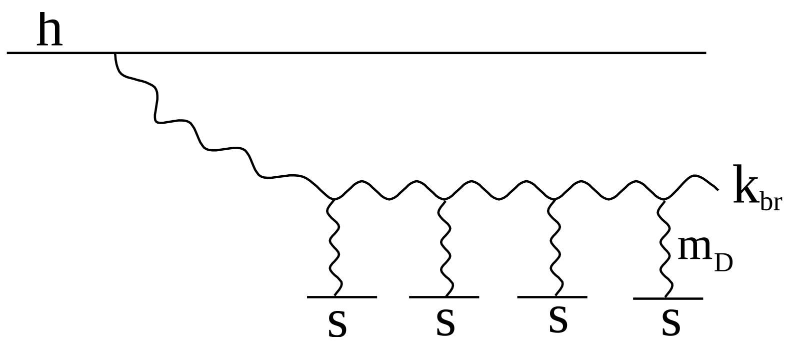
- $N_s > N_h$ at $Q_s\tau > \alpha^{-5/2}$
- soft particles thermalize after $Q_s\tau > \alpha^{-5/2}$

  ◇ $m_D \sim \alpha^{1/2}(\alpha^{1/2}Q_s)$ ◇ $N_s \sim (\alpha^{1/2}Q_s)^3$ ◇ $k_s \sim \alpha^{1/2}Q_s$

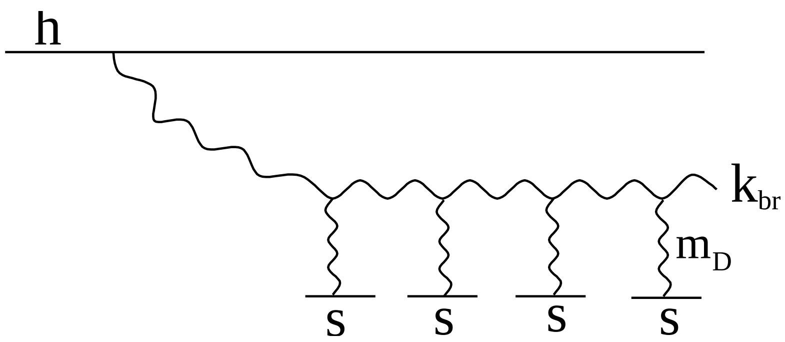
  Temperature of soft sector: $T \sim \alpha^{1/2}Q_s$

  Thermalized system: $m_D \sim \alpha^{1/2}T$, $N_s \sim T^3$, $k_s \sim T$

- Hard gluons lose energy to soft bath




◇ production of $k_{br}$ gluons LPM suppressed since $k_{br} > k_{LPM} \sim \frac{m_D^2}{N_s\sigma}$




- branching time $t_{br}$
  - ◇ $t_{br} \sim \frac{1}{\alpha} t_f$
  - ◇ formation time: $t_f \sim \frac{k_{br}}{k_t^2}$ , $\left[ v_t t_f \sim x_t \sim k_t^{-1}, \quad v_t \sim \frac{k_t}{k_{br}} \right]$
  - ◇ transv. momentum: $k_t^2 \sim N_{coll}\, m_D^2\, t_f \sim \frac{\alpha^2}{m_D^2} N_s\, m_D^2\, t_f \sim \alpha^2\, N_s\, t_f$

  $\Rightarrow t_{br} \sim \frac{k_{br}^{1/2}}{\alpha^2\, N_s^{1/2}}$

- momentum $k_{br}$ over a time $\tau$
  - ◇ $k_{br} \sim t_{br}^2\, \alpha^4\, N_s \sim \alpha^4\, \tau^2\, T^3$ $\qquad [k_{br} > k_{LPM}]$

- Energy flow to thermal bath

  $\dot{\epsilon} \sim \frac{dN(k_{br})}{d\tau}\; k_{br} \sim \frac{N_h}{t_{br}}\; k_{br} \sim \alpha^3\, Q_s^2\, T^3$

- Temperature of soft bath

  ◇ $\dot{\epsilon} \sim \frac{dT^4}{d\tau} \quad \Rightarrow \quad T \sim \alpha^3\, Q_s^2\, \tau$

  $\Rightarrow$ temperature increases as system expands

- whole system thermalizes when

  ◇ $N_h\, Q_s \sim T^4 \quad \Rightarrow \quad Q_s \tau_{eq} \sim \alpha^{-13/5}$

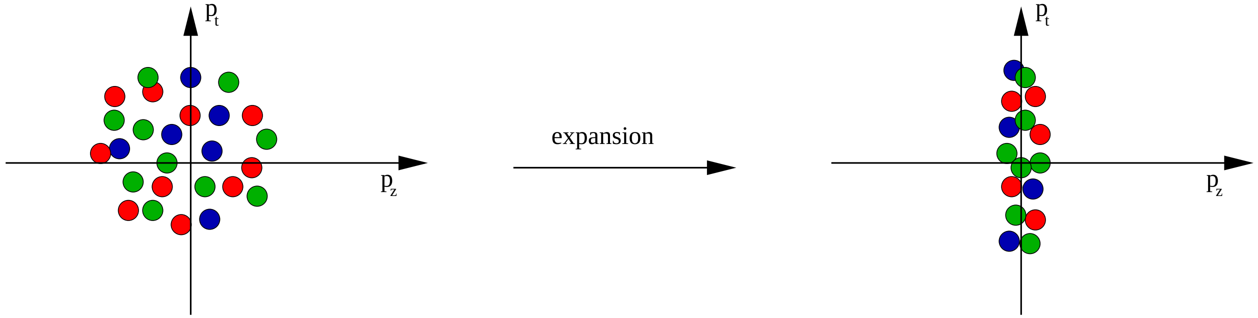
  ◇ estimated thermalization time: $\sim 2 - 3\,\mathrm{fm}$.

- final temperature: $T \sim Q_s \alpha^{2/5}$

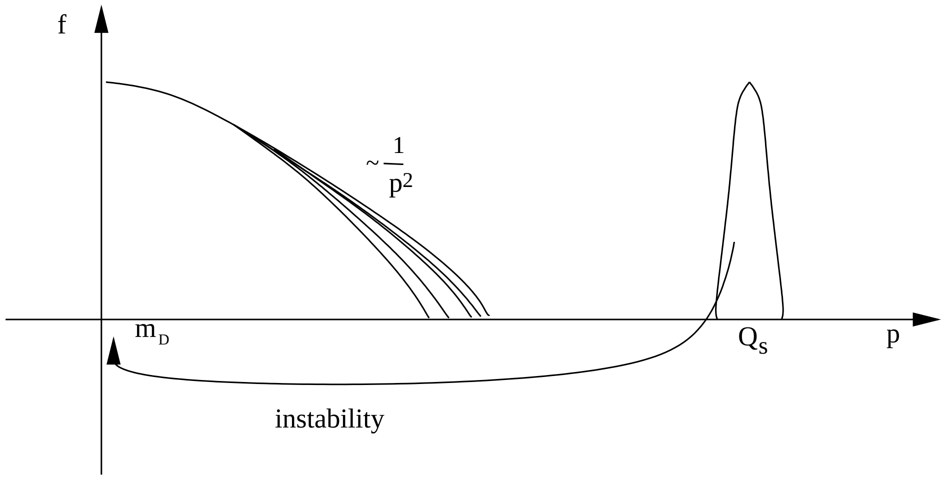
# Plasma instabilities

[Arnold, Lenaghan, Moore, Yaffe], [Romatschke, Strickland], [Dumitru, Nara], [Mrowczynski 97]

- longitudinal expansion ▷ anisotropic momentum distribution




▷ Plasma instabilities:




◇ soft modes ($\sim m_D$) produced exponentially

◇ typical time scale: $t_{inst} \sim \frac{1}{m_D} \sim \sqrt{\frac{\tau}{Q_s}}$

◇ instability fastest mechanism: $t_{inst} \ll t_{coll} \ll \tau$

▷ early stage of bottom-up szenario changes, Bethe-Heitler ⇒ instability

# Lower bound for thermalization time

[Arnold,Lenaghan 04]

- Requirements:

  ◇ $\tau_{eq} \geq t_{typ} \sim \frac{1}{\alpha^2 T}$ $\quad [t_{typ} \sim \frac{1}{\sigma N} \sim \frac{1}{\alpha^2/T^2 T^3} \sim \frac{1}{\alpha^2 T}]$

  ◇ energy conservation: $N_h\, Q_s \sim T^4$ $\quad \triangleright \quad T \sim \frac{Q_s}{(\alpha\, Q_s\, \tau_{eq})^{1/4}}$

- Bound: $Q_s\, \tau_{eq} \geq \alpha^{-7/3}$

  ▷ close to bottom-up thermalization time: $\alpha^{-13/5}$

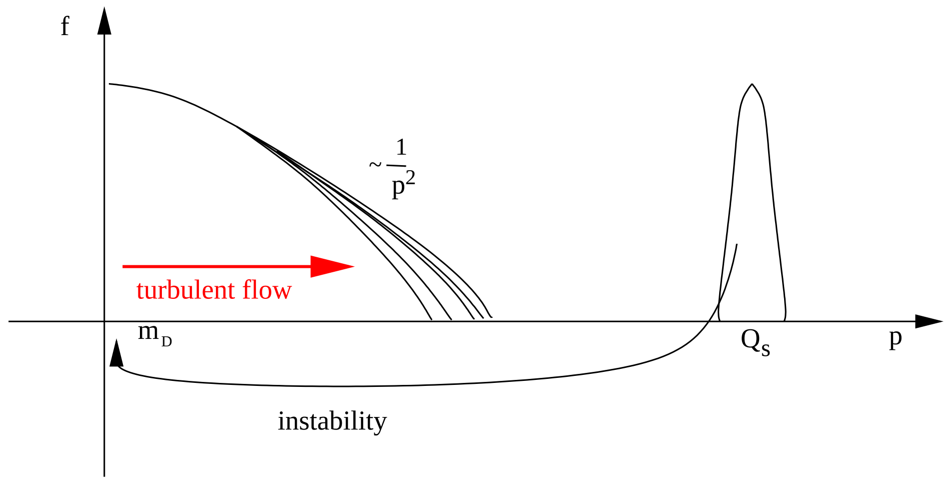
  ▷ no fast thermalization due to instability

- sytem may match onto late stages of bottom-up szenario

  [Mueller,Shoshi,Wong 05]

# Thermalization and wave turbulence

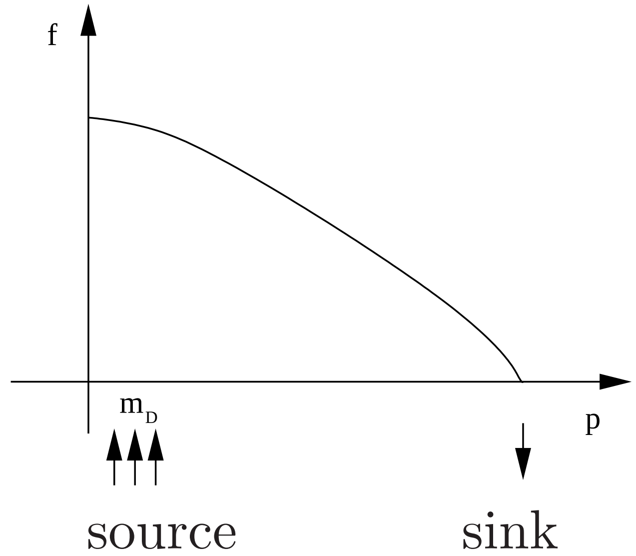
- Heavy ion collisions:




  ◇ approx. constant flow of energy
  ◇ turbulent-like spectrum

- Similar problem studied by Zakharov, L'vov and Falkovich ($\phi^4$ theory):

  ◇ rate of inflow of energy: $\dot{\epsilon}_0 = m_0^5/\alpha$
  ◇ spatially homogenous, spherically symmetric



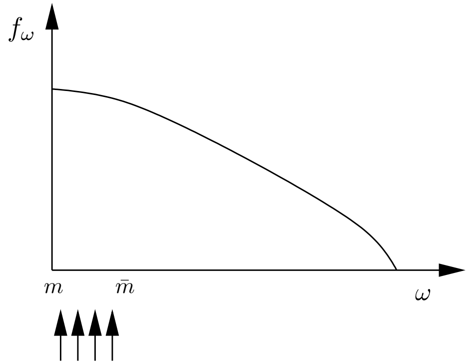

  ◇ $\phi^4$ theory:
  ◇ interaction local in momentum
  ▷ $f_p \sim \frac{1}{\alpha}\left(\frac{m_0}{p}\right)^{5/3}$ from $\dot{\epsilon}_0 \sim \dot{\epsilon}_p$

where $\dot{\epsilon}_p = (4\pi p^2 \dot{p} f_p)p$ and $\dot{p} \sim p/\lambda \sim pN_p f_p \sigma \sim p(p^3 f_p) f_p \alpha^2/p^2$

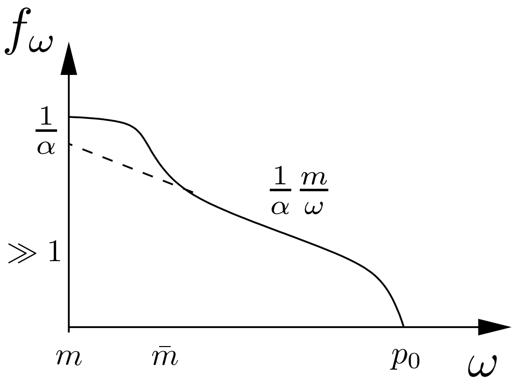
# Kolmogorov wave turbulence in QCD

[Mueller, Shoshi, Wong 2006]

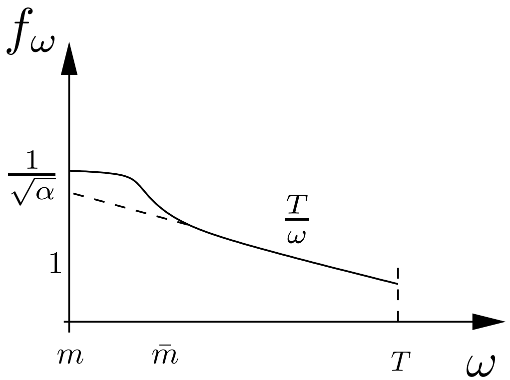
- Input:
  - ◇ $\dot{\epsilon}_0 = \frac{m_0^5}{\alpha}$, in modes $m < \omega < \bar{m}$
  - ◇ spatially homogenous, sperically symmetric
  - ◇ Boltzmann equation: $\frac{df(p)}{dt} = C[f]$



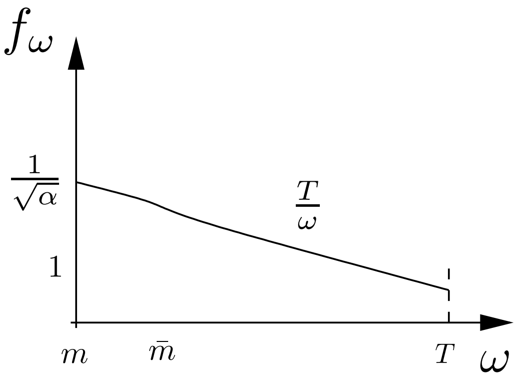

- Results:
  - ◇ spectrum, $f_p \sim \frac{1}{p}$ at $p \gg m$




$1 < m_0\, t < \alpha^{-7/5}$




$\alpha^{-7/5} < m_0\, t < \alpha^{-9/5}$



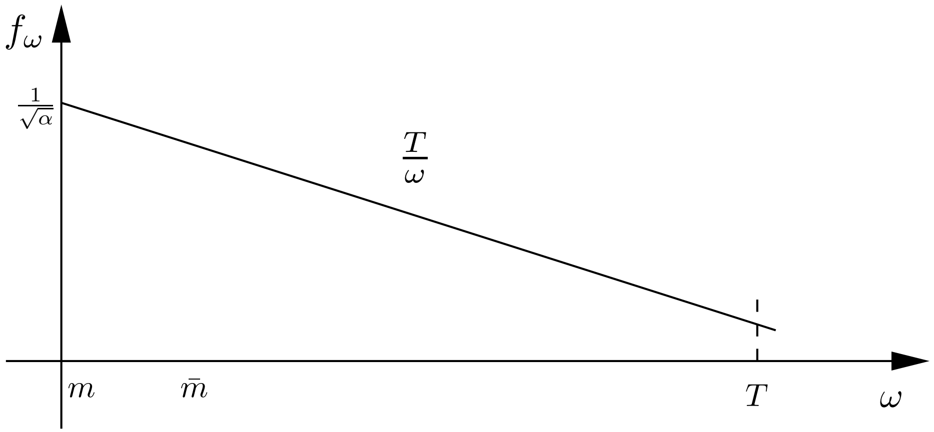

$\alpha^{-9/5} < m_0\, t$

◇ In QCD, energy transfer from soft to hard directly!

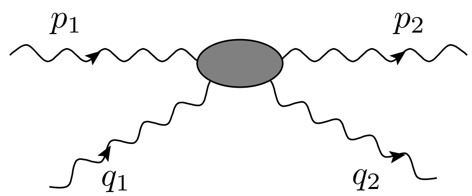
# Back-up transparencies

# A. The late time regime, $\alpha^{-9/5} < m_0 t$

- System close to equilibrium:
  - ◇ spectrum: $f_\omega \approx \frac{1}{e^{\omega/T}-1}$, $f_\omega \sim T/\omega$ at $\omega/T \ll 1$
  - ◇ Temperature: $T \simeq m_0 \left(\frac{m_0 t}{g_E \alpha}\right)^{1/4}$ , from $\epsilon(t) = \frac{m_0^5}{\alpha} t \simeq g_E T^4$
  - ◇ plasma frequency: $m = \sqrt{\frac{4\pi\alpha N_c}{9}}\, T$



- elastic interactions:




$q_1, q_2 \sim m$

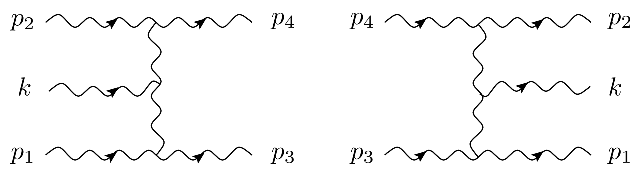
◇ Boltzmann equation

$$\dot{\epsilon}^{el} \sim \frac{m_0^5}{\alpha} \alpha^{9/4}(m_0 t)^{5/4} \left[\omega_1 - \omega_2 + \frac{T}{f_{q_2}} - \frac{T}{f_{q_1}}\right] \frac{1}{m} \frac{p}{T}$$

▷ $\alpha^{9/4}(m_0 t)^{5/4} > 1$ or $m_0 t > \alpha^{-9/5}$ ▷ close to equilibrium ($f_q \sim T/\omega$)

▷ energy transferred directly from soft to hard!

- inelastic interactions:

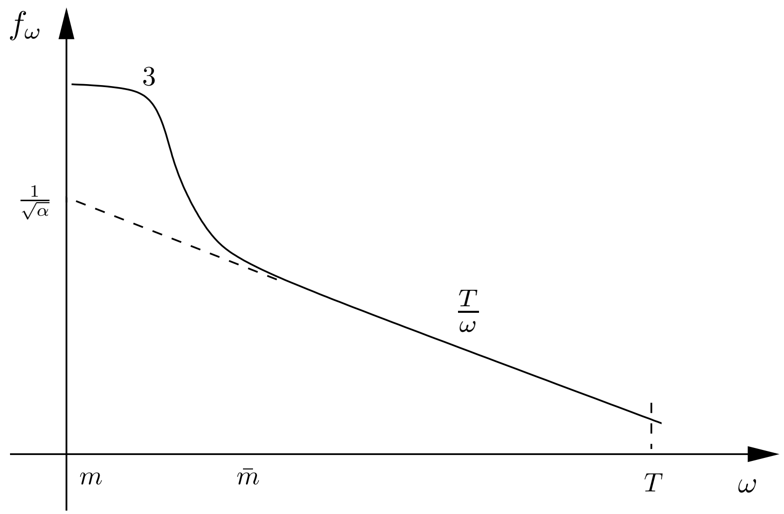



◇ $\dot{\epsilon}_k^{in} \sim \frac{m_0^5}{\alpha} \alpha^{9/4}(m_0 t)^{5/4} \left[\omega - \frac{T}{f_k}\right] \frac{1}{m}$

▷ $f_k$ close to equilibrium

# B. Intermediate time regime, $\alpha^{-7/5} < m_0 t < \alpha^{-9/5}$

- Spectrum:




late time formula: $\dot{\epsilon}^{in}_{\omega} \sim \frac{m_0^5}{\alpha}\,\alpha^{9/4}(m_0 t)^{5/4}\left[\omega - \frac{T}{f_\omega}\right]\frac{1}{m}\,[\sqrt{\alpha} f_\omega]$

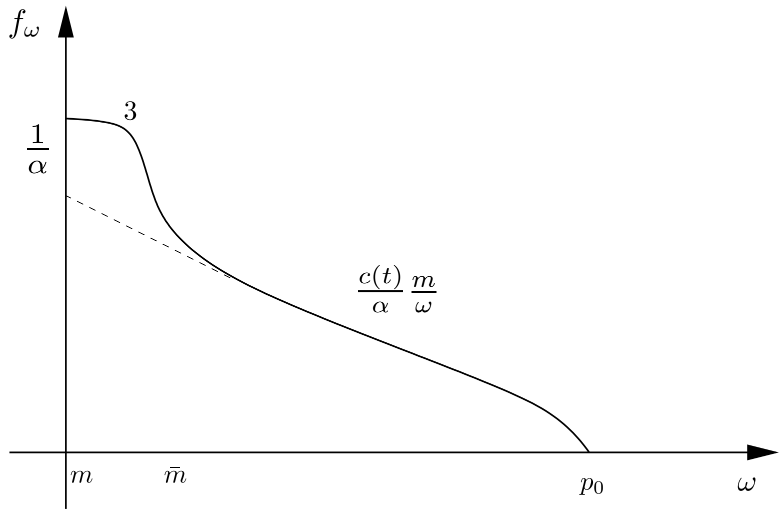
◇ at $m_0 t < \alpha^{-9/5}$ , $\dot{\epsilon}^{in}_{\omega} < \frac{m_0^5}{\alpha}$ unless $f_w > 1/\sqrt{\alpha}$ in region $m < \omega < \bar{m}$

◇ $f_\omega$ has equilibrium distribution when $\omega > \bar{m}$

◇ max. occupation: $f_{\omega_3} \sim \frac{1}{\sqrt{\alpha}}\,\frac{1}{\left(\alpha^{9/5} m_0 t\right)^5}$ , $k_3/m \sim (\alpha^{9/5}\, m_0 t)^{5/4}$

# C. The early time regime, $1 < m_0 t < \alpha^{-7/5}$

- Spectrum:




◇ hardest momenta, $p_0$, not equilibrated ($\lambda_{p_0}/t \sim 1$)

◇ From

◇ $\dot{f}(t) \sim m^3 \nabla^2 f(p)$ Boltzmann eq.

◇ $\frac{m_0^5}{\alpha} t \sim f_{p_0} p_0^4$ energy conservation

◇ $m^2 \sim \alpha f_{p_0} p_0^2$ screeening mass

▷ $f_\omega \simeq \frac{c(t)m}{\alpha\omega}$ at $\omega \gg m$

where $c(t) \sim (m_0 t)^{-5/4}$, $m \sim m_0 (m_0 t)^{1/14}$, $p_0 \sim m_0 (m_0 t)^{3/7}$

▷ max. occupancy: $f_{\omega_3} \simeq \frac{1}{\alpha}(m_0 t)^{15/14}$ , $k_3/m \sim (m_0 t)^{-5/4}$

# Kolmogorov wave turbulence in QCD

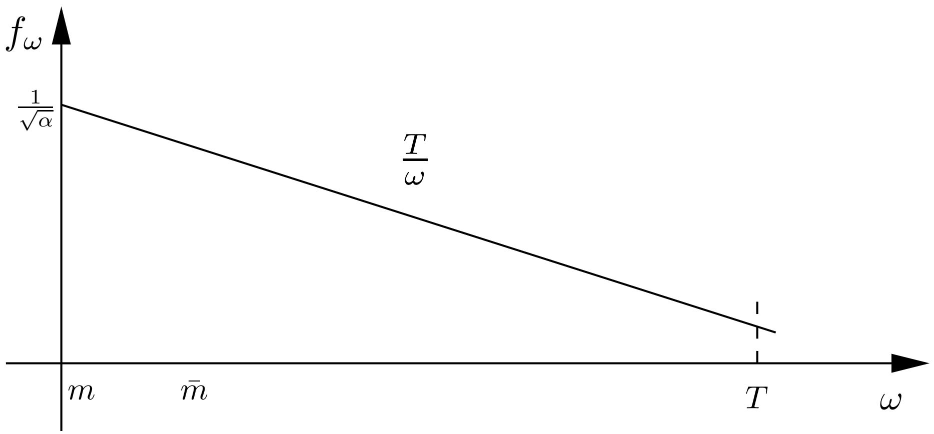
[Mueller, Shoshi, Wong 2006]

- QCD sytem:
  - ◇ constant rate of incoming energy: $\dot{\epsilon}_0 = \frac{m_0^5}{\alpha}$
  - ◇ spatially homogenous, sperically symmetric in momentum space
  - ◇ energy inflow in modes $m < \omega < \bar{m}$, $m$ plasma frequency
- A. The late time regime, $\alpha^{-9/5} < m_0 t$

◇ Close to equilibrium: $f_\omega \approx \frac{1}{e^{\omega/T}-1} \sim T/\omega$ at $\omega/T \ll 1$

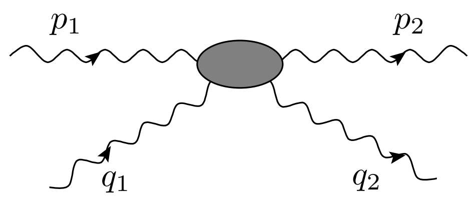
◇ Temperature: $T \simeq m_0 \left(\frac{m_0 t}{g_E \alpha}\right)^{1/4}$, from $\epsilon(t) = \frac{m_0}{\alpha} t \simeq g_E T^4$

◇ plasma frequency: $m = \sqrt{\frac{4\pi\alpha N_c}{9}}\, T$



# Energy transfer from soft to hard

- elastic interactions:




$q_1, q_2 \sim m$

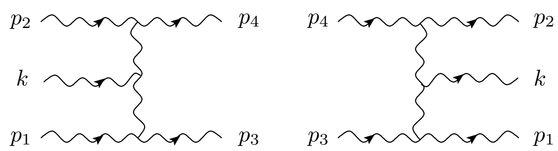
◇ Boltzmann equation

$$\dot{\epsilon}^{el} \sim \frac{m_0^5}{\alpha} \alpha^{9/4}(m_0 t)^{5/4} \left[\omega_1 - \omega_2 + \frac{T}{f_{q_2}} - \frac{T}{f_{q_1}}\right] \frac{1}{m} \frac{p}{T}$$

▷ $\alpha^{9/4}(m_0 t)^{5/4} > 1$ or $m_0 t > \alpha^{-9/5}$ ▷ close to equilibrium ($f_q \sim T/\omega$)

▷ energy transferred directly from soft to hard!

- inelastic interactions:

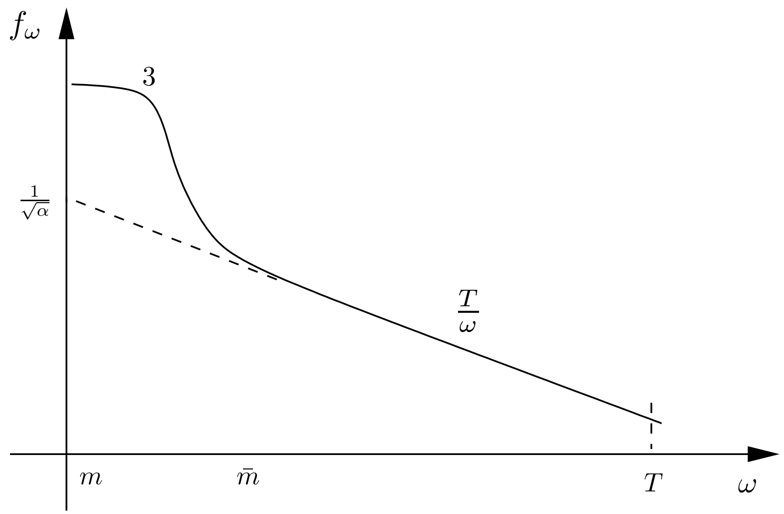



◇ $\dot{\epsilon}_k^{in} \sim \frac{m_0^5}{\alpha} \alpha^{9/4}(m_0 t)^{5/4} \left[\omega - \frac{T}{f_k}\right] \frac{1}{m}$

## Interaction non-local in momentum in QCD!

## B. Intermediate time regime, $\alpha^{-7/5} < m_0 t < \alpha^{-9/5}$

- Spectrum:




late time formula: $\dot{\epsilon}_k^{in} \sim \frac{m_0^5}{\alpha}\, \alpha^{9/4} (m_0 t)^{5/4} \left[\omega - \frac{T}{f_k}\right] \frac{1}{m}$

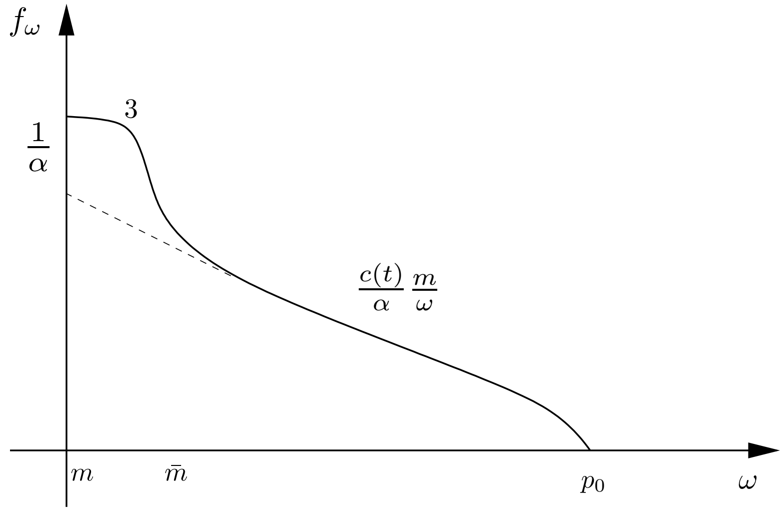
◇ at $m_0 t < \alpha^{-9/5}$ $\quad \dot{\epsilon}_k^{in} < \frac{m_0^5}{\alpha}$ unless $f_w > 1/\sqrt{\alpha}$ in region $m < \omega < \bar{m}$

◇ $f_\omega$ has equilibrium distribution when $\omega > \bar{m}$

◇ max. occupation: $f_{\omega_3} \sim \frac{1}{\sqrt{\alpha}} \frac{1}{\left(\alpha^{9/5} m_0 t\right)^5}$

# C. The early time regime, $1 < m_0 t < \alpha^{-7/5}$

- Spectrum:




from Boltzmann equation and energy conservation:

◇ hardest momenta, $p_0$, not equilibrated $(\lambda/t \sim 1)$

◇ at $\omega > \bar{m}$, $f_\omega \simeq \frac{c(t)m}{\alpha\omega}$ where

$$c(t) \sim (m_0 t)^{-5/4}, \quad m \sim m_0 (m_0 t)^{1/14}, \quad p_0 \sim m_0 (m_0 t)^{3/7}$$

◇ max. occupancy: $f_{\omega_3} \simeq \frac{1}{\alpha}(m_0 t)^{15/14}$