The first fm/c of a nucleus-nucleus collision in the Color Glass Condensate framework

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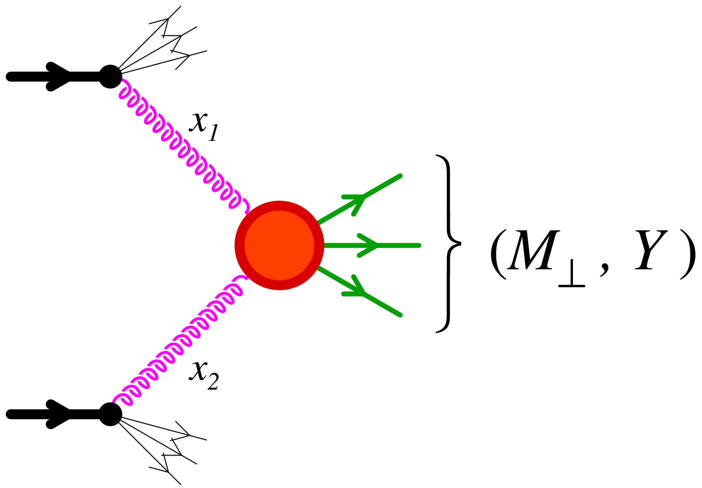
Introduction

Infrared and collinear divergences

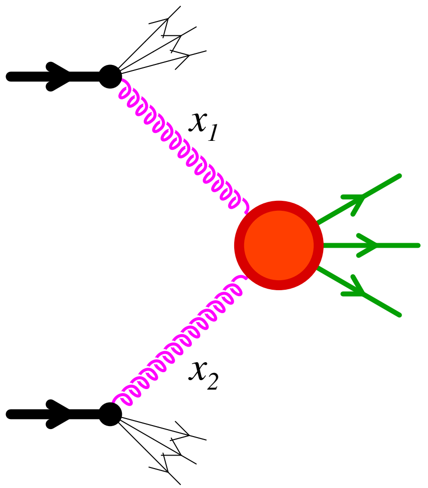
- Calculation of some process at LO :



$$\begin{cases} x_1 = M_\perp\, e^{+Y}/\sqrt{s} \\ x_2 = M_\perp\, e^{-Y}/\sqrt{s} \end{cases}$$

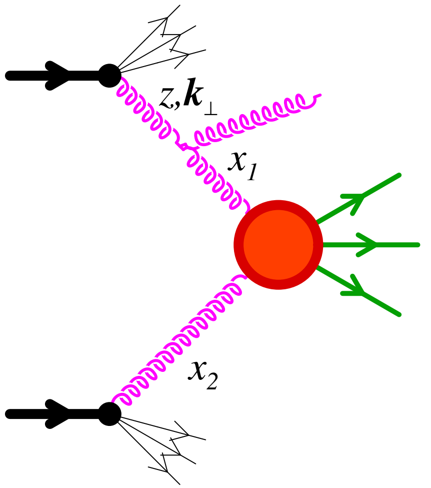
Infrared and collinear divergences

- Calculation of some process at LO :



$$\left.\right\}\ (M_\perp\,,\,Y) \qquad \begin{cases} x_1 = M_\perp\, e^{+Y}/\sqrt{s} \\ x_2 = M_\perp\, e^{-Y}/\sqrt{s} \end{cases}$$

- Radiation of an extra gluon :



$$\left.\right\}\ (M_\perp\,,\,Y) \qquad \Longrightarrow \qquad \alpha_s \int_{x_1} \frac{dz}{z} \int^{M_\perp} \frac{d^2\vec{\boldsymbol{k}}_\perp}{\boldsymbol{k}_\perp^2}$$

Factorization

- Logs of $M_\perp$ can be resummed by :
  - promoting $f(x_1)$ to $f(x_1, M_\perp^2)$
  - letting $f(x_1, M_\perp^2)$ evolve with $M_\perp$ according to the DGLAP equation

$$\frac{\partial f(x,M^2)}{\partial \ln(M^2)} = \alpha_s(M^2)\int_x^1 \frac{dz}{z} P(x/z) \otimes f(z,M^2)$$

  ▷ collinear factorization

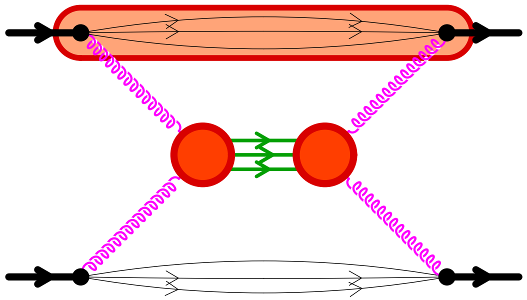
- Logs of x_1 can be resummed by :
  - promoting $f(x_1)$ to a non integrated distribution $\varphi(x_1, \vec{\boldsymbol{k}}_\perp)$
  - letting $\varphi(x_1, \vec{\boldsymbol{k}}_\perp)$ evolve with x_1 according to the BFKL equation

$$\frac{\partial \varphi(x,k_\perp)}{\partial \ln(1/x)} = \alpha_s \int \frac{d^2\vec{\boldsymbol{p}}_\perp}{(2\pi)^2} K(\vec{\boldsymbol{k}}_\perp, \vec{\boldsymbol{p}}_\perp) \otimes \varphi(x, \vec{\boldsymbol{p}}_\perp)$$

  ▷ $k_\perp$-factorization

Higher twist corrections

- Leading twist :



▷ 2-point function in the projectile ▷ gluon number

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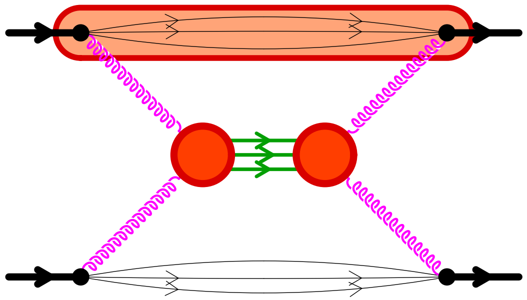
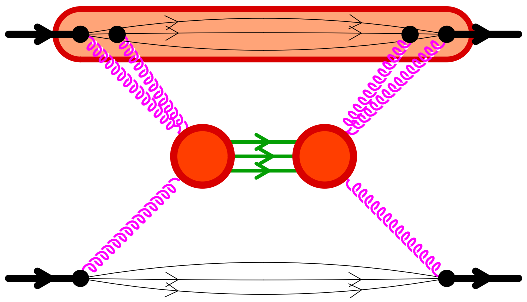


▷ 2-point function in the projectile ▷ gluon number

- Higher twist contributions :



▷ 4-point function in the projectile ▷ higher correlation
▷ multiple scatterings in the projectile

Higher twist corrections

- Power counting : rescattering corrections are suppresssed by inverse powers of the typical mass scale in the process :

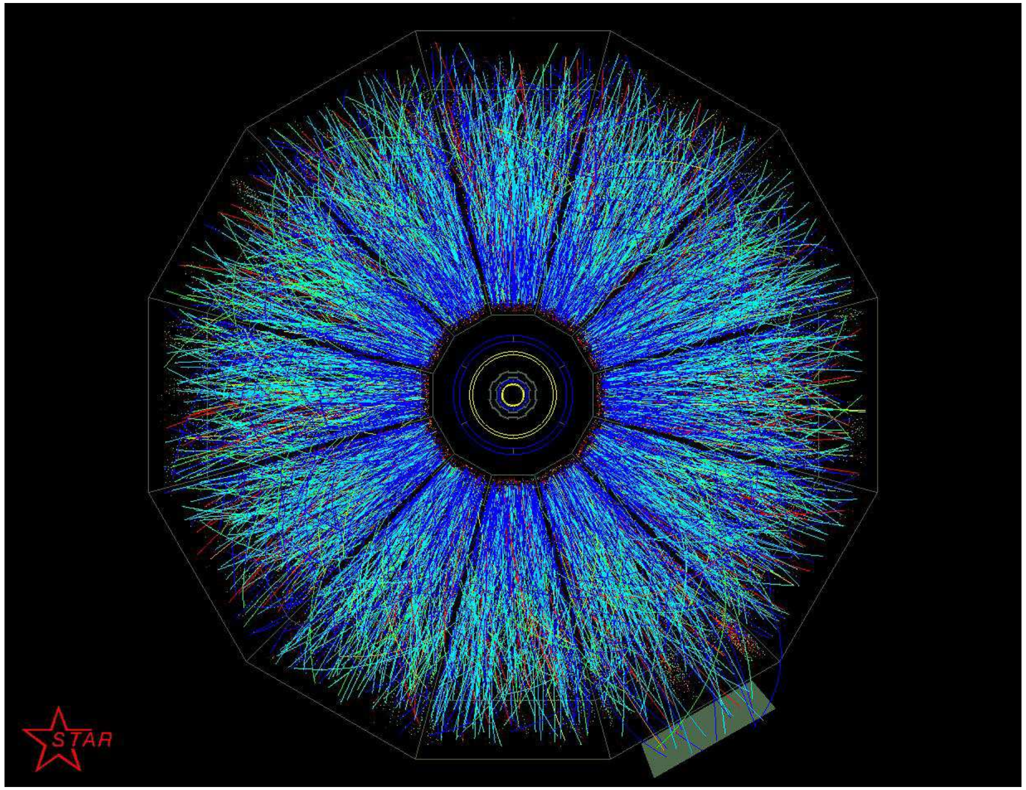
$$\left[\frac{\mu^2}{M_\perp^2}\right]^n$$

- The parameter μ^2 has a factor of α_s, and a factor proportional to the gluon density $\triangleright$ rescatterings are important at high density
- Relative order of magnitude :

$$\frac{\text{2 scatterings}}{\text{1 scattering}} \sim \frac{Q_s^2}{M_\perp^2} \quad \text{with} \quad Q_s^2 \sim \alpha_s \frac{xG(x,Q_s^2)}{\pi R^2}$$

- When this ratio becomes ~ 1, all the rescattering corrections become important
- These effects are not accounted for in DGLAP or BFKL

Higher twist corrections



- 99% of the multiplicity below $p_\perp \sim 2$ GeV
- Q_s^2 might be as large as 5 GeV^2 at the LHC ($\sqrt{s} = 5.5$ TeV)
  $\triangleright$ multiple scatterings are important, and one should also resum logs of $1/x$

Goals

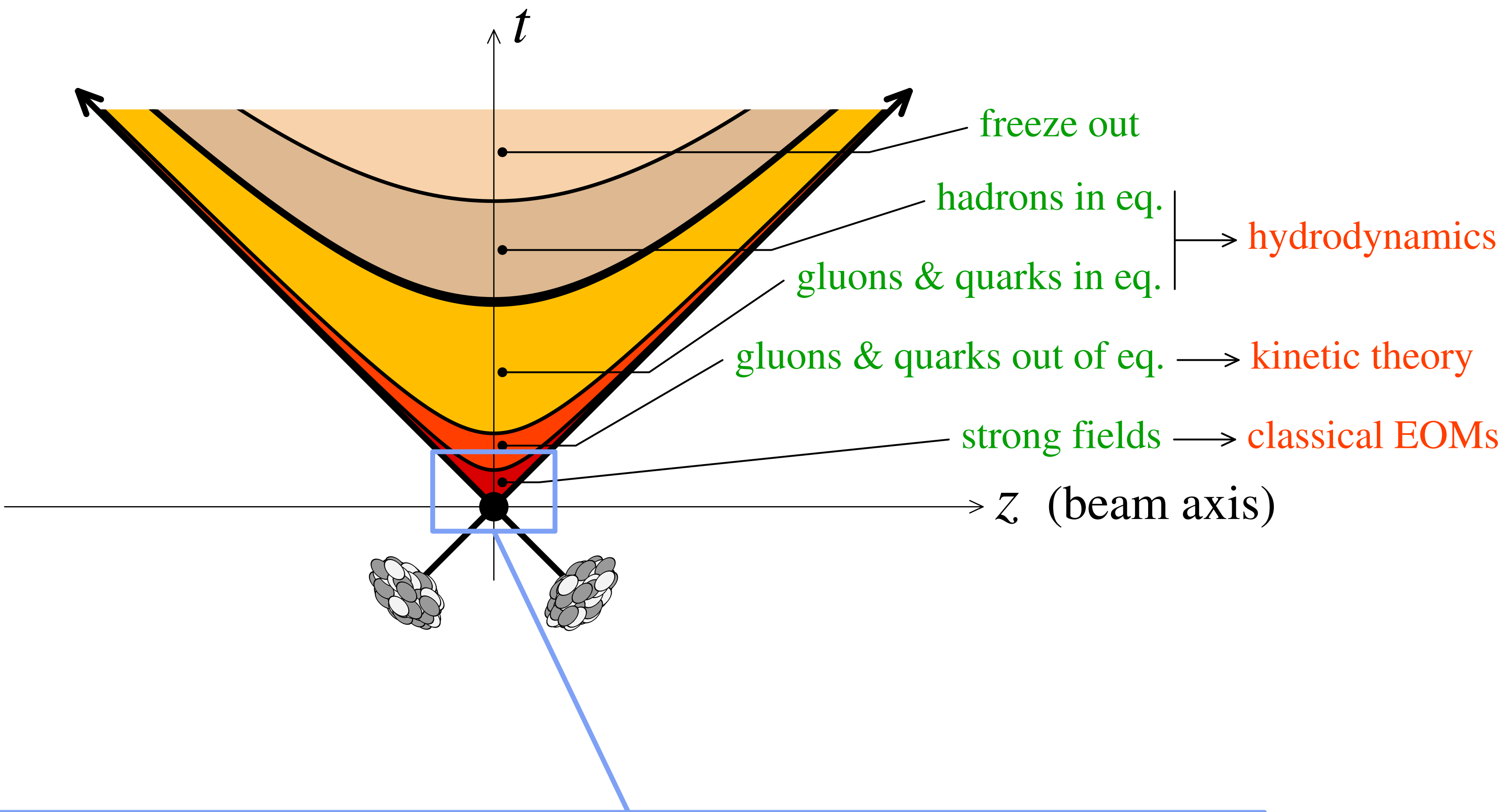
- The Color Glass Condensate framework provides the technology for resumming all the $[\alpha_s \ln(1/x)]^m [Q_s/\boldsymbol{p}_\perp]^n$ corrections
- Generalize the concept of “parton distribution”

  ▷ This was achieved by describing hadrons as a collection of color sources with a density $\rho(\vec{\boldsymbol{x}}_\perp)$. Their distribution $W[\rho]$ contains all the information we need

  ▷ These distributions should be universal, with non-perturbative information relegated into the initial condition for the evolution

  ▷ If logs of $1/x$ show up in loop corrections, one should be able to factor them out into the evolution of these distributions
- There may possibly be extra divergences associated with the evolution of the final state

Initial Conditions



- calculate the initial production of semi-hard particles
- prepare the stage for kinetic theory or hydrodynamics

Outline

- Basic principles and bookkeeping
- Inclusive gluon spectrum at leading order
- Loop corrections, factorization, instabilities
- Less inclusive quantities

- Baltz, FG, Peshier, McLerran, nucl-th/0101024
- FG, Venugopalan, hep-ph/0601209, 0605246
- Fukushima, FG, McLerran, hep-ph/0610416

+ work in progress with Lappi, Venugopalan

Basic principles

Degrees of freedom and their interplay

McLerran, Venugopalan (1994), Iancu, Leonidov, McLerran (2001)

- Soft modes have a large occupation number
  ▷ they are described by a classical color field A^μ that obeys Yang-Mills's equation:

$$[D_\nu, F^{\nu\mu}]_a = J^\mu_a$$

- The source term J^μ_a comes from the faster partons. The hard modes, slowed down by time dilation, are described as frozen color sources ρ_a. Hence :

$$J^\mu_a = \delta^{\mu +}\delta(x^-)\rho_a(\vec{\boldsymbol{x}}_\perp) \qquad\qquad (x^- \equiv (t-z)/\sqrt{2})$$

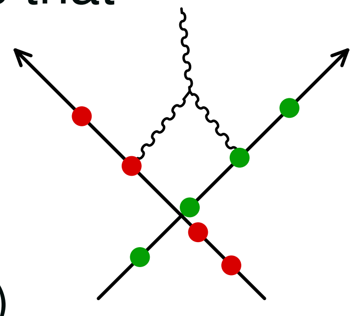
- The color sources ρ_a are random, and described by a distribution functional $W_Y[\rho]$, with Y the rapidity that separates "soft" and "hard". Evolution equation (JIMWLK) :

$$\frac{\partial W_Y[\rho]}{\partial Y} = \mathcal{H}[\rho]\;\; W_Y[\rho]$$

Description of hadronic collisions

- In DIS, only one of the two projectiles is described by the CGC (the other is elementary) ▷ the Yang-Mills equations can be solved analytically, and the interaction of the virtual photon with the color field is given by the eikonal limit

- For hadron-hadron collisions, there are two sources that contribute to the color current :



$$J^\mu \equiv \delta^{\mu +}\delta(x^-)\,\rho_1(\vec{\boldsymbol{x}}_\perp) + \delta^{\mu -}\delta(x^+)\,\rho_2(\vec{\boldsymbol{x}}_\perp)$$

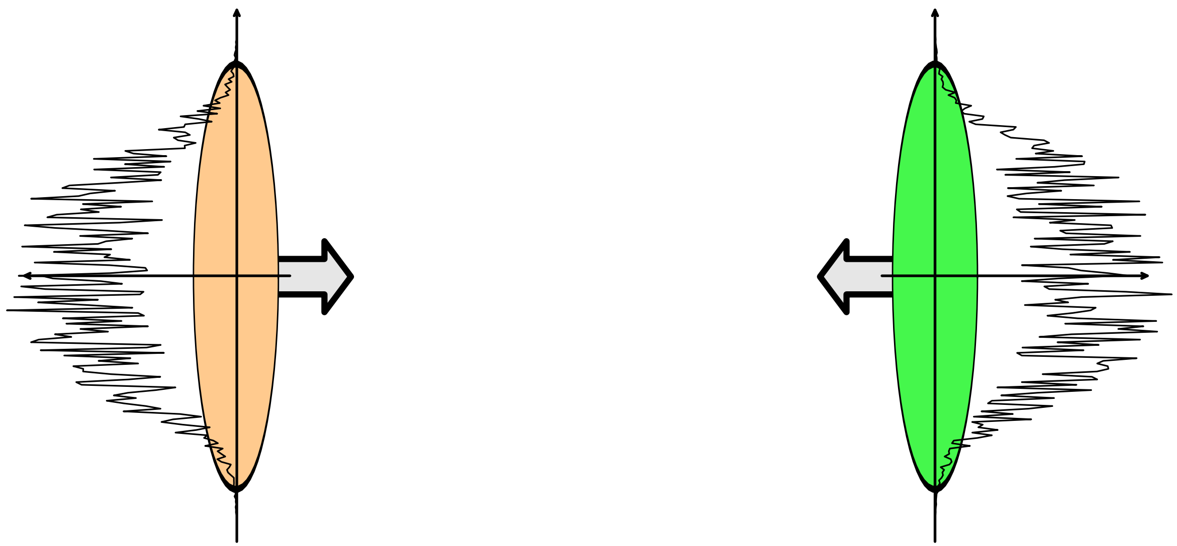
(Note: the boundary condition depend on the observable)

- Average over the sources ρ_1, ρ_2

$$\langle \mathcal{O}_Y \rangle = \int [D\rho_1]\,[D\rho_2]\; W_{Y_{\rm beam}-Y}[\rho_1]\; W_{Y+Y_{\rm beam}}[\rho_2]\; \mathcal{O}[\rho_1,\rho_2]$$

- Can this procedure – and in particular the above factorization formula – be justified ?

Description of hadronic collisions



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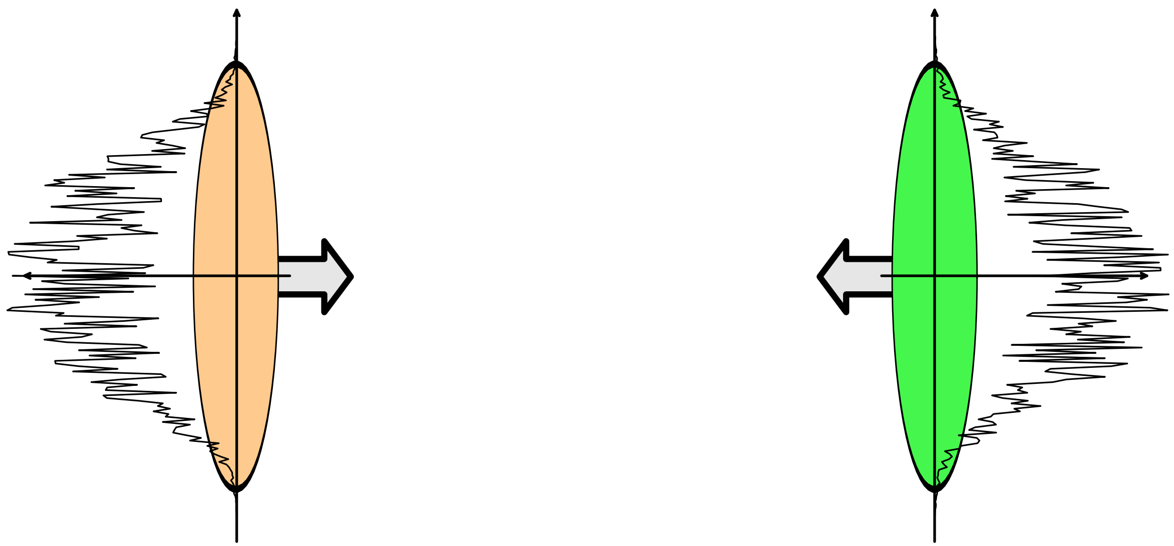

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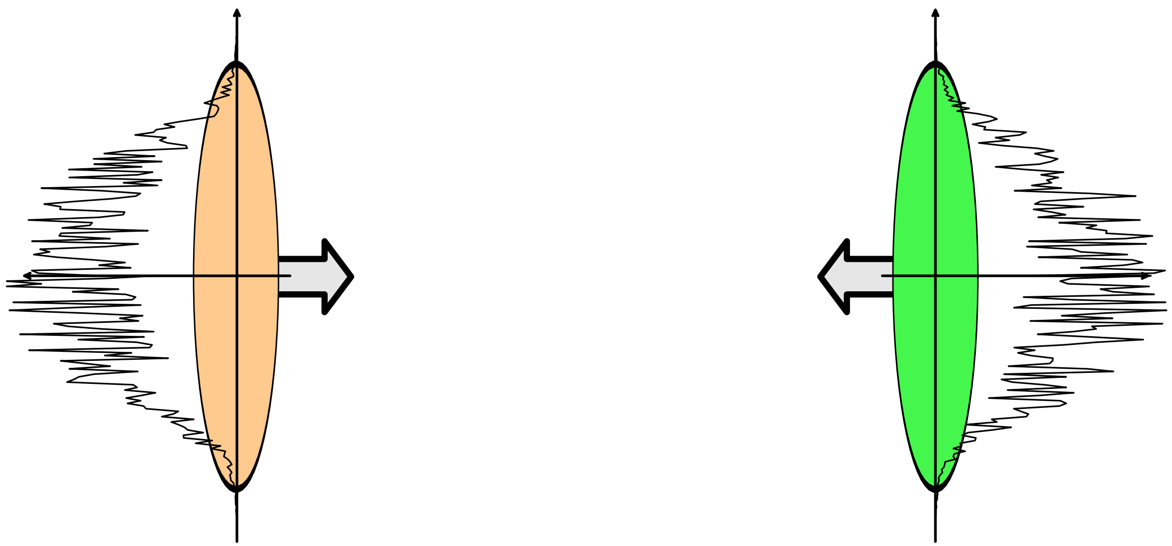

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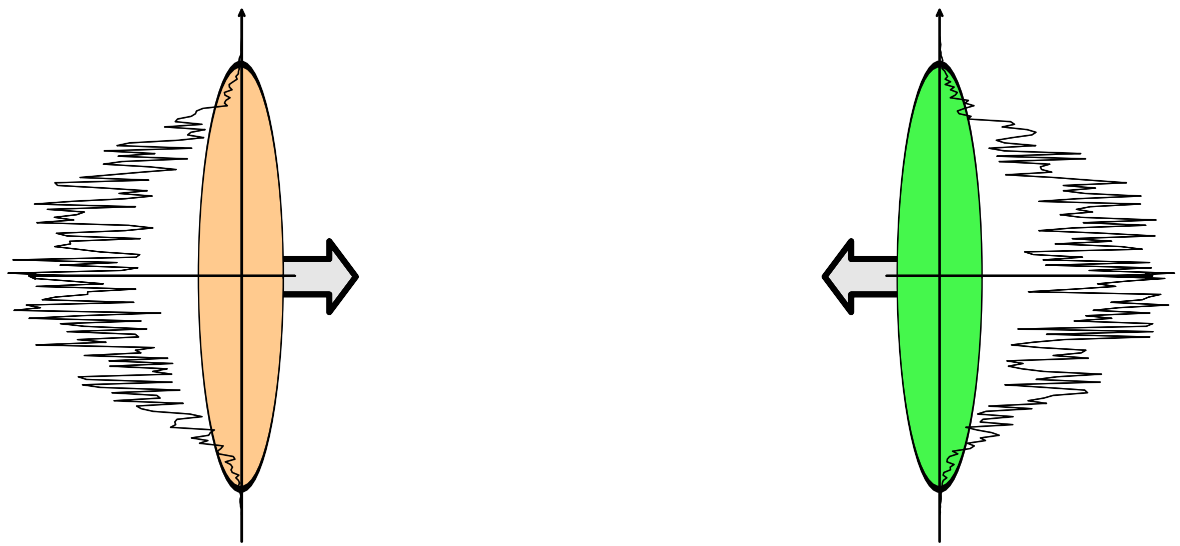

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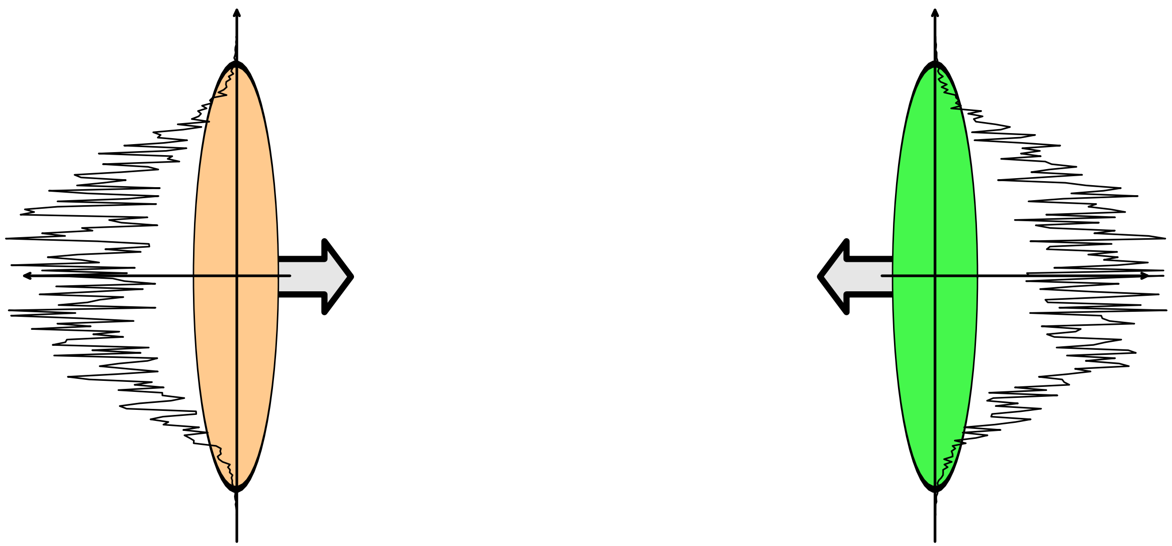

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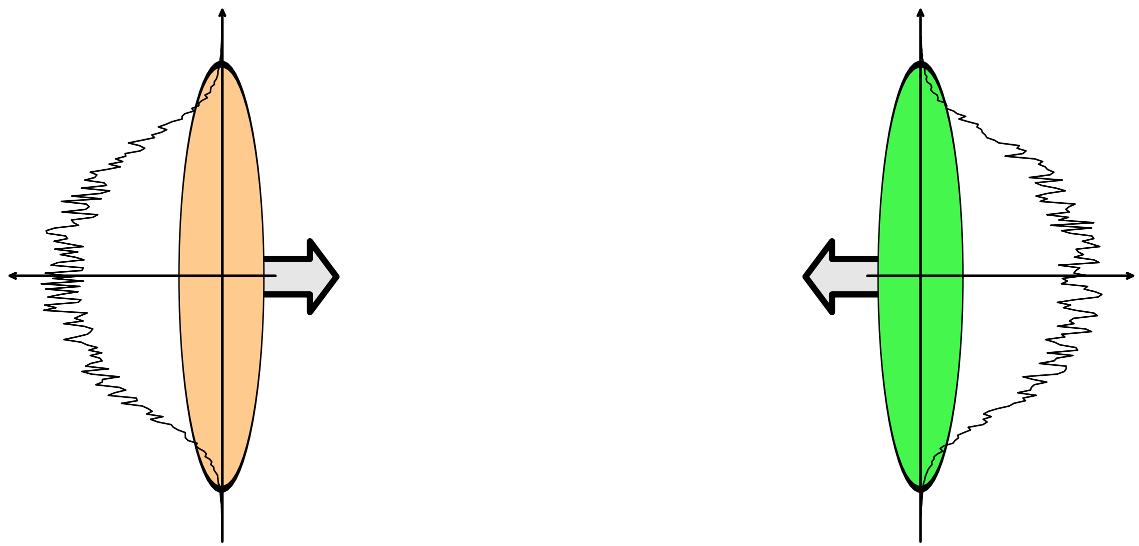
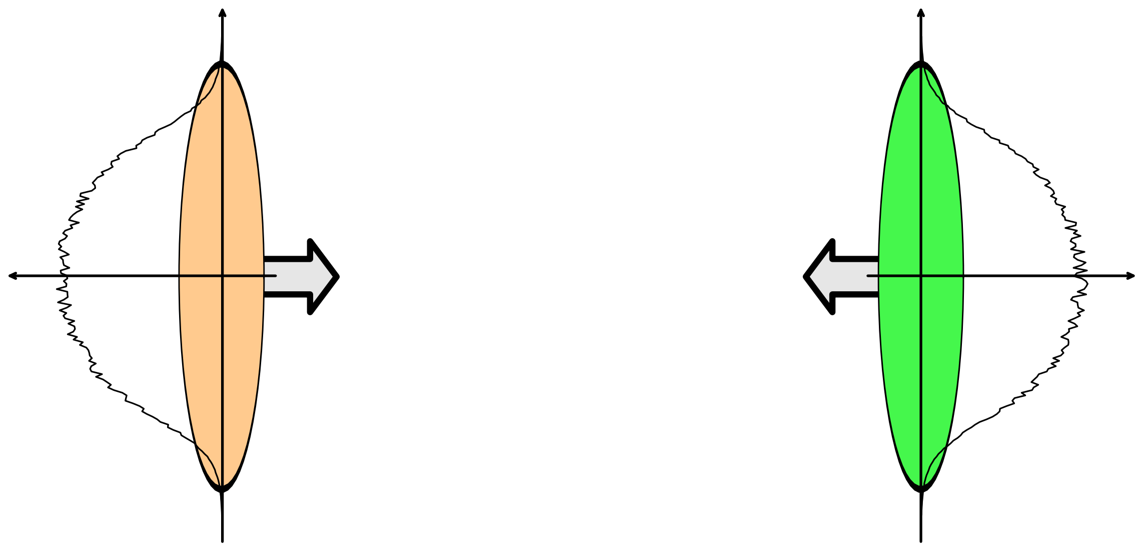


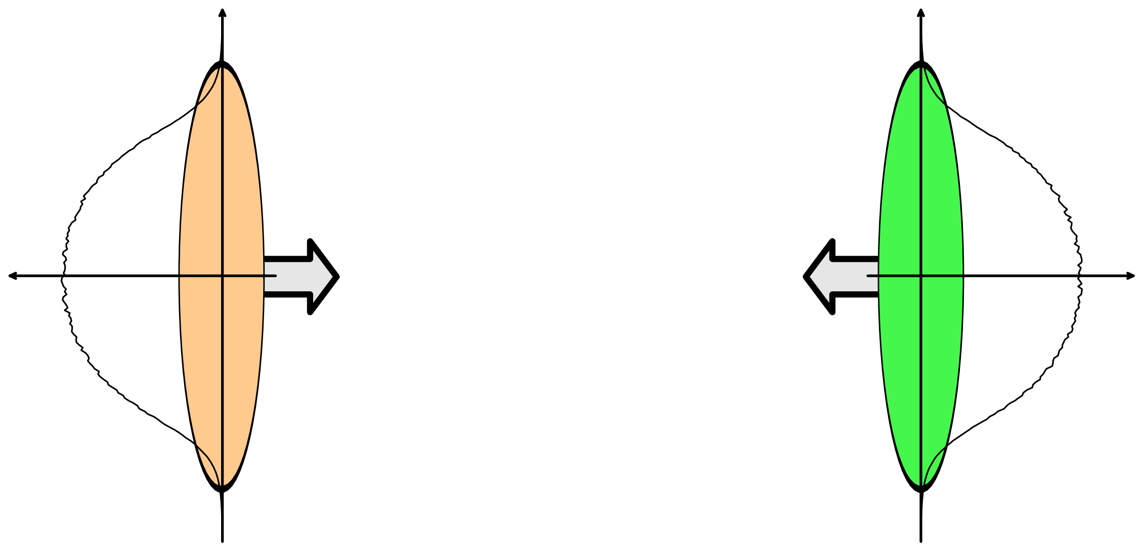
10 configurations

Description of hadronic collisions



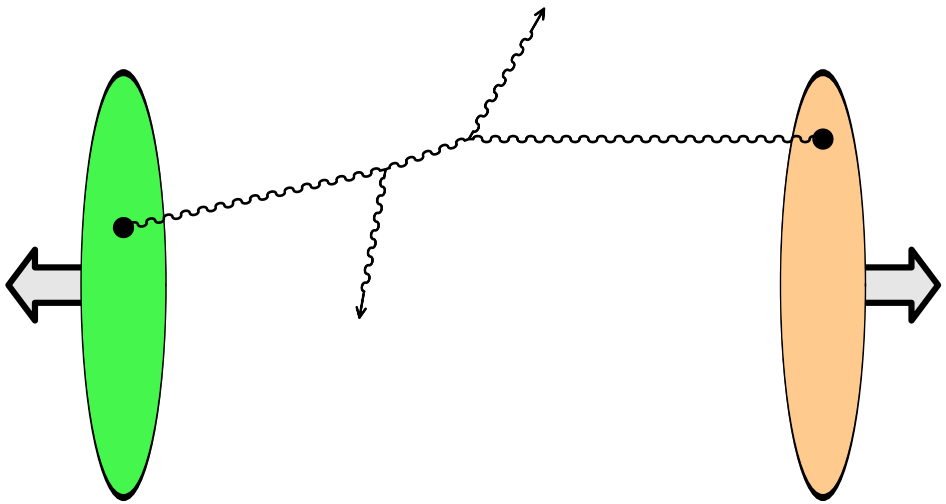
100 configurations

Description of hadronic collisions



1000 configurations

Main issues



- Dilute regime : one source in each projectile interact

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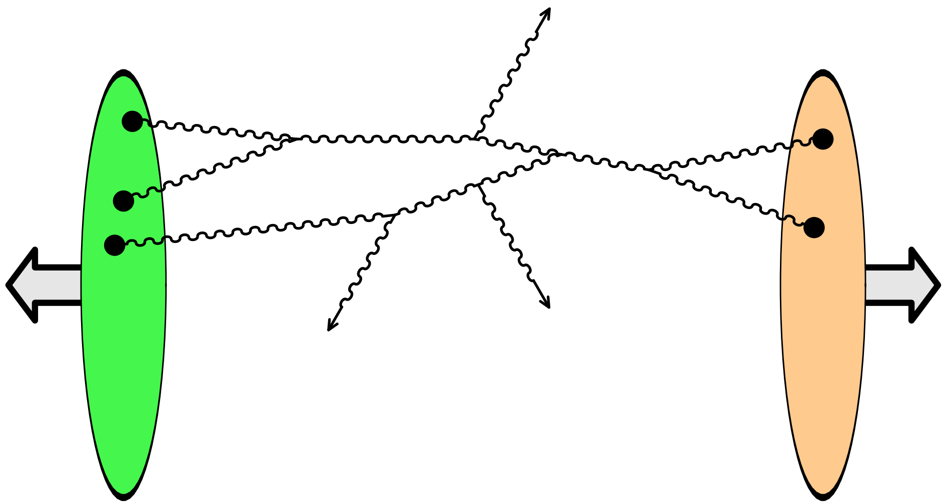

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- Dense regime : non linearities are important

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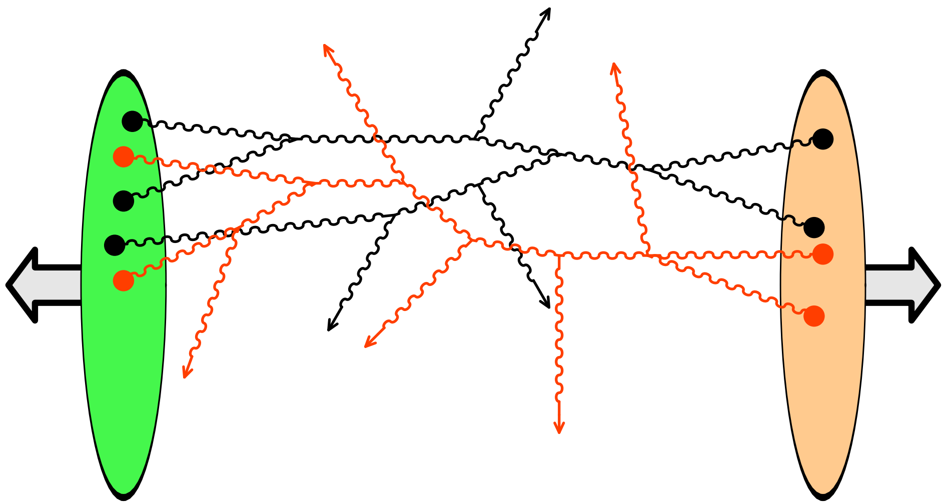

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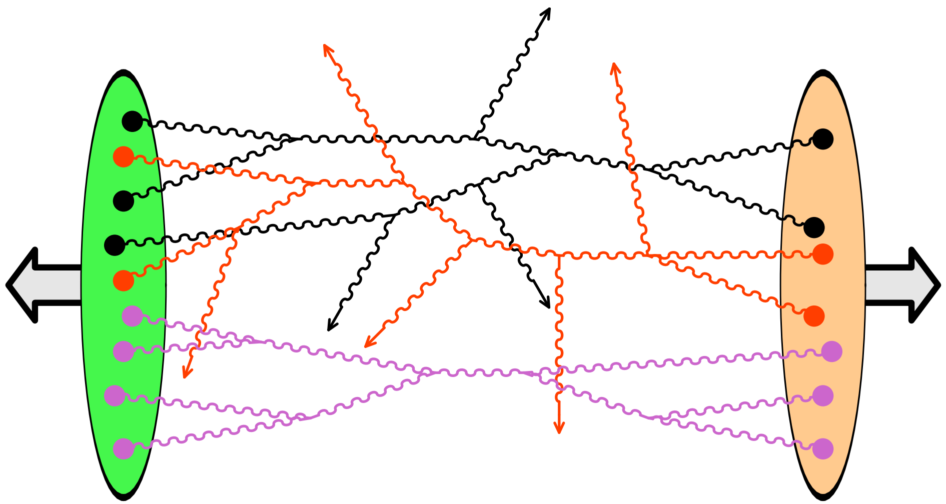

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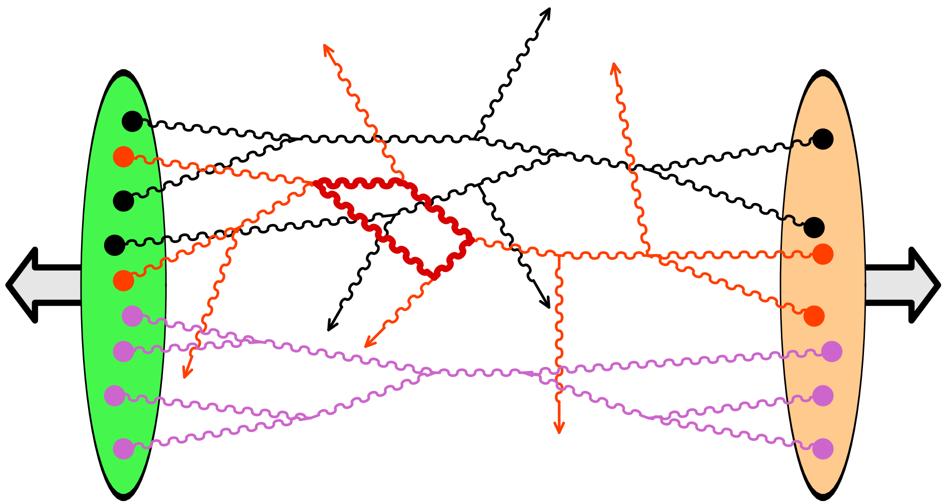


- Dilute regime : one source in each projectile interact
- Dense regime : non linearities are important
- There can be many simultaneous disconnected diagrams
- Some of them may not produce anything (vacuum diagrams)
- All these diagrams can have loops (not at LO though)

Field theories with sources

- To avoid encumbering the discussion with unessential (for now) details, we first consider a scalar field theory with a ϕ^3 coupling, coupled to a source $j(x)$:

$$\mathcal{L} \equiv \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 + j\phi$$

- From the Heisenberg field operator $\phi(x)$, one can define two free fields $\phi_{\rm in}(x)$ and $\phi_{\rm out}(x)$, which coincide with $\phi(x)$ respectively at $t = -\infty$ and $t = +\infty$:

$$\begin{aligned}
\phi(x) &= U(-\infty, x^0)\phi_{\rm in}(x)U(x^0, -\infty) \\
\phi(x) &= U(+\infty, x^0)\phi_{\rm out}(x)U(x^0, +\infty)
\end{aligned}$$

with $\quad U(t_2, t_1) \equiv T \exp i \int_{t_1}^{t_2} d^4x \, \mathcal{L}_{\rm int}(\phi_{\rm in/out}(x))$

- These free fields have a simple Fourier decomposition :

$$\phi_{\rm in/out}(x) = \int \frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3 2E_k}\left[a_{\rm in/out}(\vec{\boldsymbol{k}})e^{-ik\cdot x} + a^\dagger_{\rm in/out}(\vec{\boldsymbol{k}})e^{+ik\cdot x}\right]$$

Reduction formulas

- In order to express transition amplitudes in terms of field operators, we need the following relations :

$$a_{\rm in/out}(\vec{\boldsymbol{k}}) = i\int d^3\vec{\boldsymbol{x}}\; e^{ik\cdot x}\,\Big[\partial_0 - iE_k\Big]\;\phi_{\rm in/out}(x)$$

- When calculating particle production in this theory, the initial state is the vacuum $\big|0_{\rm in}\big\rangle$

- Production of a single particle :

$$\big\langle \boldsymbol{p}_{\rm out}\big|0_{\rm in}\big\rangle = \frac{1}{Z^{1/2}}\int d^4x\; e^{ip\cdot x}\;(\square_x+m^2)\big\langle 0_{\rm out}\big|\phi(x)\big|0_{\rm in}\big\rangle$$

- Production of a two particles :

$$\begin{aligned}\big\langle \vec{\boldsymbol{p}}\vec{\boldsymbol{q}}_{\rm out}\big|0_{\rm in}\big\rangle &= \frac{1}{Z}\int d^4x\, d^4y\; e^{iq\cdot y} e^{ip\cdot x}\\ &\quad\times(\square_x+m^2)(\square_y+m^2)\big\langle 0_{\rm out}\big|T\phi(x)\phi(y)\big|0_{\rm in}\big\rangle\end{aligned}$$

Perturbative expansion

- In order to calculate $\big\langle 0_{\rm out}\big|T\phi(x_1)\cdots\phi(x_n)\big|0_{\rm in}\big\rangle$ in perturbation theory, we start from :

$$\big\langle 0_{\rm out}\big|T\phi(x_1)\cdots\phi(x_n)\big|0_{\rm in}\big\rangle =$$
$$= \big\langle 0_{\rm in}\big|T\phi_{\rm in}(x_1)\cdots\phi_{\rm in}(x_n)e^{i\int_{-\infty}^{+\infty}d^4x\mathcal{L}_{\rm int}(\phi_{\rm in})}\big|0_{\rm in}\big\rangle$$

  and we expand the exponential to the desired order

- This expansion generates vacuum-vacuum diagrams, whose sum appears as a multiplicative prefactor.

- If $j=0$, the sum of the vacuum-vacuum diagrams, $\big\langle 0_{\rm out}\big|0_{\rm in}\big\rangle$, is a pure phase and can be disregarded from squared amplitudes. This is not the case here

Vacuum-vacuum diagrams

- If at least one transition amplitude $\big\langle\vec{\boldsymbol{p}}\cdots{}_{\rm out}\big|0_{\rm in}\big\rangle$ is non-zero, then the vacuum-to-vacuum transition amplitude $\big\langle 0_{\rm out}\big|0_{\rm in}\big\rangle$ is non trivial. Indeed,

$$\sum_\alpha \left|\big\langle\alpha_{\rm out}\big|0_{\rm in}\big\rangle\right|^2=1 \qquad \text{(unitarity)}$$

implies $\left|\big\langle 0_{\rm out}\big|0_{\rm in}\big\rangle\right|^2<1$ if at least one term with $\alpha\neq 0$ is non-zero

- A source $j(x)$ that describes a single projectile does not produce particles. Indeed, it is static in the rest-frame of this projectile, and therefore can only have space-like modes

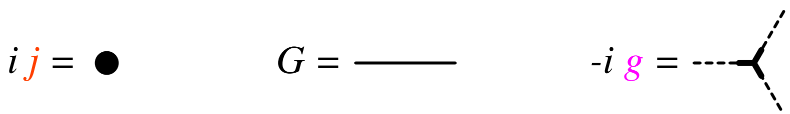
- A source $j(x)\equiv j_1(x)+j_2(x)$ describing two projectiles moving in opposite directions can produce particles

Vacuum-vacuum diagrams

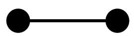
- The sum of all the vacuum-vacuum diagrams in $\left\langle 0_{\rm out} \middle| 0_{\rm in} \right\rangle$ is equal to the exponential of the sum of the connected ones

$$\left\langle 0_{\rm out} \middle| 0_{\rm in} \right\rangle = e^{iV[j]}$$

- Let us denote :



- The perturbative expansion of $iV[j]$ starts with :



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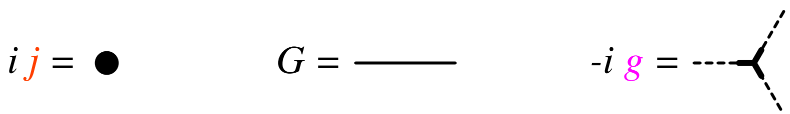

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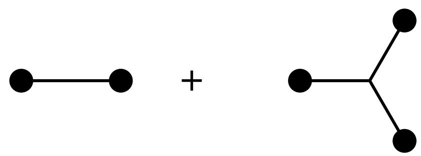

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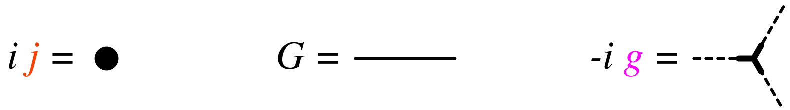

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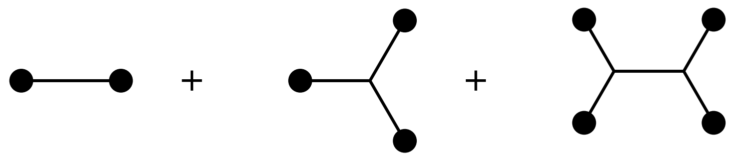

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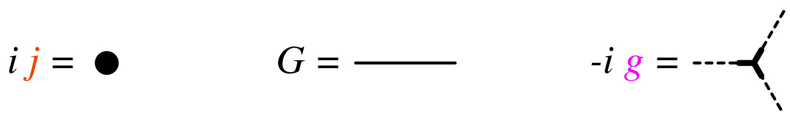

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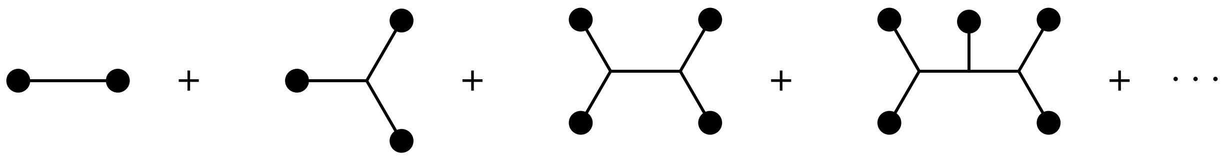

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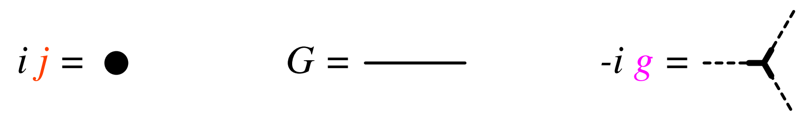

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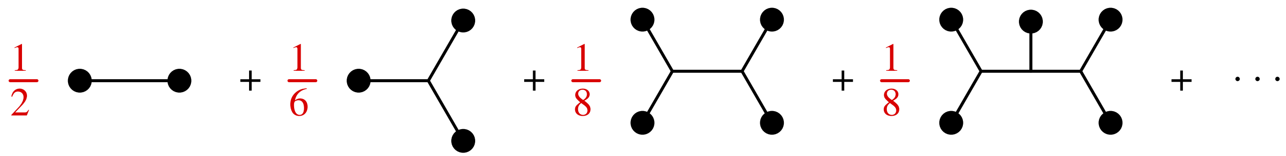

Note : each graph Γ comes with a symmetry factor $1/S_\Gamma$, where S_Γ is the order of its symmetry group

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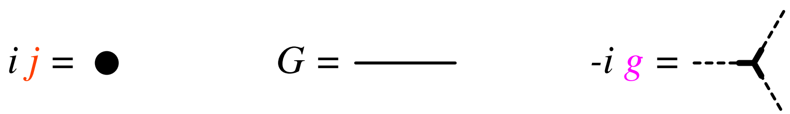

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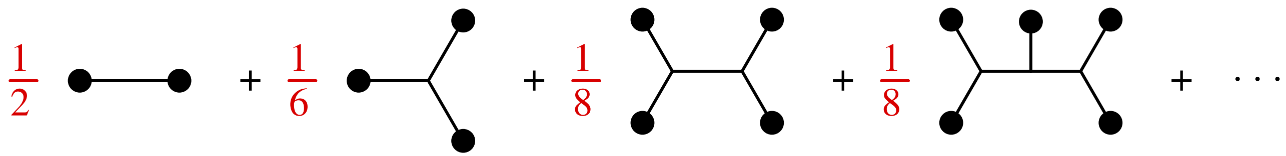

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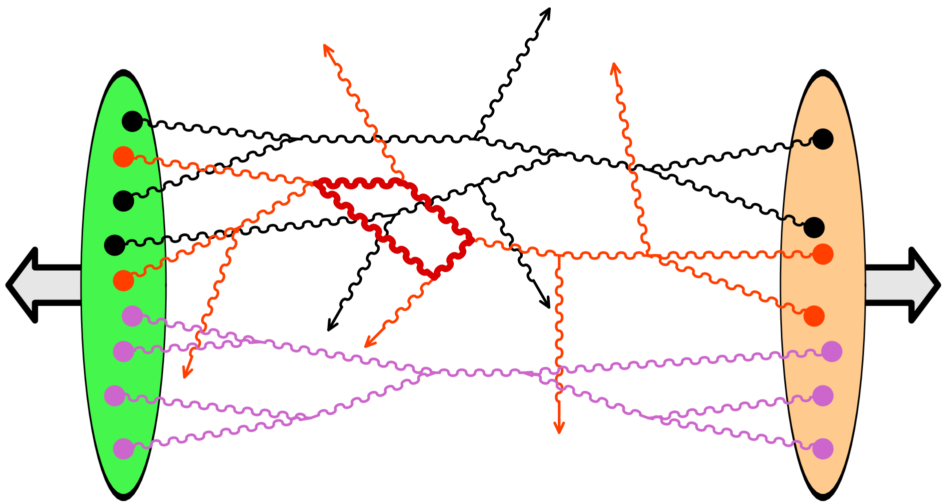
- Note : $\langle 0_{\rm out} | 0_{\rm in} \rangle$ can be seen as a generating functional :

$$\langle 0_{\rm out} | T\phi(x_1)\cdots\phi(x_n) | 0_{\rm in} \rangle = \frac{\delta}{i\delta j(x_1)} \cdots \frac{\delta}{i\delta j(x_n)} \; e^{iV[j]}$$

Power counting



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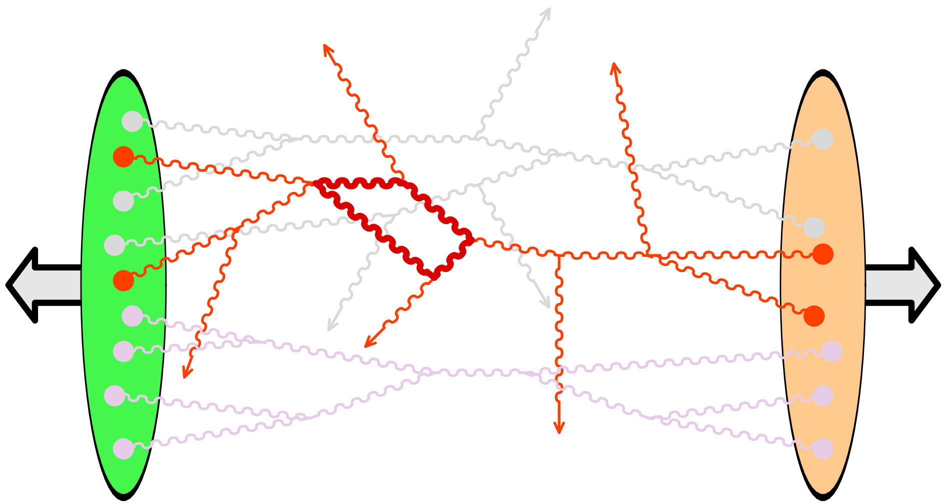
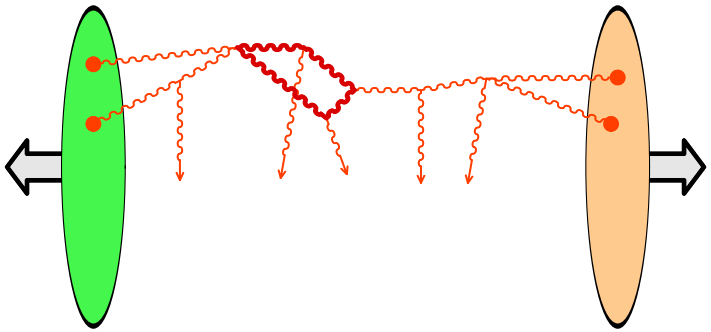


- In the saturated regime, the sources are of order $1/g$
- The order of each disconnected diagram is given by :

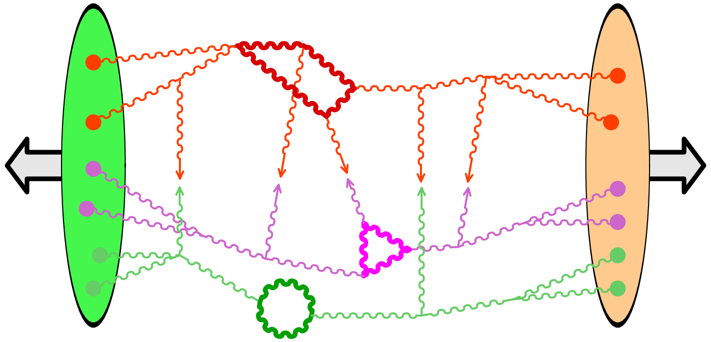
$$\frac{1}{g^2}\; g^{\#\text{ produced gluons}}\; g^{2(\#\text{ loops})}$$

- The total order of a graph is the product of the orders of its disconnected subdiagrams ▷ quite messy...

Bookkeeping

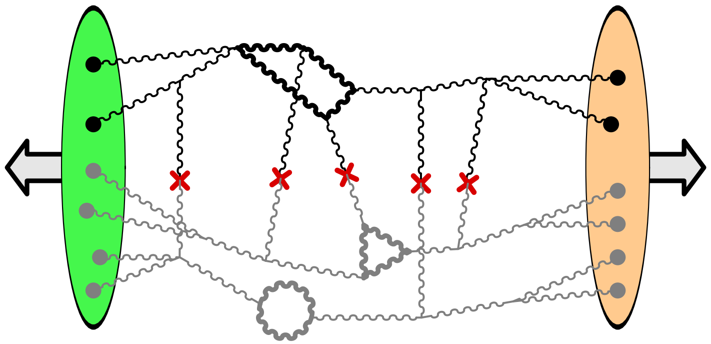


Bookkeeping



- Consider squared amplitudes (including interference terms) rather than the amplitudes themselves

Bookkeeping



- Consider squared amplitudes (including interference terms) rather than the amplitudes themselves
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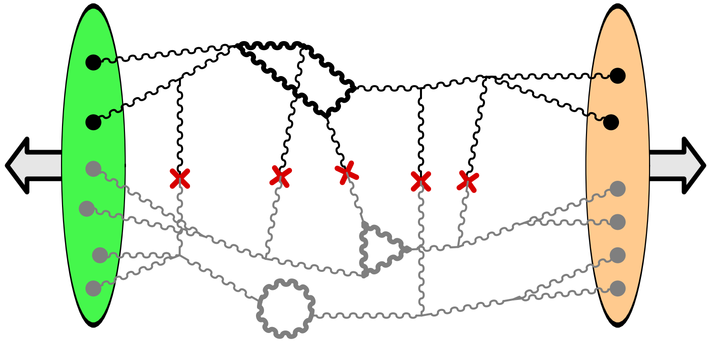


- Consider squared amplitudes (including interference terms) rather than the amplitudes themselves
- See them as cuts through vacuum diagrams
- Consider only the simply connected ones, thanks to :

$$\sum \left(\begin{matrix} \text{all the vacuum} \\ \text{diagrams} \end{matrix} \right) = \exp \left\{ \sum \left(\begin{matrix} \text{simply connected} \\ \text{vacuum diagrams} \end{matrix} \right) \right\}$$

- Simpler power counting for connected vacuum diagrams :

$$\frac{1}{g^2} \quad g^{2(\#\ \text{loops})}$$

Bookkeeping

- The probability of producing exactly n particles in the collision of the two hadrons is given by :

$$P_n = \frac{1}{n!}\int \frac{d^3\vec{\boldsymbol{p}}_1}{(2\pi)^3 2E_1}\cdots\frac{d^3\vec{\boldsymbol{p}}_n}{(2\pi)^3 2E_n}\left|\left\langle \vec{\boldsymbol{p}}_1\cdots\vec{\boldsymbol{p}}_{n\,\mathrm{out}}\middle|0_{\mathrm{in}}\right\rangle\right|^2$$

- The reduction formula can be written as :

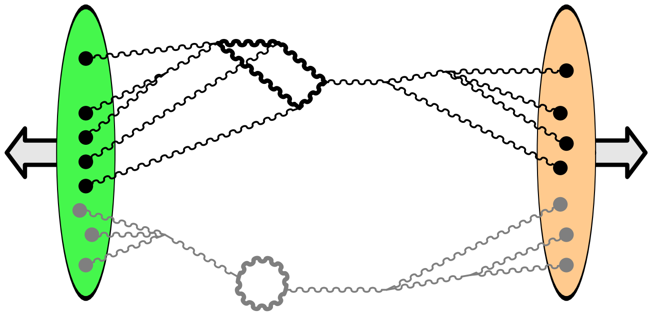
$$\left\langle \vec{\boldsymbol{p}}_1\cdots\vec{\boldsymbol{p}}_{n\,\mathrm{out}}\middle|0_{\mathrm{in}}\right\rangle = \frac{1}{Z^{n/2}}\int\left[\prod_{i=1}^{n} d^4x_i\; e^{ip_i\cdot x_i}\;(\Box_i + m^2)\frac{\delta}{i\delta j(x_i)}\right] e^{iV[j]}$$

and we have $$P_n = \frac{1}{n!}\,\mathcal{D}^n\; e^{iV[j_+]}\; e^{-iV^*[j_-]}\Big|_{j_+=j_-=j}$$

with
$$\begin{cases} \mathcal{D} \equiv \dfrac{1}{Z}\displaystyle\int_{x,y} G^0_{+-}(x,y)\,(\Box_x+m^2)(\Box_y+m^2)\,\dfrac{\delta}{\delta j_+(x)}\,\dfrac{\delta}{\delta j_-(y)} \\ G^0_{+-}(x,y) \equiv \displaystyle\int \dfrac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_p}\, e^{ip\cdot(x-y)} \end{cases}$$

Bookkeeping

- The operator $\mathcal{D}$ acts on a pair of vacuum diagrams by removing two sources and attaching a cut propagator instead :



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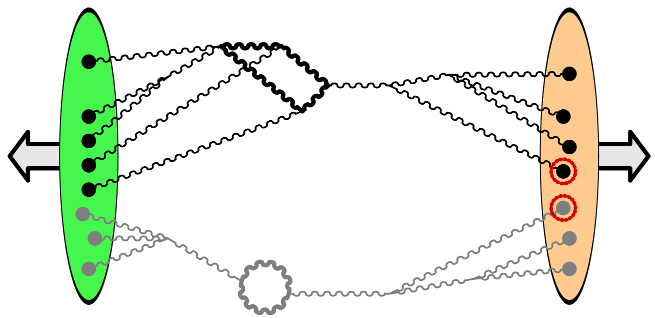

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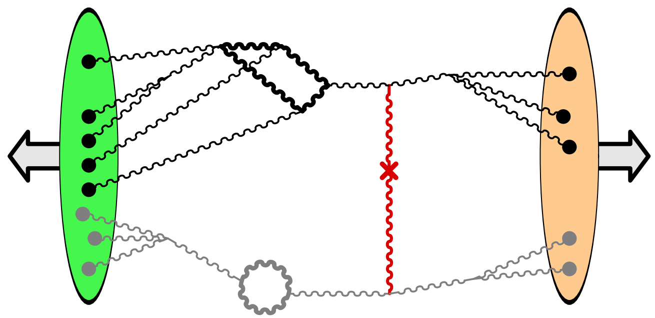

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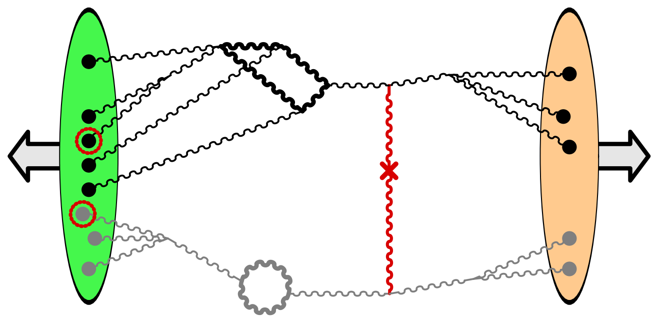

- $\mathcal{D}$ can also act directly on single diagram, if it is already cut

Bookkeeping

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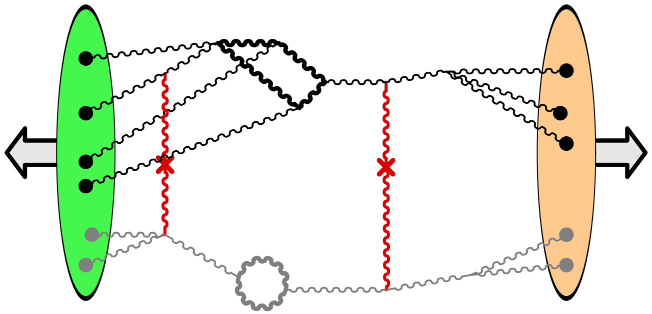

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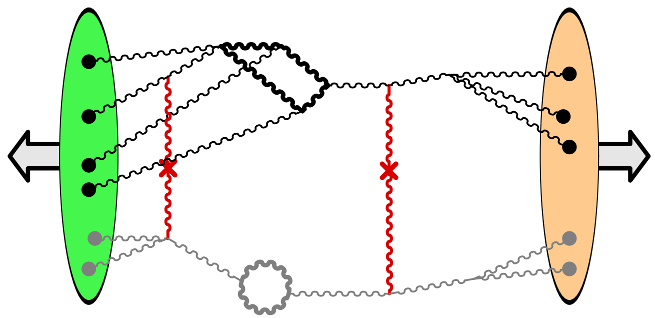


- $\mathcal{D}$ can also act directly on single diagram, if it is already cut
- The sum of all the cut vacuum diagrams, with sources j_+ on one side of the cut and j_- on the other side, can be written as :

$$\sum \left(\begin{matrix} \text{all the cut} \\ \text{vacuum diagrams} \end{matrix} \right) = e^{\mathcal{D}} \, e^{iV[j_+]} \, e^{-iV^*[j_-]}$$

▷ Note : if we set $j_+ = j_- = j$, then this is $\sum_n P_n = 1$

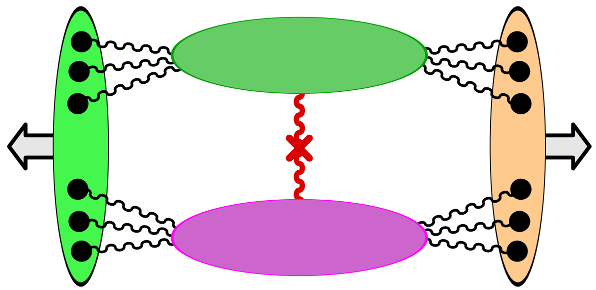
Inclusive gluon spectrum

First moment of the distribution

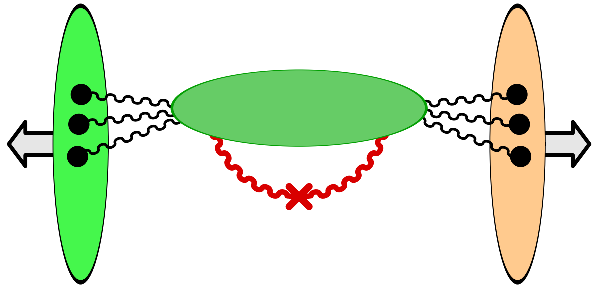
- It is easy to express the average multiplicity as :

$$\overline{N} = \sum_n n\, P_n = \mathcal{D}\left\{ e^{\mathcal{D}}\, e^{iV}\, e^{-iV^*} \right\}$$

- $\overline{N}$ is obtained by the action of $\mathcal{D}$ on the sum of all the cut vacuum diagrams. There are two kind of terms :
  - $\mathcal{D}$ picks two sources in two distinct connected cut diagrams



  - $\mathcal{D}$ picks two sources in the same connected cut diagram

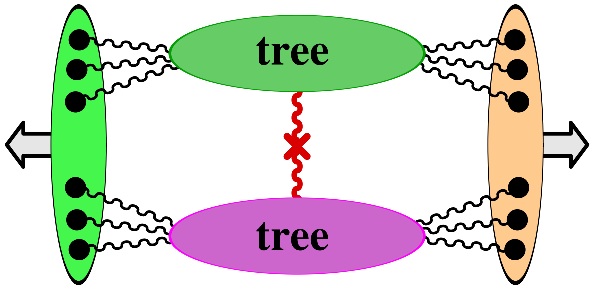


Gluon multiplicity at LO

- At LO, only tree diagrams contribute ▷ the second type of topologies can be neglected (it starts at 1-loop)

- In each blob, we must sum over all the tree diagrams, and over all the possible cuts :

$$\overline{N}_{LO} = \sum_{\text{trees}} \sum_{\text{cuts}}$$



- A major simplification comes from the following property :


$=$ **retarded** propagator

- The sum of all the tree diagrams constructed with retarded propagators is the retarded solution of Yang-Mills equations :

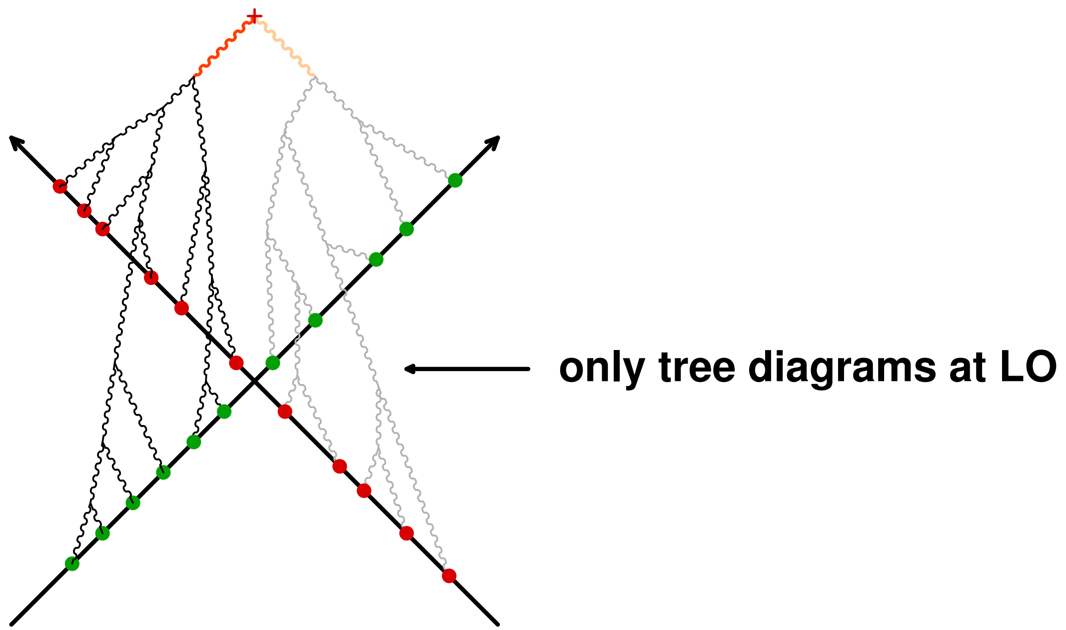
$$[D_\mu, F^{\mu\nu}] = J^\nu \qquad \text{with} \qquad A^\mu(x_0 = -\infty) = 0$$

Gluon multiplicity at LO

Krasnitz, Nara, Venugopalan (1999 – 2001), Lappi (2003)

$$\frac{d\overline{N}_{LO}}{dY\,d^2\vec{\boldsymbol{p}}_\perp} = \frac{1}{16\pi^3}\int_{x,y} e^{ip\cdot(x-y)}\,\square_x\square_y \sum_\lambda \epsilon^\mu_\lambda\epsilon^\nu_\lambda\;\mathcal{A}_\mu(x)\mathcal{A}_\nu(y)$$

- $\mathcal{A}^\mu(x) =$ retarded solution of Yang-Mills equations



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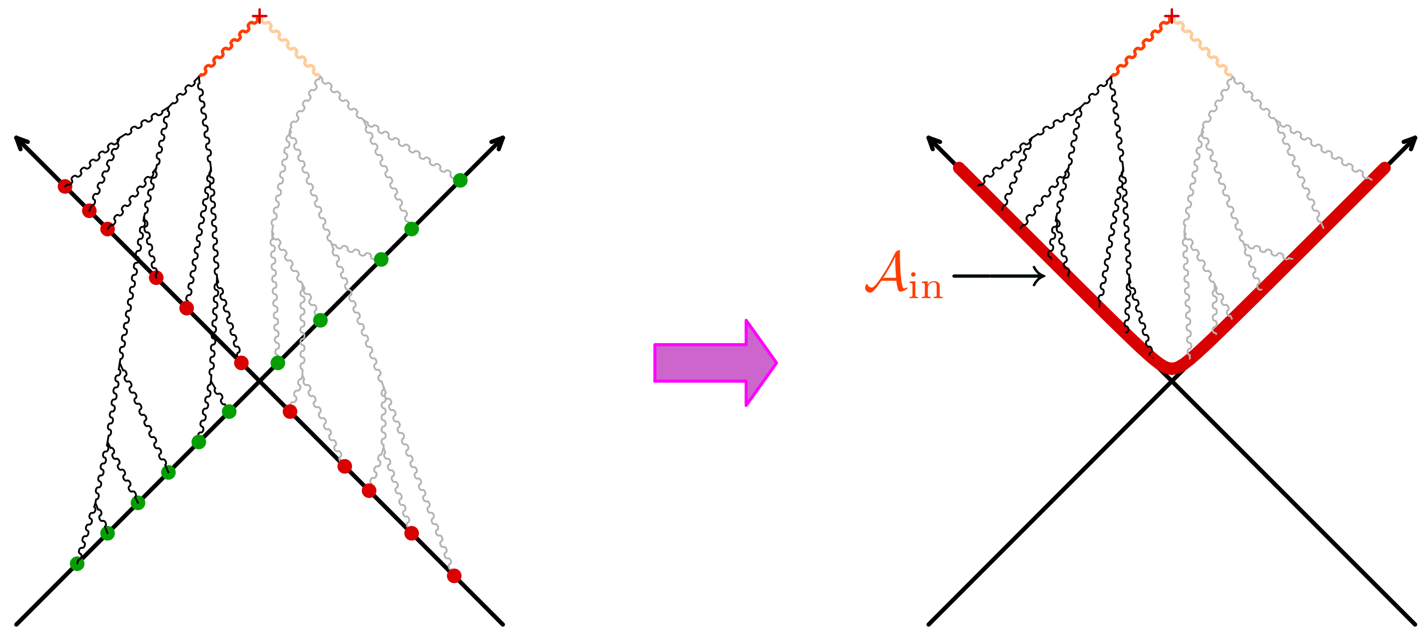
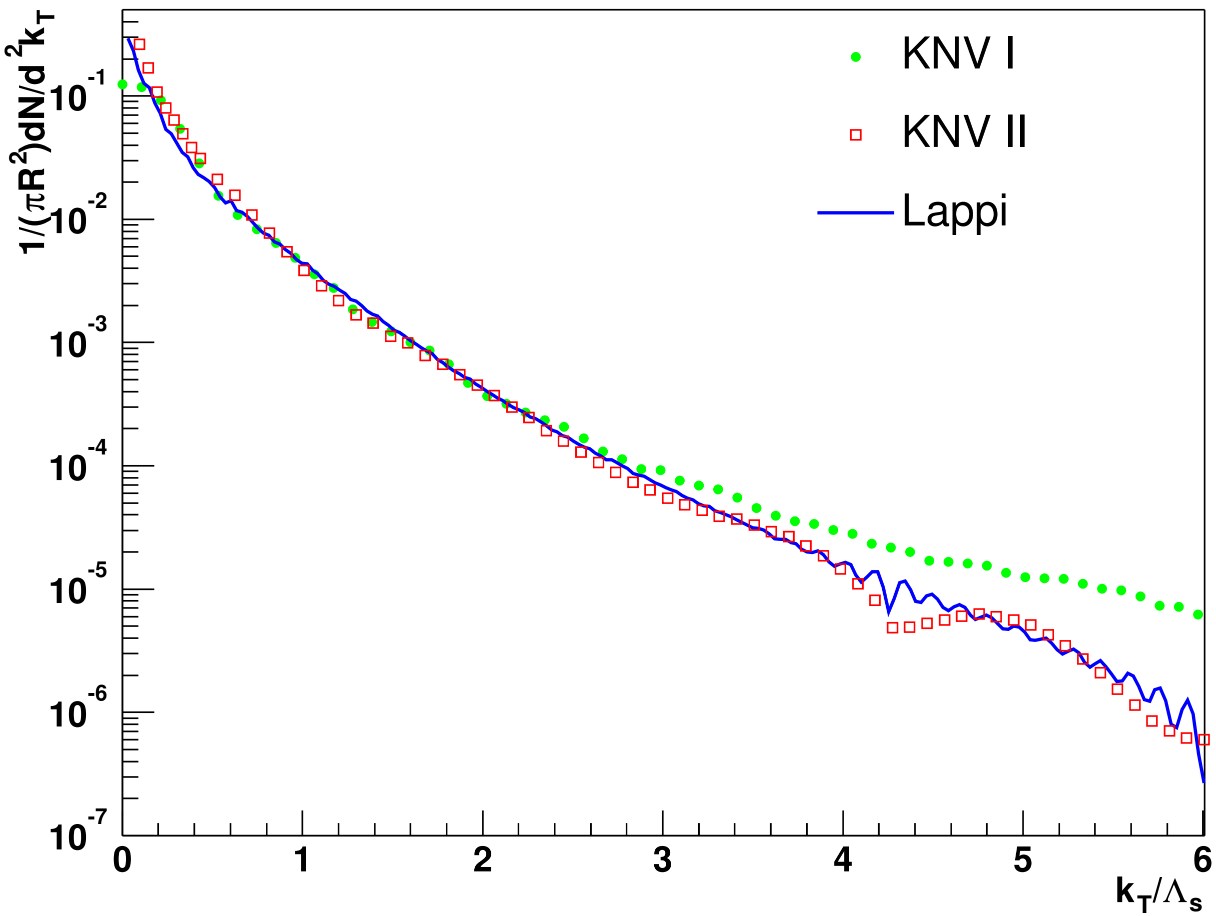
- $\mathcal{A}^\mu(x)\; = \;$ retarded solution of Yang-Mills equations
  - ▷ can be cast into an initial value problem on the light-cone

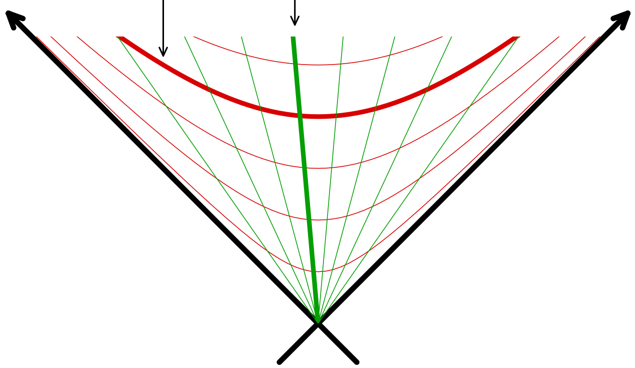


Gluon multiplicity at LO



- Lattice artefacts at large momentum
  (they do not affect much the overall number of gluons)
- Important softening at small $k_\perp$ compared to pQCD (saturation)

Initial conditions and boost invariance



- Gauge condition : $x^+ A^- + x^- A^+ = 0$

$$\Rightarrow \begin{cases} A^i(x) & = & \alpha^i(\tau,\eta,\vec{\boldsymbol{x}}_\perp) \\ A^\pm(x) & = & \pm\, x^\pm\, \beta(\tau,\eta,\vec{\boldsymbol{x}}_\perp) \end{cases}$$

- Initial values at $\tau = 0^+$: $\alpha^i(0^+,\eta,\vec{\boldsymbol{x}}_\perp)$ and $\beta(0^+,\eta,\vec{\boldsymbol{x}}_\perp)$ do not depend on the rapidity η

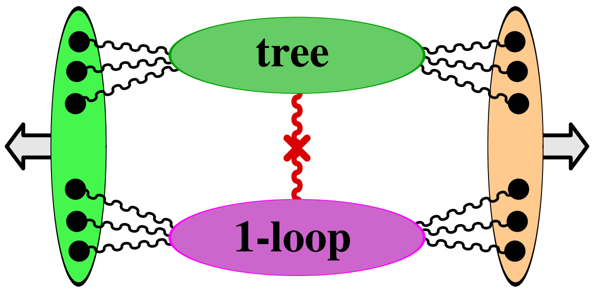
  ▷ α^i and β remain independent of η at all times (invariance under boosts in the z direction)

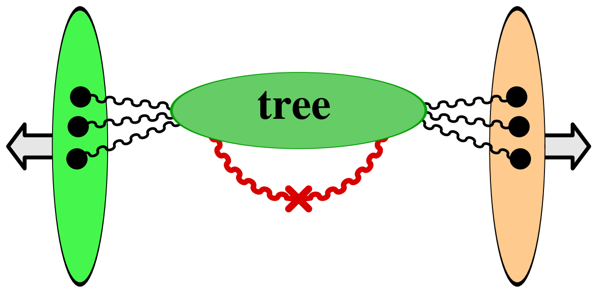
  ▷ numerical resolution performed in $1+2$ dimensions

Loop corrections

1-loop corrections to N

- 1-loop diagrams for $\overline{N}$





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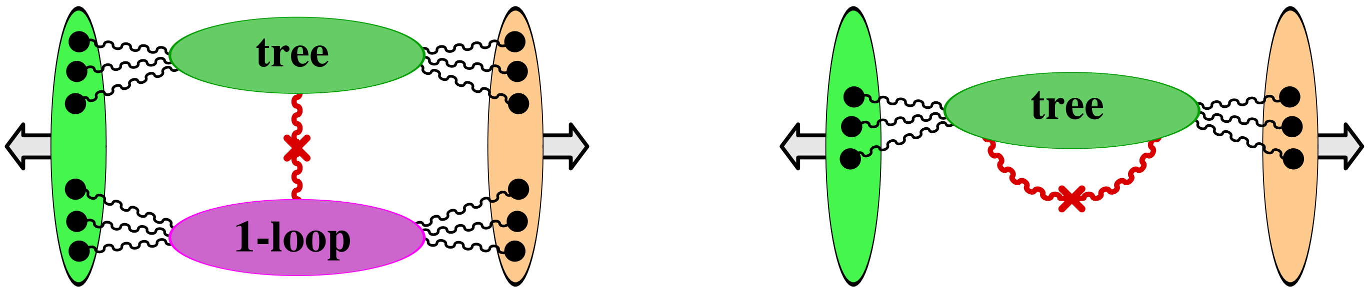
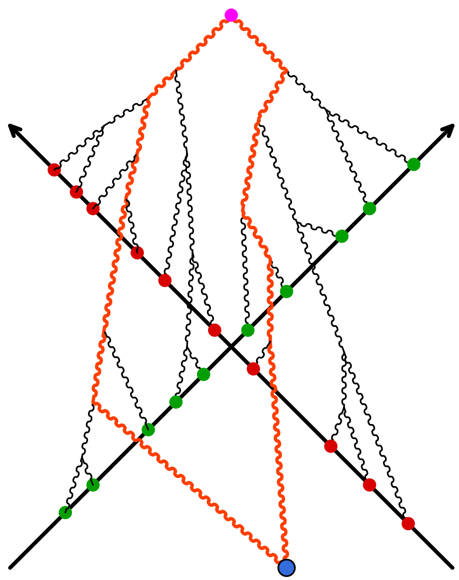
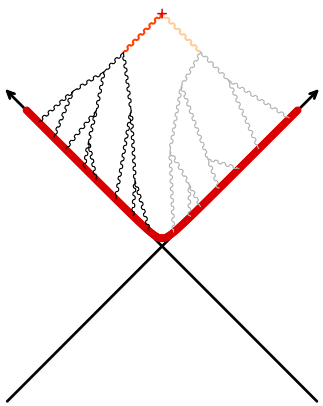


- This involves diagrams such as :



1-loop corrections to N

- The 1-loop correction to $\overline{N}$ can be written as a perturbation of the initial value problem encountered at LO :



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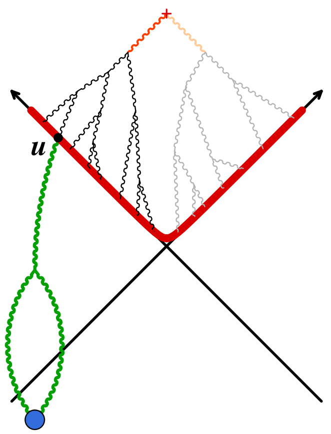

$$\delta\overline{N} = \left[\int\limits_{\vec{u} \in \text{light cone}} \delta\mathcal{A}_{\rm in}(\vec{u}) \; T_{\vec{u}} \right] \overline{N}_{LO}$$

- $\overline{N}_{LO}$ is a functional of the initial fields $\mathcal{A}_{\rm in}(\vec{u})$ on the light-cone
- $T_{\vec{u}}$ is the generator of shifts of the initial condition at the point $\vec{u}$ on the light-cone, i.e. : $T_{\vec{u}} \sim \delta/\delta\mathcal{A}_{\rm in}(\vec{u})$

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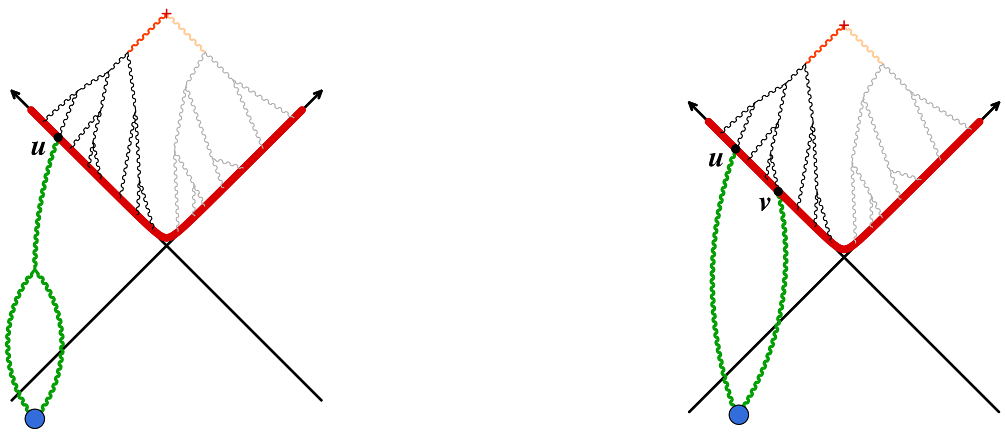

$$\delta\overline{N} = \left[\int_{\vec{u}\,\in\,\text{light cone}} \delta\mathcal{A}_{\rm in}(\vec{u})\ \boldsymbol{T}_{\vec{u}} + \int_{\vec{u},\vec{v}\,\in\,\text{light cone}} \frac{1}{2}\ \boldsymbol{\Sigma}(\vec{u},\vec{v})\ \boldsymbol{T}_{\vec{u}}\,\boldsymbol{T}_{\vec{v}} \right] \overline{N}_{LO}$$

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- $\delta\mathcal{A}_{\rm in}(\vec{u})$ and $\boldsymbol{\Sigma}(\vec{u},\vec{v})$ are in principle calculable analytically

Generalization

- Conjecture : this result can be generalized to any observable that can be written in terms of the gauge field with retarded boundary conditions, $\mathcal{O} \equiv \mathcal{O}[\mathcal{A}]$:

$$\delta\mathcal{O} = \left[\int_{\vec{u}\in\text{light cone}} \delta\mathcal{A}_{\text{in}}(\vec{u})\; \boldsymbol{T}_{\vec{u}} + \int_{\vec{u},\vec{v}\in\text{light cone}} \frac{1}{2}\,\boldsymbol{\Sigma}(\vec{u},\vec{v})\; \boldsymbol{T}_{\vec{u}}\,\boldsymbol{T}_{\vec{v}} \right] \mathcal{O}_{LO}$$

▷ whatever we conclude for the multiplicity from this formula holds true for any such observable

Divergences

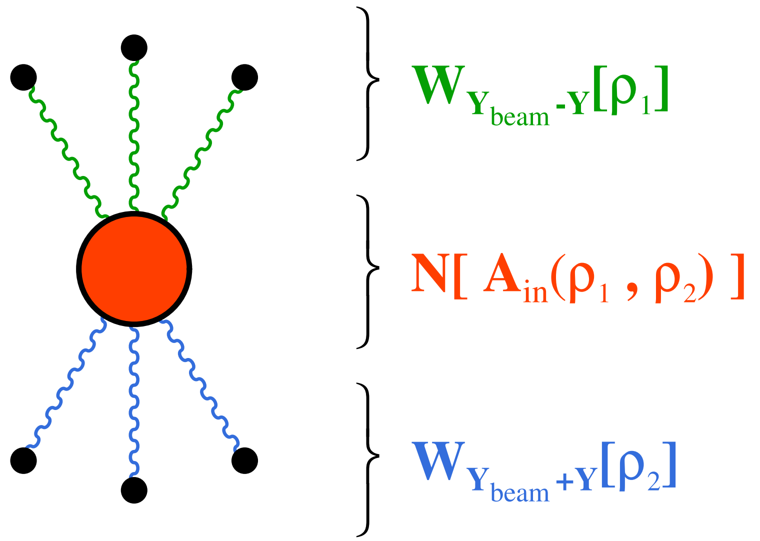
- If taken at face value, this 1-loop correction is plagued by several divergences :
  - The two coefficients $\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{x}})$ and $\boldsymbol{\Sigma}(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})$ are infinite, because of an unbounded integration over a rapidity variable
  - At late times, $\boldsymbol{T}_{\vec{\boldsymbol{x}}}\mathcal{A}(\tau,\vec{\boldsymbol{y}})$ diverges exponentially,

$$\boldsymbol{T}_{\vec{\boldsymbol{x}}}\mathcal{A}(\tau,\vec{\boldsymbol{y}}) \underset{\tau\to+\infty}{\sim} e^{\sqrt{\mu\tau}}$$

because of an instability of the classical solution of Yang-Mills equations under rapidity dependent perturbations (Romatschke, Venugopalan (2005))

Initial state factorization

- Anatomy of the full calculation :



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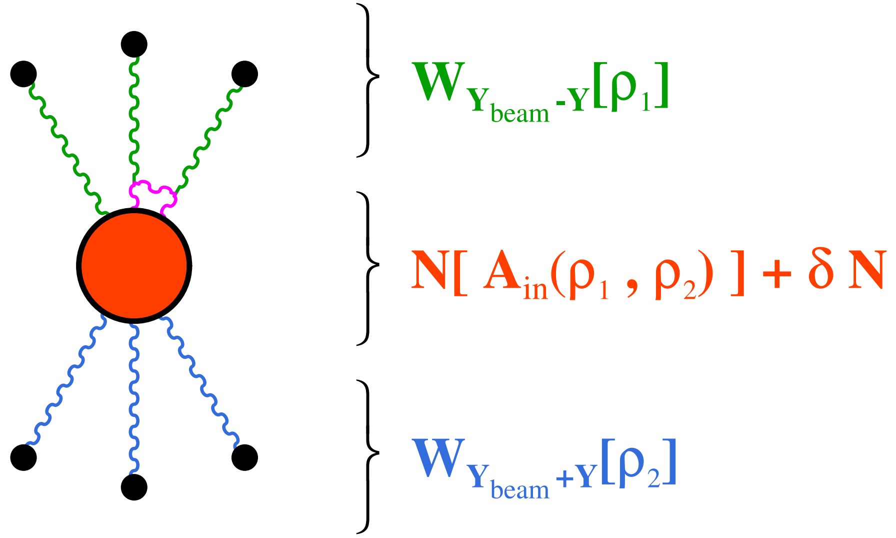

- When the observable $\overline{N}[\mathcal{A}_{\text{in}}(\rho_1, \rho_2)]$ is corrected by an extra gluon, one gets divergences of the form $\alpha_s \int dY$ in $\delta\overline{N}$
  ▷ one would like to be able to absorb these divergences into the Y dependence of the source densities $W_Y[\rho_{1,2}]$

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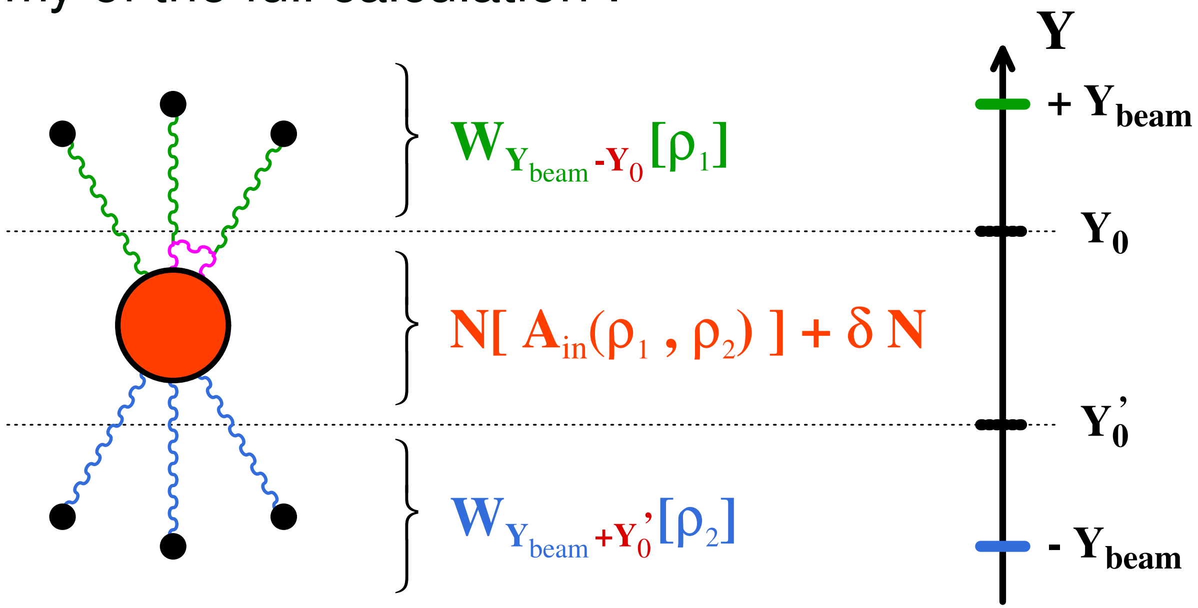


- When the observable $\overline{N}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)]$ is corrected by an extra gluon, one gets divergences of the form $\alpha_s\int dY$ in $\delta\overline{N}$
  ▷ one would like to be able to absorb these divergences into the Y dependence of the source densities $W_{Y}[\rho_{1,2}]$
- Equivalently, if one puts some arbitrary frontier Y_0 between the “observable” and the “source distributions”, the dependence on Y_0 should cancel between the various factors

Initial state factorization

- The two kind of divergences don't mix, because the divergent part of the coefficients is boost invariant.

  Given their structure, the divergent coefficients seem related to the evolution of the sources in the initial state

- In order to prove the factorization of these divergences in the initial state distributions of sources, one needs to establish :

$$\left[\delta\overline{N}\right]_{\substack{\text{divergent}\\\text{coefficients}}} = \Big[(Y_0 - Y)\,\mathcal{H}^\dagger[\rho_1] + (Y - Y_0')\,\mathcal{H}^\dagger[\rho_2]\Big]\;\overline{N}_{LO}$$

where $\mathcal{H}[\rho]$ is the Hamiltonian that governs the rapidity dependence of the source distribution $W_Y[\rho]$:

$$\frac{\partial W_Y[\rho]}{\partial Y} = \mathcal{H}[\rho]\;W_Y[\rho]$$

FG, Lappi, Venugopalan (work in progress)

Initial state factorization

- Why is it plausible ?
  - Reminder :

$$\left[\delta\overline{N}\right]_{\substack{\text{divergent}\\\text{coefficients}}} = \left\{ \int_{\vec{x}} \left[\delta\mathcal{A}_{\text{in}}(\vec{x})\right]_{\text{div}} T_{\vec{x}} + \frac{1}{2}\int_{\vec{x},\vec{y}} \left[\boldsymbol{\Sigma}(\vec{x},\vec{y})\right]_{\text{div}} T_{\vec{x}} T_{\vec{y}} \right\} \overline{N}_{LO}$$

  - Compare with the evolution Hamiltonian :

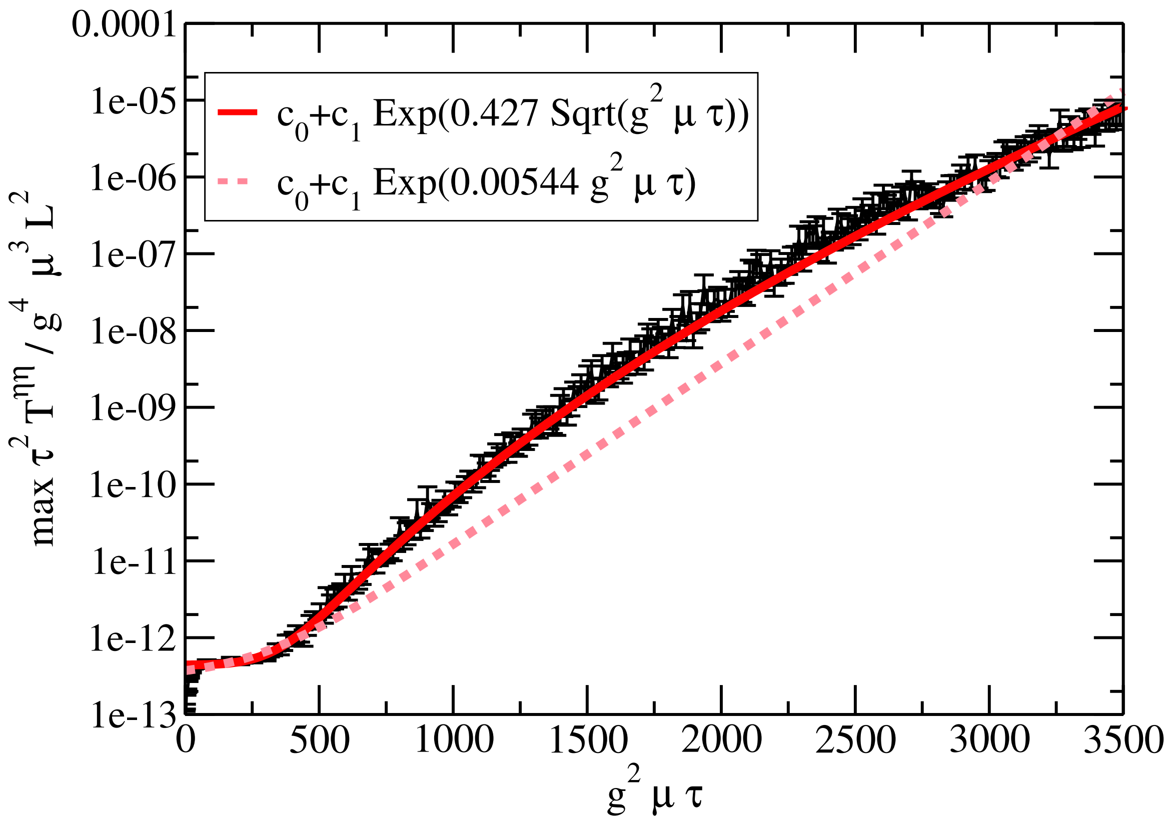
$$\mathcal{H}[\rho] = \int_{\vec{x}_\perp} \sigma(\vec{x}_\perp)\frac{\delta}{\delta\rho(\vec{x}_\perp)} + \frac{1}{2}\int_{\vec{x}_\perp,\vec{y}_\perp} \chi(\vec{x}_\perp,\vec{y}_\perp)\frac{\delta^2}{\delta\rho(\vec{x}_\perp)\delta\rho(\vec{y}_\perp)}$$

- The coefficients σ and χ in the Hamiltonian are well known. There is a well defined calculation that will tell us if it works...

Unstable modes

Romatschke, Venugopalan (2005)

- Rapidity dependent perturbations to the classical fields grow like $\exp(\#\sqrt{\tau})$ until the non-linearities become important :



Unstable modes

- The coefficient $\delta\mathcal{A}_{\rm in}(\vec{x})$ is boost invariant.
  Hence, the divergences due to the unstable modes all come from the quadratic term in $\delta\overline{N}$:

$$\left[\delta\overline{N}\right]_{\substack{\rm unstable\\ \rm modes}} = \left\{\frac{1}{2}\int_{\vec{x},\vec{y}} \boldsymbol{\Sigma}(\vec{x},\vec{y})\, \boldsymbol{T}_{\vec{x}}\boldsymbol{T}_{\vec{y}}\right\} \overline{N}_{LO}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)]$$

- When summed to all orders, this becomes a certain functional $Z[\boldsymbol{T}_{\vec{x}}]$:

$$\left[\delta\overline{N}\right]_{\substack{\rm unstable\\ \rm modes}} = Z[\boldsymbol{T}_{\vec{x}}]\;\overline{N}_{LO}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)]$$

Unstable modes

- This can be arranged in a more intuitive way :

$$\left[\delta\overline{N}\right]_{\substack{\text{unstable}\\\text{modes}}} = \int [Da]\;\widetilde{Z}[a(\vec{\boldsymbol{x}})]\;e^{i\int_{\vec{\boldsymbol{x}}} a(\vec{\boldsymbol{x}})\,\boldsymbol{T}_{\vec{\boldsymbol{x}}}}\;\overline{N}_{LO}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)]$$

$$= \int [Da]\;\widetilde{Z}[a(\vec{\boldsymbol{x}})]\;\overline{N}_{LO}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)+a]$$

▷ summing these divergences simply requires to add fluctuations to the initial condition for the classical problem
▷ the fact that $\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{x}})$ does not contribute implies that the distribution of fluctuations is real

- Interpretation :

Despite the fact that the fields are coupled to strong sources, the classical approximation alone is not good enough, because the classical solution has unstable modes that can be triggered by the quantum fluctuations

Unstable modes

Fukushima, FG, McLerran (2006)

- By a different method, one obtains Gaussian fluctuations characterized by :

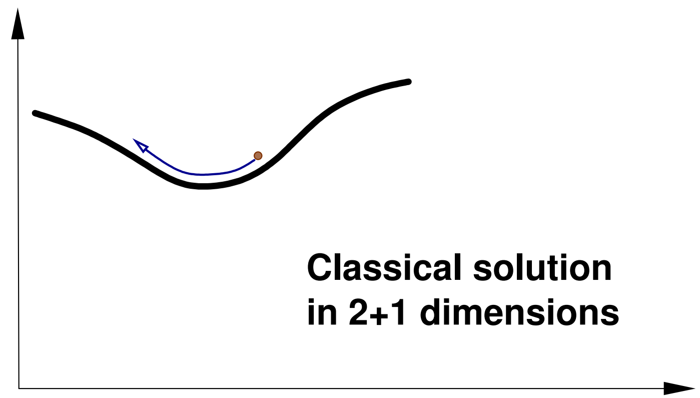
$$\big\langle a_i(\eta,\vec{\boldsymbol{x}}_\perp)\, a_j(\eta',\vec{\boldsymbol{x}}'_\perp)\big\rangle = $$

$$= \frac{1}{\tau\sqrt{-(\partial_\eta/\tau)^2-\boldsymbol{\partial}_\perp^2}}\left[\delta_{ij}+\frac{\partial_i\partial_j}{(\partial_\eta/\tau)^2}\right]\delta(\eta-\eta')\,\delta(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{x}}'_\perp)$$

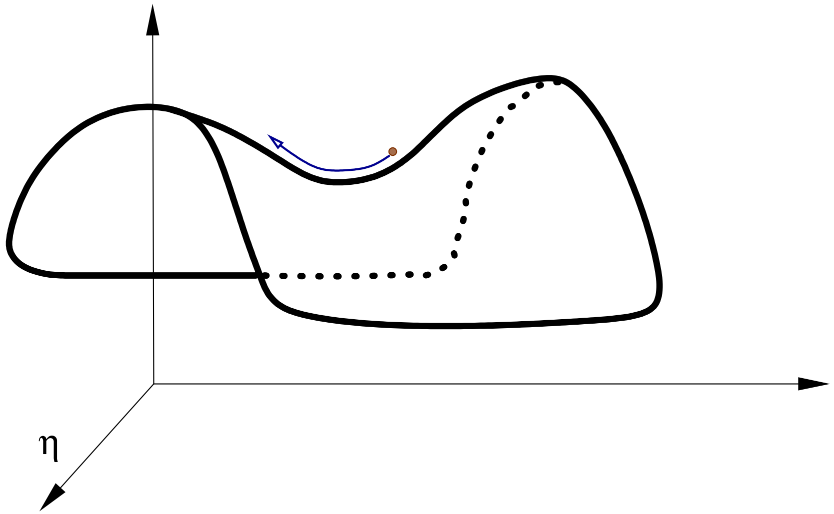
$$\big\langle e^i(\eta,\vec{\boldsymbol{x}}_\perp)\, e^j(\eta',\vec{\boldsymbol{x}}'_\perp)\big\rangle = $$

$$= \tau\sqrt{-(\partial_\eta/\tau)^2-\boldsymbol{\partial}_\perp^2}\left[\delta_{ij}-\frac{\partial_i\partial_j}{(\partial_\eta/\tau)^2+\boldsymbol{\partial}_\perp^2}\right]\delta(\eta-\eta')\,\delta(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{x}}'_\perp)$$

Unstable modes



Unstable modes



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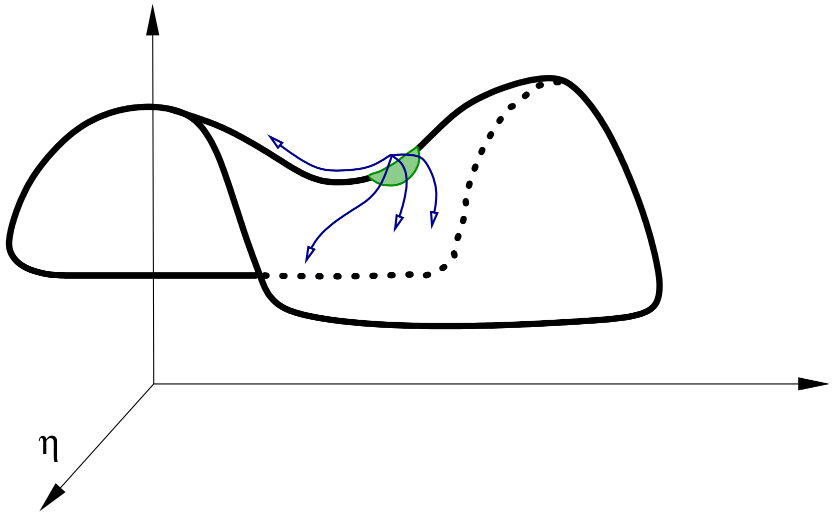
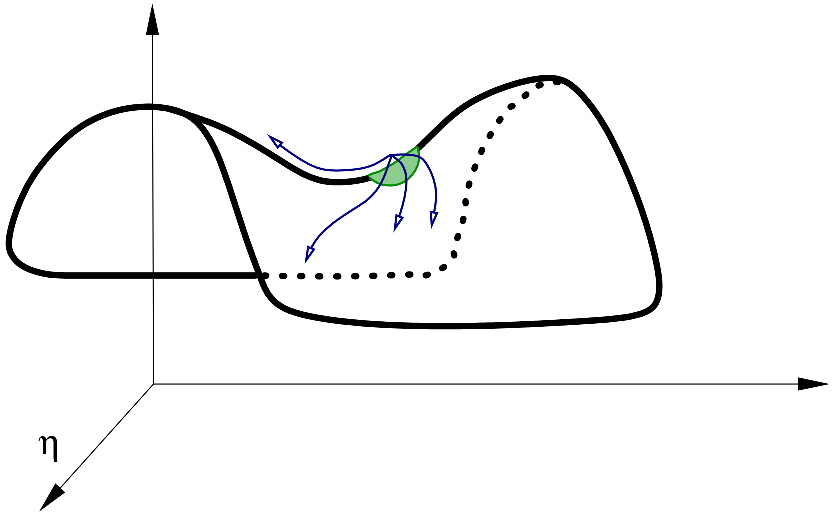


Unstable modes



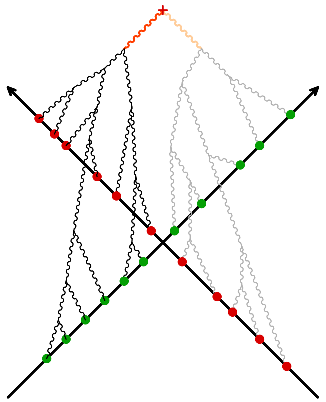
- Combining everything, one should write

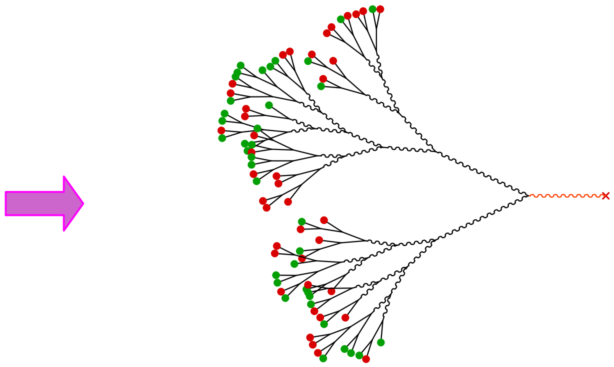
$$\frac{d\overline{N}}{dY\,d^2\vec{\boldsymbol{p}}_\perp} = \int [D\rho_1]\,[D\rho_2]\;\; W_{Y_{\rm beam}-Y}[\rho_1]\; W_{Y_{\rm beam}+Y}[\rho_2]$$
$$\times \int [Da]\;\; \widetilde{Z}[a]\;\; \frac{d\overline{N}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)+a]}{dY\,d^2\vec{\boldsymbol{p}}_\perp}$$

▷ This formula resums (all?) the divergences that occur at one loop

Unstable modes – Interpretation

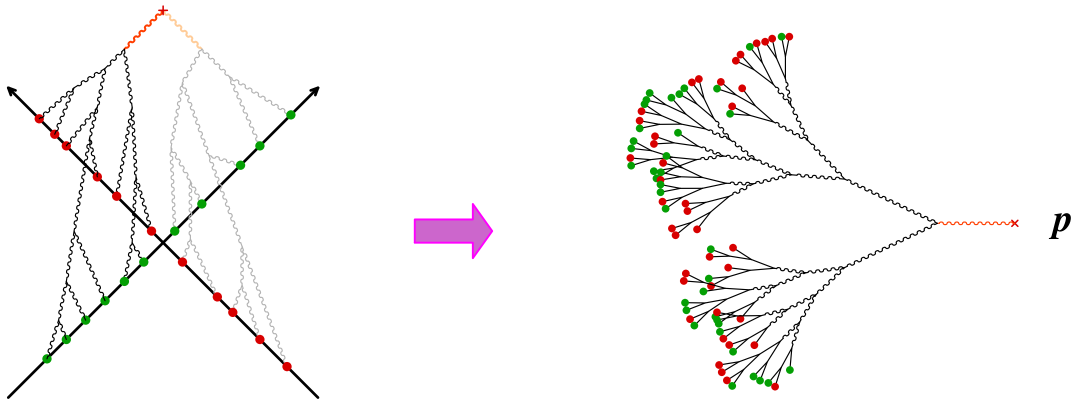
- Tree level :



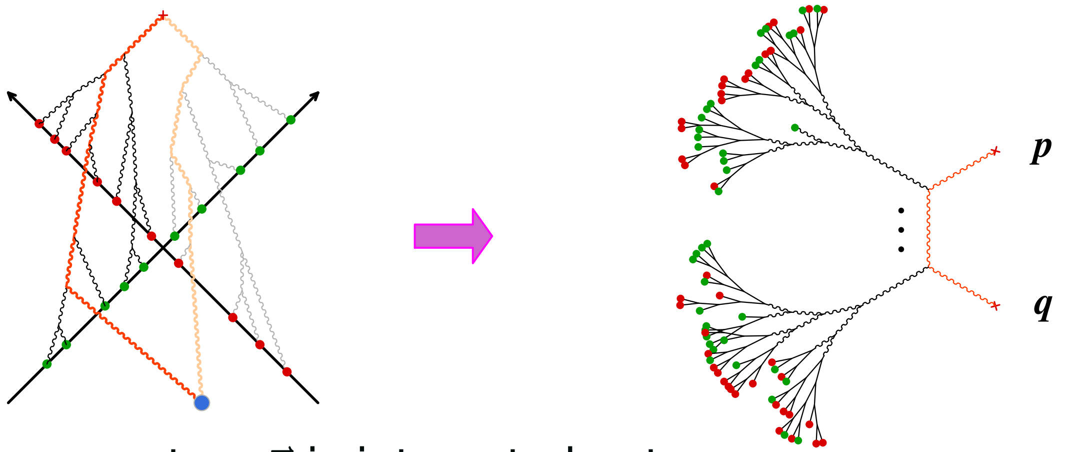


Unstable modes – Interpretation

- Tree level :



- One loop ▷ gluon pairs (includes Schwinger pairs):



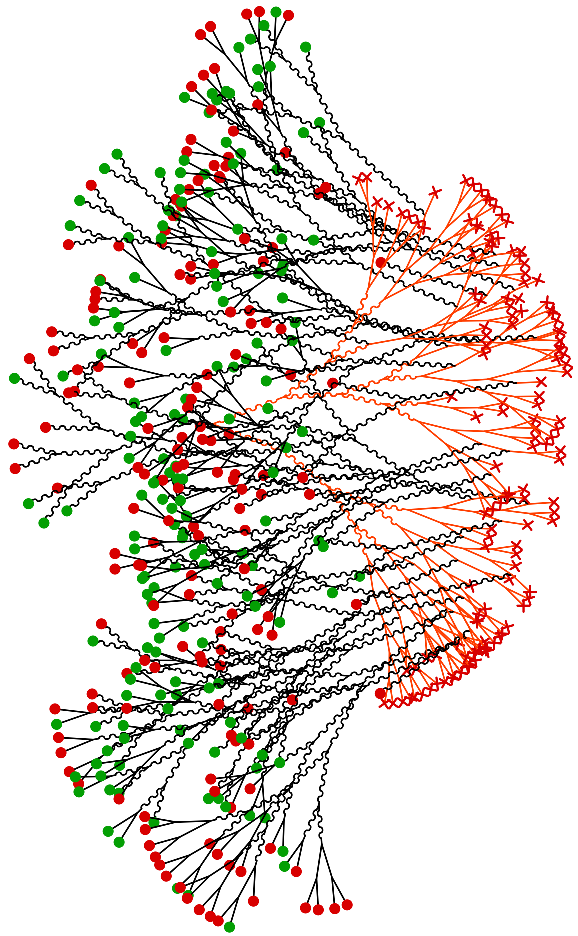
▷ The momentum $\vec{q}$ is integrated out

▷ If $\alpha_s^{-1} \lesssim |y_{\boldsymbol{p}} - y_{\boldsymbol{q}}|$, the correction is absorbed in $W[\rho_{1,2}]$

▷ If $|y_{\boldsymbol{p}} - y_{\boldsymbol{q}}| \lesssim \alpha_s^{-1}$: gluon splitting in the final state

Unstable modes – Interpretation

- After summing the fluctuations, things may look like this :



▷ these splittings may help to fight against the expansion ?

Note : the timescale for this process is $\tau \sim Q_s^{-1} \ln^2(1/\alpha_s)$

Summary

Summary

- When the parton densities in the projectiles are large, the study of particle production becomes rather involved

  ▷ non-perturbative techniques that resum all-twist contributions are needed

- At Leading Order, the inclusive gluon spectrum can be calculated from the classical solution with retarded boundary conditions on the light-cone

- At Next-to-Leading Order, the gluonic corrections can be seen as a perturbation of the initial value problem encountered at LO

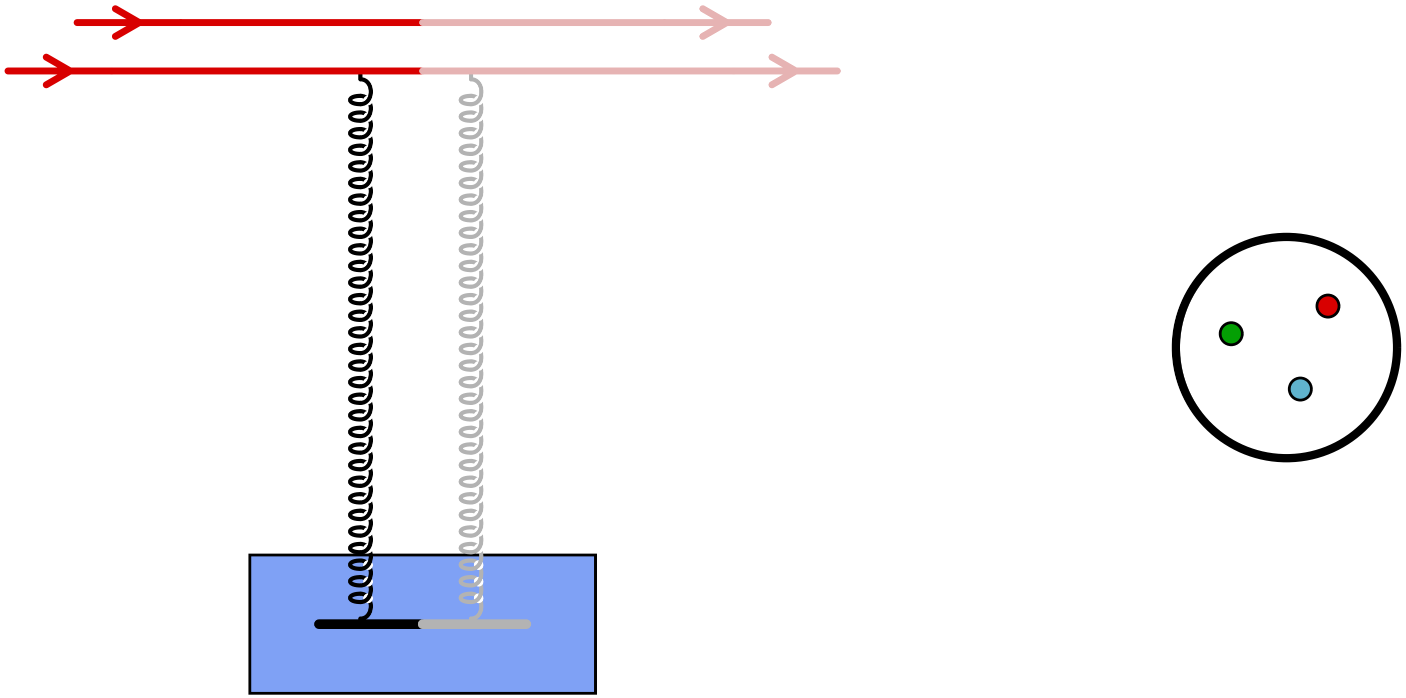
- Resummation of the leading divergences to all orders :

  ▷ Evolution with Y of the distribution of sources

  ▷ Quantum fluctuations on top of initial condition for the classical solution in the forward light-cone

Extra bits

Parton evolution

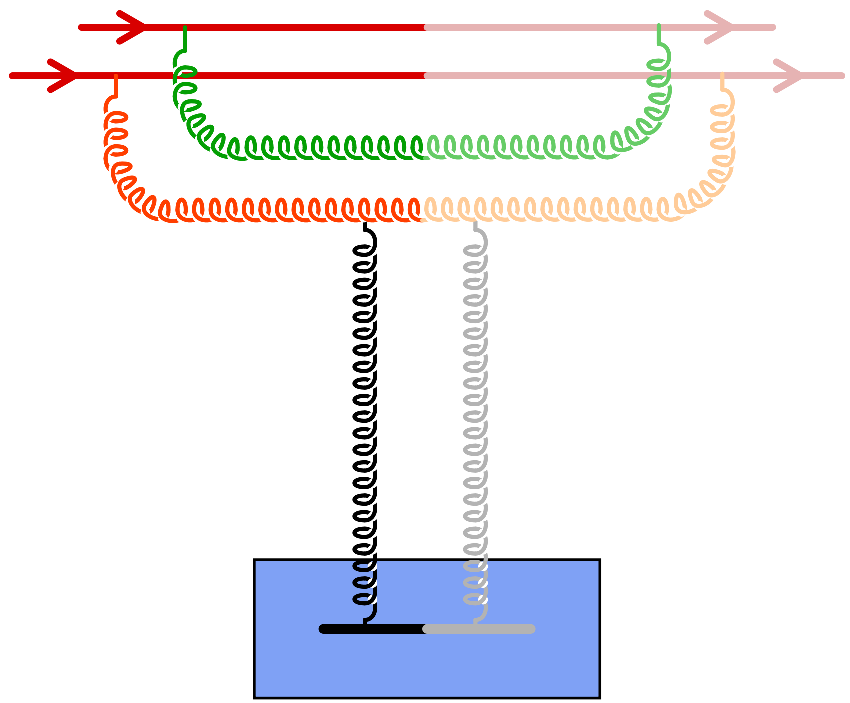


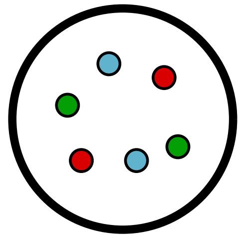
▷ assume that the projectile is big, e.g. a nucleus, and has many valence quarks (only two are represented)

▷ on the contrary, consider a small probe, with few partons

▷ at low energy, only valence quarks are present in the hadron wave function

Parton evolution



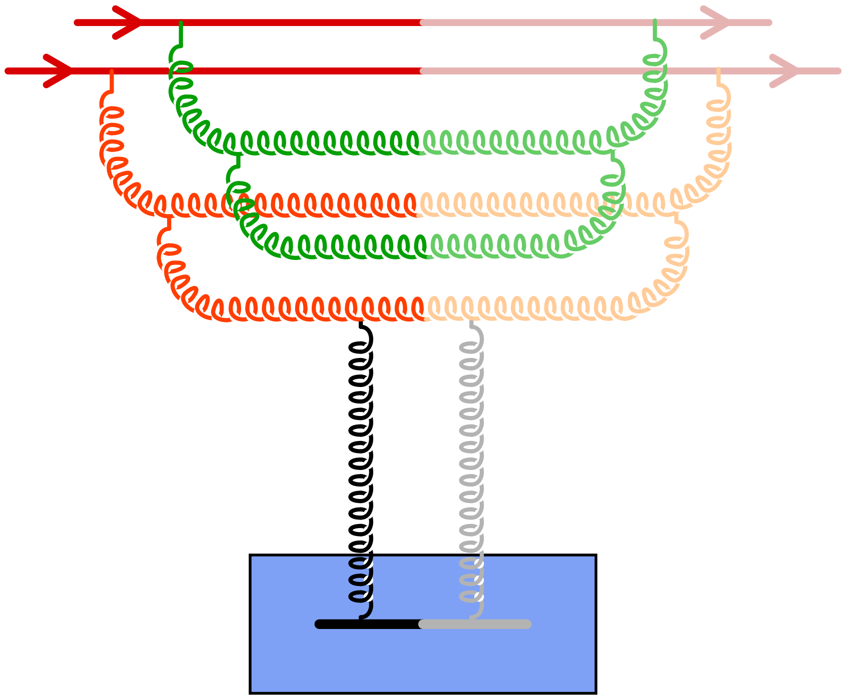


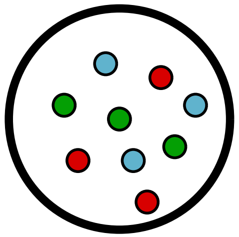
▷ when energy increases, new partons are emitted

▷ the emission probability is $\alpha_s \int \frac{dx}{x} \sim \alpha_s \ln(\frac{1}{x})$, with x the longitudinal momentum fraction of the gluon

▷ at small-x (i.e. high energy), these logs need to be resummed

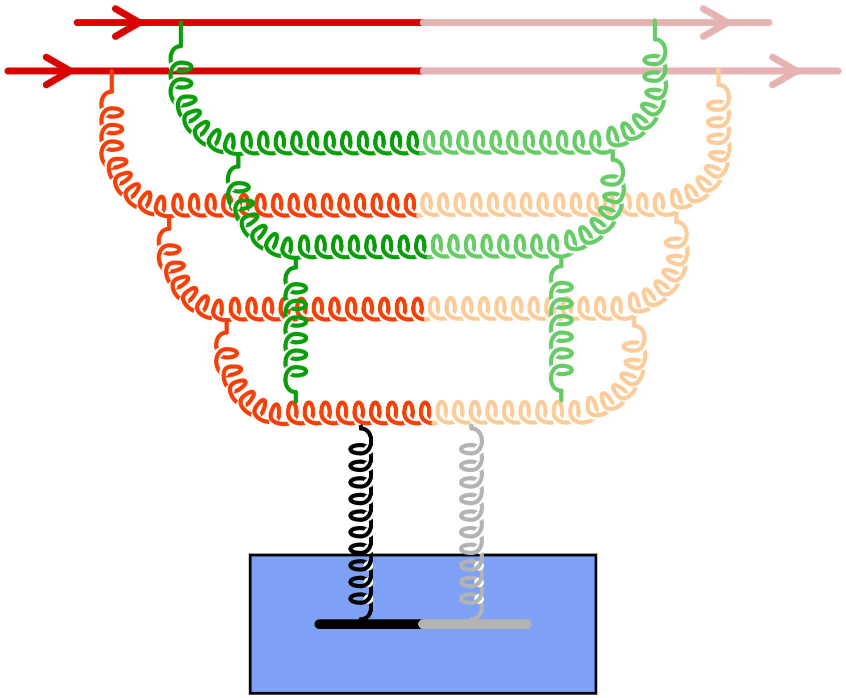
Parton evolution

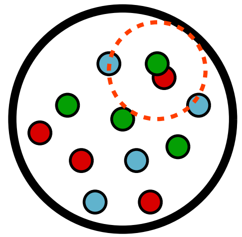




▷ as long as the density of constituents remains small, the evolution is linear: the number of partons produced at a given step is proportional to the number of partons at the previous step (BFKL)

Parton evolution





▷ eventually, the partons start overlapping in phase-space

▷ parton recombination becomes favorable

▷ after this point, the evolution is non-linear:
the number of partons created at a given step depends non-linearly on the number of partons present previously

Saturation criterion

Gribov, Levin, Ryskin (1983)

- Number of gluons per unit area:

$$\rho \sim \frac{xG_A(x,Q^2)}{\pi R_A^2}$$

- Recombination cross-section:

$$\sigma_{gg\to g} \sim \frac{\alpha_s}{Q^2}$$

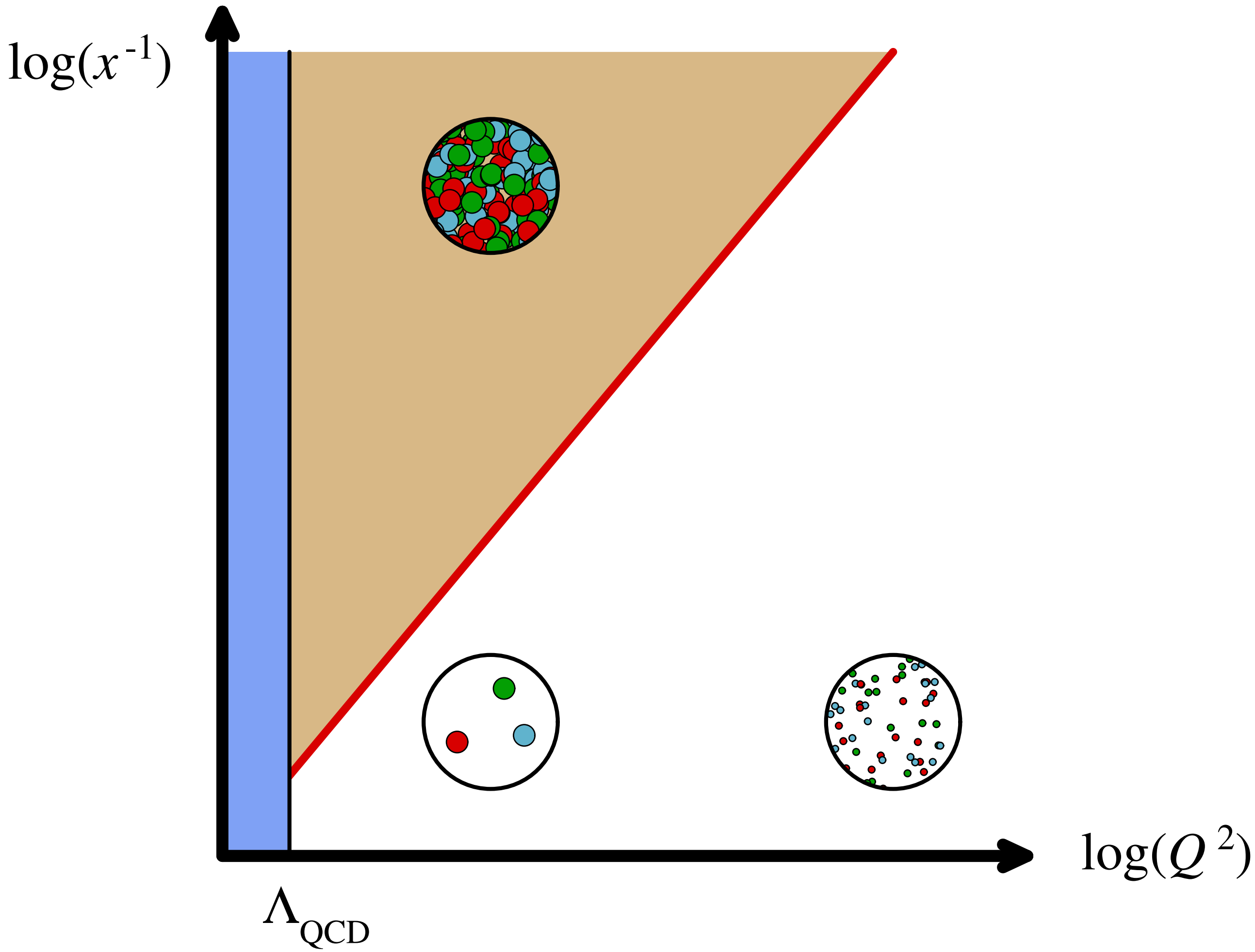
- Recombination happens if $\rho\sigma_{gg\to g} \gtrsim 1$, i.e. $Q^2 \lesssim Q_s^2$, with:

$$Q_s^2 \quad \sim \quad \frac{\alpha_s xG_A(x,Q_s^2)}{\pi R_A^2} \quad \sim \quad A^{1/3}\frac{1}{x^{0.3}}$$

- At saturation, the phase-space density is:

$$\frac{dN_g}{d^2\vec{\boldsymbol{x}}_\perp d^2\vec{\boldsymbol{p}}_\perp} \sim \frac{\rho}{Q^2} \sim \frac{1}{\alpha_s}$$

Saturation domain

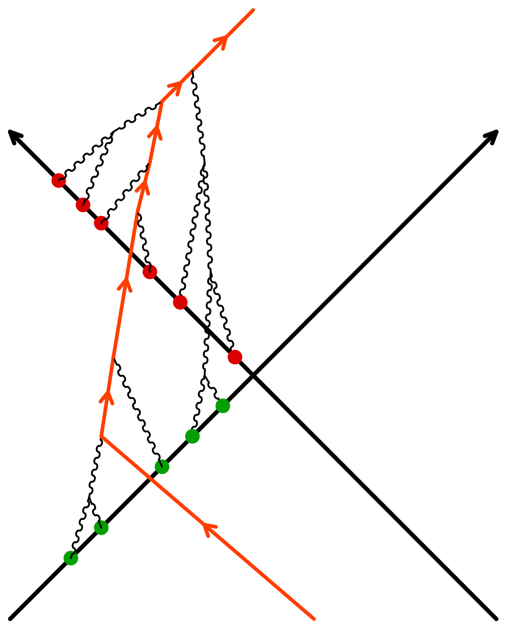


Quark production

FG, Kajantie, Lappi (2004, 2005)

$$E_{\boldsymbol{p}} \frac{d\langle n_{\rm quarks} \rangle}{d^3\vec{\boldsymbol{p}}} = \frac{1}{16\pi^3} \int_{x,y} e^{ip\cdot(x-y)} \,\partial\!\!\!/_x \partial\!\!\!/_y \,\langle \overline{\psi}(x)\psi(y)\rangle$$

- Dirac equation in the classical color field :



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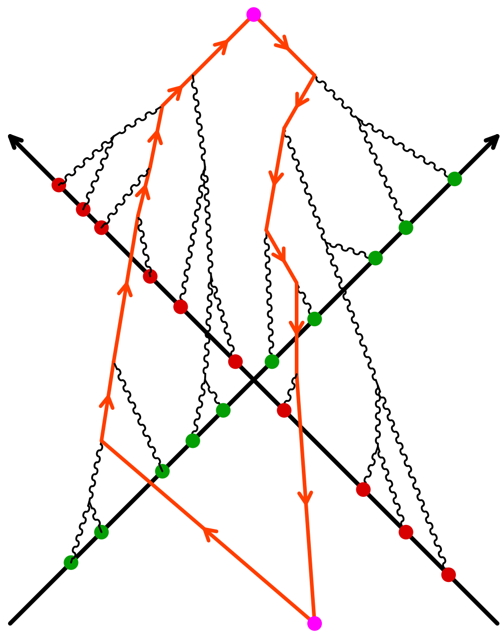
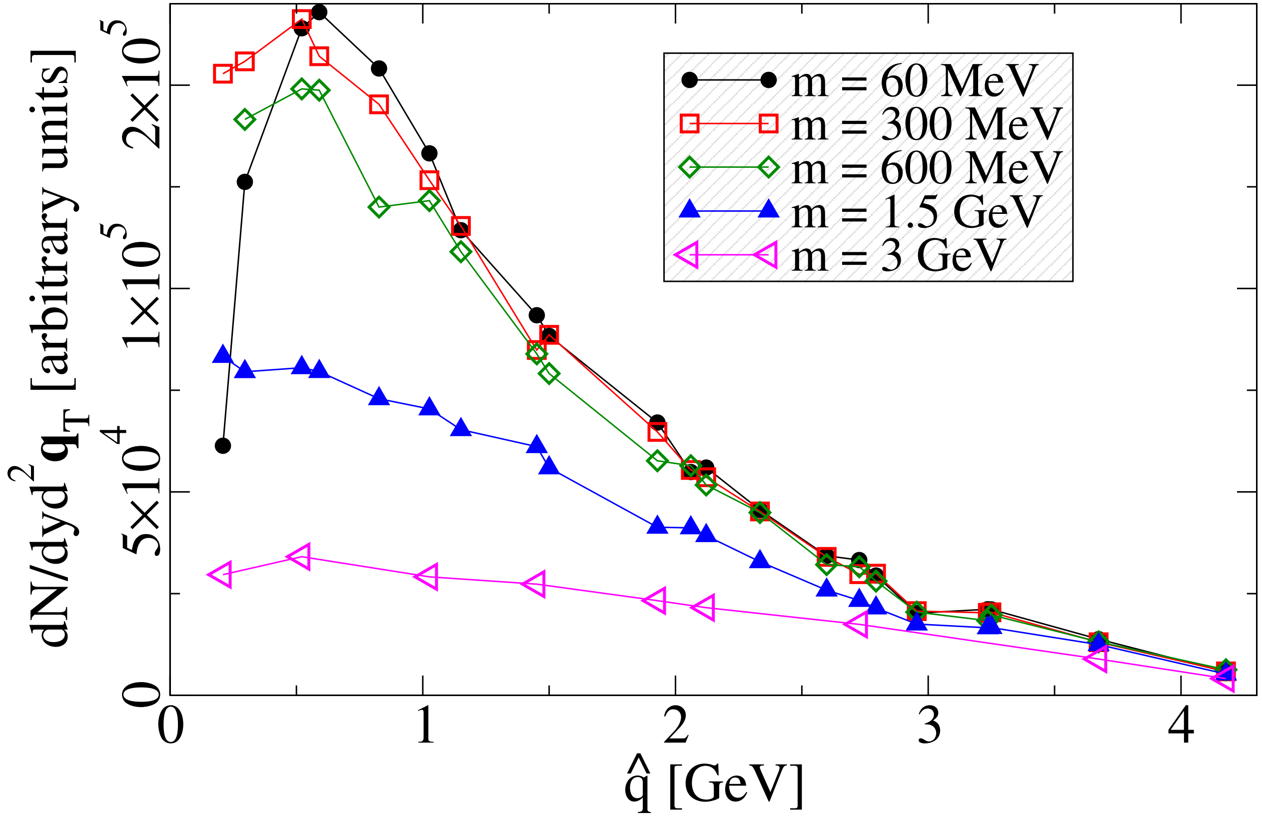
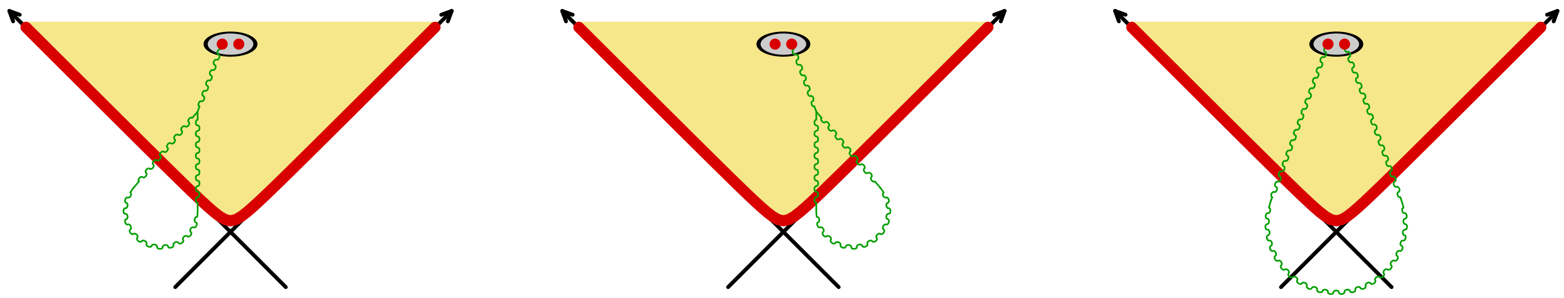


Spectra for various quark masses



Sketch of a proof – I



- The first two terms involve :

$$\delta\mathcal{A}(x) \equiv \frac{g}{2}\int d^4z \sum_{\epsilon=\pm} \epsilon\, \boldsymbol{G}_{+\epsilon}(x,z)\boldsymbol{G}_{\epsilon\epsilon}(z,z)$$

- The third term involves $\boldsymbol{G}_{+-}(x,y)$
- The propagators $\boldsymbol{G}_{\pm\pm}$ are propagators in the background $\mathcal{A}$, in the Schwinger-Keldysh formalism. They obey :

$$\begin{cases} \boldsymbol{G}_{+-} = \boldsymbol{G}_R {G^0_R}^{-1} G^0_{+-} {G^0_A}^{-1} \boldsymbol{G}_A \\ \boldsymbol{G}_{\pm\pm} = \frac{1}{2}\left[\boldsymbol{G}_R {G^0_R}^{-1}(G^0_{+-}+G^0_{-+}){G^0_A}^{-1}\boldsymbol{G}_A \pm (\boldsymbol{G}_R+\boldsymbol{G}_A)\right] \end{cases}$$

$\boldsymbol{G}_{R,A}$ = retarded/advanced propagators in the background $\mathcal{A}$

Sketch of a proof – II

- $\boldsymbol{G}_{++}$ and $\boldsymbol{G}_{--}$ are only needed with equal endpoints
  - ▷ they are both equal to

$$\boldsymbol{G}_{++}(z,z)=\boldsymbol{G}_{--}(z,z)=\frac{1}{2}\left[\boldsymbol{G}_{_R}G_{_R}^{0\ -1}(G_{+-}^0+G_{-+}^0)G_{_A}^{0\ -1}\boldsymbol{G}_{_A}\right](z,z)$$

▷ thus, $\delta\mathcal{A}$ can be simplified into :

$$\begin{aligned}\delta\mathcal{A}(x) &= \frac{g}{2}\int d^4z\left[\boldsymbol{G}_{++}(x,z)-\boldsymbol{G}_{+-}(x,z)\right]\boldsymbol{G}_{++}(z,z)\\ &= \frac{g}{2}\int d^4z\;\boldsymbol{G}_{_R}(x,z)\boldsymbol{G}_{++}(z,z)\end{aligned}$$

- $\boldsymbol{G}_{_R}G_{_R}^{0\ -1}G_{+-}^0G_{_A}^{0\ -1}\boldsymbol{G}_{_A}$ can be written as :

$$\left[\boldsymbol{G}_{_R}G_{_R}^{0\ -1}G_{+-}^0G_{_A}^{0\ -1}\boldsymbol{G}_{_A}\right](x,y)=\int\frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}}\;\zeta_{\vec{\boldsymbol{p}}}(x)\zeta^*_{\vec{\boldsymbol{p}}}(y)\;,$$

with $\left[\square_x+m^2+g\mathcal{A}(x)\right]\zeta_{\vec{\boldsymbol{p}}}(x)=0$ and $\lim_{x_0\to-\infty}\zeta_{\vec{\boldsymbol{p}}}(x)=e^{ip\cdot x}$

Sketch of a proof – III

- Green's formulas :

$$\mathcal{A}(x) = \int_\Omega d^4z\, G^0_R(x,z)\Big[j(z) - \frac{g}{2}\mathcal{A}^2(z)\Big] + \int_{\rm LC} d^3\vec{\boldsymbol{u}}\, G^0_R(x,u)\Big[n\cdot\overrightarrow{\partial}_u - n\cdot\overleftarrow{\partial}_u\Big]\mathcal{A}_{\rm in}(\vec{\boldsymbol{u}})$$

$$\delta\mathcal{A}(x) = \int_\Omega d^4z\, \boldsymbol{G}_R(x,z)\,\frac{g}{2}\boldsymbol{G}_{++}(z,z) + \int_{\rm LC} d^3\vec{\boldsymbol{u}}\, \boldsymbol{G}_R(x,u)\Big[n\cdot\overrightarrow{\partial}_u - n\cdot\overleftarrow{\partial}_u\Big]\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{u}})$$

$$\zeta_{\vec{\boldsymbol{p}}}(x) = \int_{\rm LC} d^3\vec{\boldsymbol{u}}\, \boldsymbol{G}_R(x,u)\Big[n\cdot\overrightarrow{\partial}_u - n\cdot\overleftarrow{\partial}_u\Big]\zeta_{\vec{\boldsymbol{p}}\,{\rm in}}(\vec{\boldsymbol{u}})$$

$$\boldsymbol{G}_R(x,y) = G^0_R(x,y) + g\int_\Omega d^4z\, G^0_R(x,z)\mathcal{A}(z)\boldsymbol{G}_R(z,y)$$

Sketch of a proof – IV

- Thanks to the operator

$$a_{\rm in}(\vec{\boldsymbol u})\cdot \boldsymbol{T}_{\vec{\boldsymbol u}} \equiv a_{\rm in}(\vec{\boldsymbol u})\frac{\delta}{\delta \mathcal{A}_{\rm in}(\vec{\boldsymbol u})} + \Big[(n\cdot\partial_u)a_{\rm in}(\vec{\boldsymbol u})\Big]\frac{\delta}{\delta(n\cdot\partial_u)\mathcal{A}_{\rm in}(\vec{\boldsymbol u})}\ ,$$

we can write

$$\zeta_{\vec{\boldsymbol p}}(x) = \int\limits_{\vec{\boldsymbol u}\in{\rm LC}} \Big[\zeta_{\vec{\boldsymbol p}\,{\rm in}}(\vec{\boldsymbol u})\cdot \boldsymbol{T}_{\vec{\boldsymbol u}}\Big]\,\mathcal{A}(x)$$

$$\delta\mathcal{A}(x) = \int_\Omega d^4z\, \boldsymbol{G}_R(x,z)\,\frac{g}{2}\boldsymbol{G}_{++}(z,z) + \int\limits_{\vec{\boldsymbol u}\in{\rm LC}} \Big[\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol u})\cdot \boldsymbol{T}_{\vec{\boldsymbol u}}\Big]\,\mathcal{A}(x)$$

▷ from the classical field $\mathcal{A}(x)$, the operator $a_{\rm in}(\vec{\boldsymbol u})\cdot \boldsymbol{T}_{\vec{\boldsymbol u}}$ builds the fluctuation $a(x)$ whose initial condition on the light-cone is $a_{\rm in}(\vec{\boldsymbol u})$

- The 3rd diagram can directly be written as :

$$\int \frac{d^3\vec{\boldsymbol p}}{(2\pi)^3 2E_{\boldsymbol p}} \int\limits_{\vec{\boldsymbol u},\vec{\boldsymbol v}\in{\rm LC}} \Big[\Big[\zeta_{\vec{\boldsymbol p}\,{\rm in}}(\vec{\boldsymbol u})\cdot \boldsymbol{T}_{\vec{\boldsymbol u}}\Big]\,\mathcal{A}(x)\Big]\Big[\Big[\zeta^*_{\vec{\boldsymbol p}\,{\rm in}}(\vec{\boldsymbol v})\cdot \boldsymbol{T}_{\vec{\boldsymbol v}}\Big]\,\mathcal{A}(y)\Big]$$

Sketch of a proof – V

- One can finally prove that

$$\int_\Omega d^4z\, \boldsymbol{G}_R(x,z)\, \frac{g}{2}\boldsymbol{G}_{++}(z,z) =$$

$$= \frac{1}{2}\int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}} \int\limits_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\in\mathrm{LC}} \Big[\zeta_{\vec{\boldsymbol{p}}\,\mathrm{in}}(\vec{\boldsymbol{u}})\cdot \boldsymbol{T}_{\vec{\boldsymbol{u}}}\Big]\Big[\zeta^*_{\vec{\boldsymbol{p}}\,\mathrm{in}}(\vec{\boldsymbol{v}})\cdot \boldsymbol{T}_{\vec{\boldsymbol{v}}}\Big]\,\mathcal{A}(x)$$

$\triangleright$

$$\delta\mathcal{A}(x) = \Bigg[\int\limits_{\vec{\boldsymbol{u}}\in\mathrm{LC}} \Big[\delta\mathcal{A}_{\mathrm{in}}(\vec{\boldsymbol{u}})\cdot \boldsymbol{T}_{\vec{\boldsymbol{u}}}\Big]$$

$$+\frac{1}{2}\int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}} \int\limits_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\in\mathrm{LC}} \Big[\zeta_{\vec{\boldsymbol{p}}\,\mathrm{in}}(\vec{\boldsymbol{u}})\cdot \boldsymbol{T}_{\vec{\boldsymbol{u}}}\Big]\Big[\zeta^*_{\vec{\boldsymbol{p}}\,\mathrm{in}}(\vec{\boldsymbol{v}})\cdot \boldsymbol{T}_{\vec{\boldsymbol{v}}}\Big]\Bigg]\,\mathcal{A}(x)$$

- This leads to the announced formula for $\delta\overline{N}$, with

$$\boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}}\, \zeta_{\vec{\boldsymbol{p}}\,\mathrm{in}}(\vec{\boldsymbol{u}})\,\zeta^*_{\vec{\boldsymbol{p}}\,\mathrm{in}}(\vec{\boldsymbol{v}})$$

Longitudinal expansion

- For a system finite in the η direction, the gluons will have a longitudinal velocity tied to their space-time rapidity

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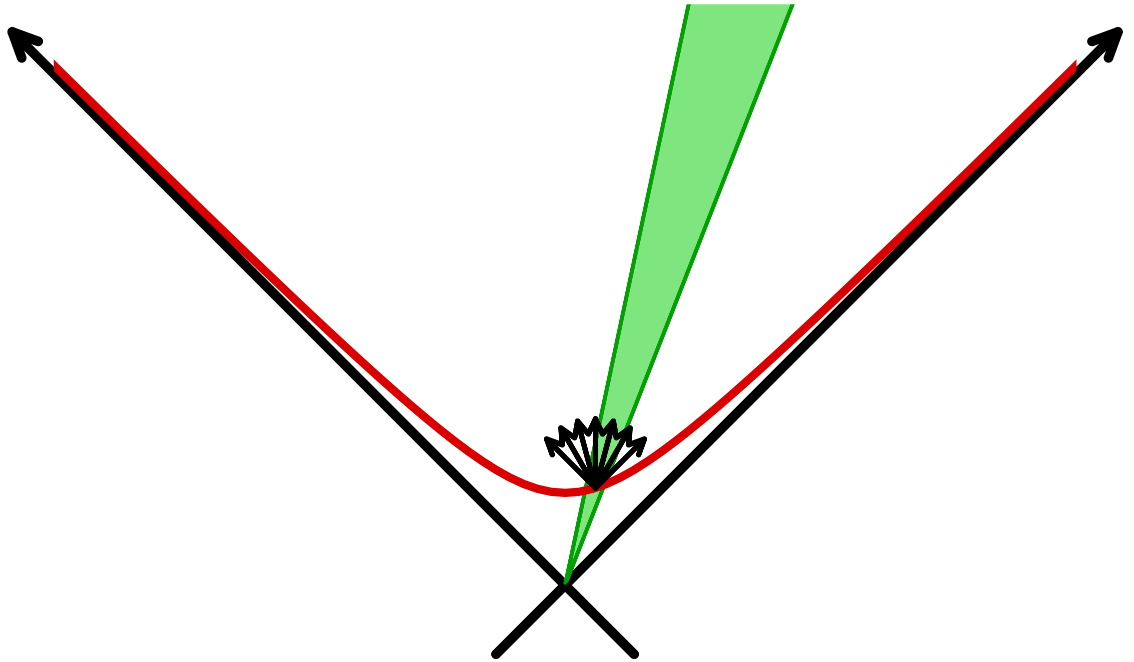

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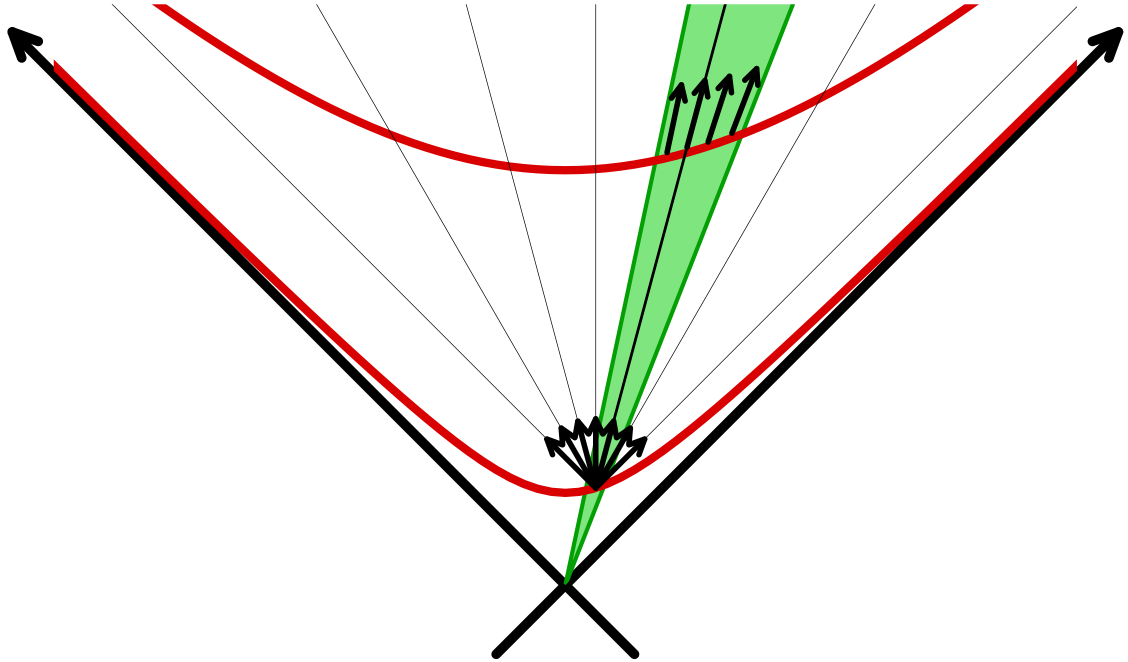


▷ at late times : if particles fly freely, only one longitudinal velocity can exist at a given η : $v_z = \tanh(\eta)$

▷ the expansion of the system is the main obstacle to local isotropy

Generating function

- Let P_n be the probability of producing n particles
- Define the generating function :

$$F(z) \equiv \sum_{n=0}^{\infty} P_n\, z^n$$

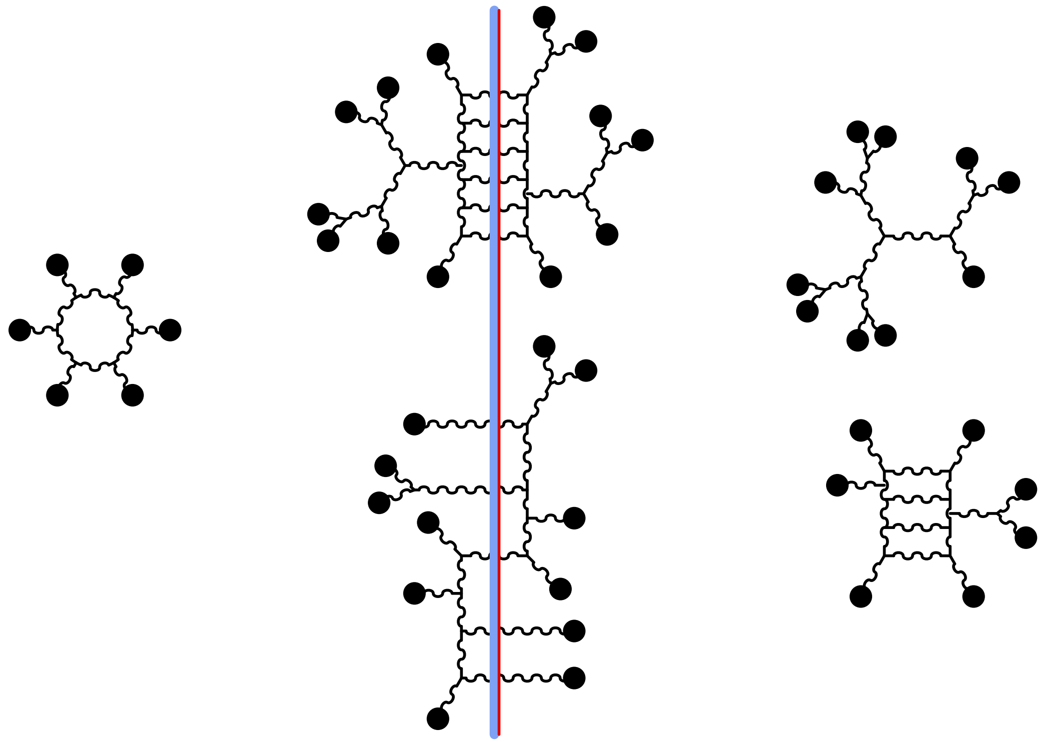
- From unitarity, $F(1) = \sum_{n=0}^{\infty} P_n = 1$. Thus, we can write

$$\ln(F(z)) \equiv \sum_{r=1}^{\infty} b_r\,(z^r - 1)$$

- At the moment, we need to know only very little about the b_r :
  - $F(z)$ is a sum of diagrams that may or may not be connected
  - $\ln(F(z))$ involves only connected diagrams. Hence, the b_r's are given by certain sums of connected diagrams
  - Every diagram in b_r produces r particles

Generating function

- Example : typical term in the coefficient of z^{11}, with contributions from b_5 and b_6 :



Distribution of connected subdiagrams

- From this form of the generating function, one gets :

$$P_n = \underbrace{\sum_{p=0}^{n} e^{-\sum_r b_r} \frac{1}{p!} \sum_{\alpha_1+\cdots+\alpha_p=n} b_{\alpha_1} \cdots b_{\alpha_n}}$$

probability of producing n particles in p cut subdiagrams

- Summing on n, we get the probability of p cut subdiagrams :

$$R_p = \frac{1}{p!} \left[\sum_{r=1}^{\infty} b_r \right]^p e^{-\sum_r b_r}$$

Note : Poisson distribution of average $\left\langle N_{\text{subdiagrams}} \right\rangle = \sum_r b_r$

- By expanding the exponential, we get the probability of having p cut subdiagrams out of a total of m :

$$R_{p,m} = \frac{(-1)^{m-p}}{(m-p)!\,p!} \left[\sum_{r=1}^{\infty} b_r \right]^m$$

AGK identities

- The quantities $R_{p,m}$ obey the following relations :

$$\forall m \geq 2\,,\quad \sum_{p=1}^{m} p\, R_{p,m} = 0\,,$$

$$\forall m \geq 3\,,\quad \sum_{p=1}^{m} p(p-1)\, R_{p,m} = 0\,, \cdots$$

- Interpretation : contributions with more than 1 subdiagram cancel in the average number of cut subdiagrams, etc...
- Correspondence with the original relations by Abramovsky-Gribov-Kancheli :
  - The original derivation is formulated in the framework of reggeon effective theories
  - Dictionary: *reggeon* $\longrightarrow$ *subdiagram*
  - These identities are more general than “reggeons”, and are valid for any kind of subdiagrams

Limitations

- The AGK relations, obtained by "integrating out" the number of produced particles, describe the combinatorics of connected diagrams
  - ▷ by doing that, a lot of information has been discarded
- For instance, to compute the average number of produced particles, one would write :

$$\langle n \rangle = \underbrace{\Big\langle N_{\rm subdiagrams} \Big\rangle}_{\sum_r b_r} \times \underbrace{\Big\langle \#\ \text{of particles per diagram} \Big\rangle}_{\text{requires a more detailed description}}$$

Generating function

- One can encode the information about all the probabilities P_n in a generating function defined as :

$$F(z) \equiv \sum_{n=0}^{\infty} P_n \; z^n$$

- From the expression of P_n in terms of the operator $\mathcal{D}$, we can write :

$$F(z) \quad = \quad e^{z\mathcal{D}} \; e^{iV} \; e^{-iV^*}$$

- Reminder :
  - $e^{\mathcal{D}} \; e^{iV} \; e^{-iV^*}$ is the sum of all the cut vacuum diagrams
  - The cuts are produced by the action of $\mathcal{D}$

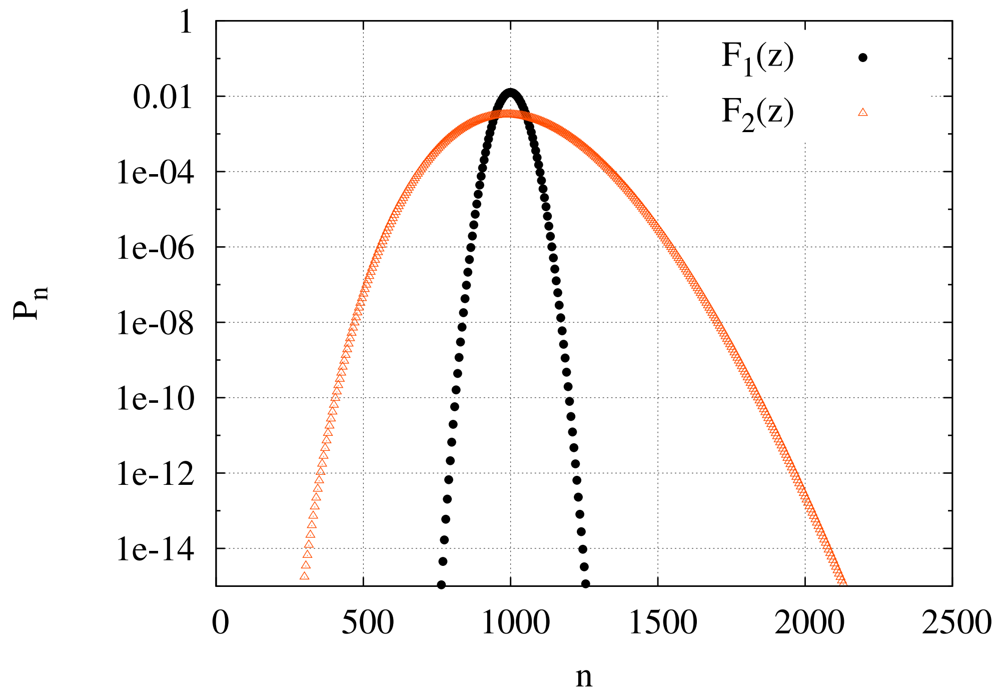
- Therefore, $F(z)$ is the sum of all the cut vacuum diagrams in which each cut line is weighted by a factor z

What would it be good for ?

- Let us pretend that we know the generating function $F(z)$.
  We could get the probability distribution as follows :

$$P_n = \frac{1}{2\pi}\int_0^{2\pi} d\theta \; e^{-in\theta} \; F(e^{i\theta})$$

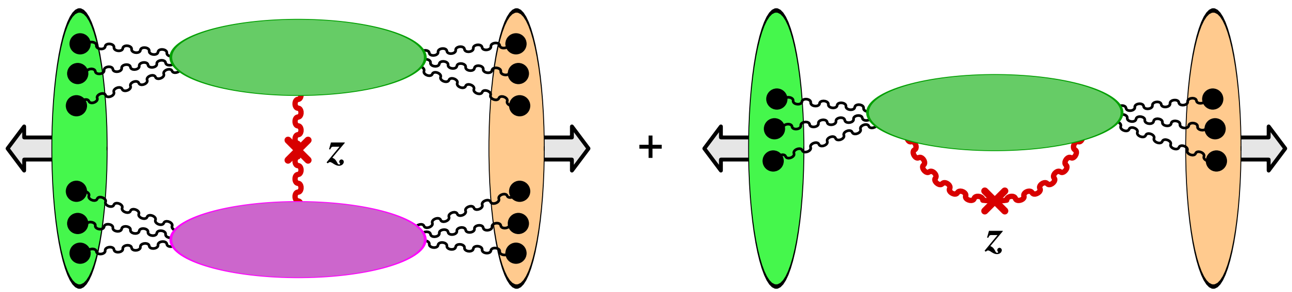
Note : this is trivial to evaluate numerically by a FFT :



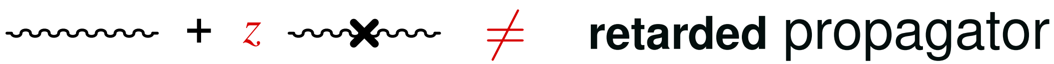
F(z) at Leading Order

- We have : $F'(z) = \mathcal{D}\left\{e^{z\mathcal{D}}\, e^{iV}\, e^{-iV^*}\right\}$
- By the same arguments as in the case of $\overline{N}$, we get :

$$\frac{F'(z)}{F(z)} =$$



- The major difference is that the cut graphs that must be evaluated have a factor z attached to each cut line
- At tree level (LO), we can write $F'(z)/F(z)$ in terms of solutions of the classical Yang-Mills equations, but these solutions are not retarded anymore, because :



F(z) at Leading Order

- The derivative F'/F has an expression which is formally identical to that of $\overline{N}$,

$$\frac{F'(z)}{F(z)} = \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}} \int_{x,y} e^{ip\cdot(x-y)}\, \square_x \square_y \sum_\lambda \epsilon_\lambda^\mu \epsilon_\lambda^\nu \; A_\mu^{(+)}(x) A_\nu^{(-)}(y)\,,$$

  with $A_\mu^{(\pm)}(x)$ two solutions of the Yang-Mills equations

- If one decomposes these fields into plane-waves,

$$\boldsymbol{A}_\mu^{(\varepsilon)}(x) \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}} \left\{ f_+^{(\varepsilon)}(x^0,\vec{\boldsymbol{p}}) e^{-ip\cdot x} + f_-^{(\varepsilon)}(x^0,\vec{\boldsymbol{p}}) e^{ip\cdot x} \right\}$$

  the boundary conditions are :

$$f_+^{(+)}(-\infty,\vec{\boldsymbol{p}}) = f_-^{(-)}(-\infty,\vec{\boldsymbol{p}}) = 0$$

$$f_+^{(-)}(+\infty,\vec{\boldsymbol{p}}) = z\, f_+^{(+)}(+\infty,\vec{\boldsymbol{p}}) \quad , \quad f_-^{(+)}(+\infty,\vec{\boldsymbol{p}}) = z\, f_-^{(-)}(+\infty,\vec{\boldsymbol{p}})$$

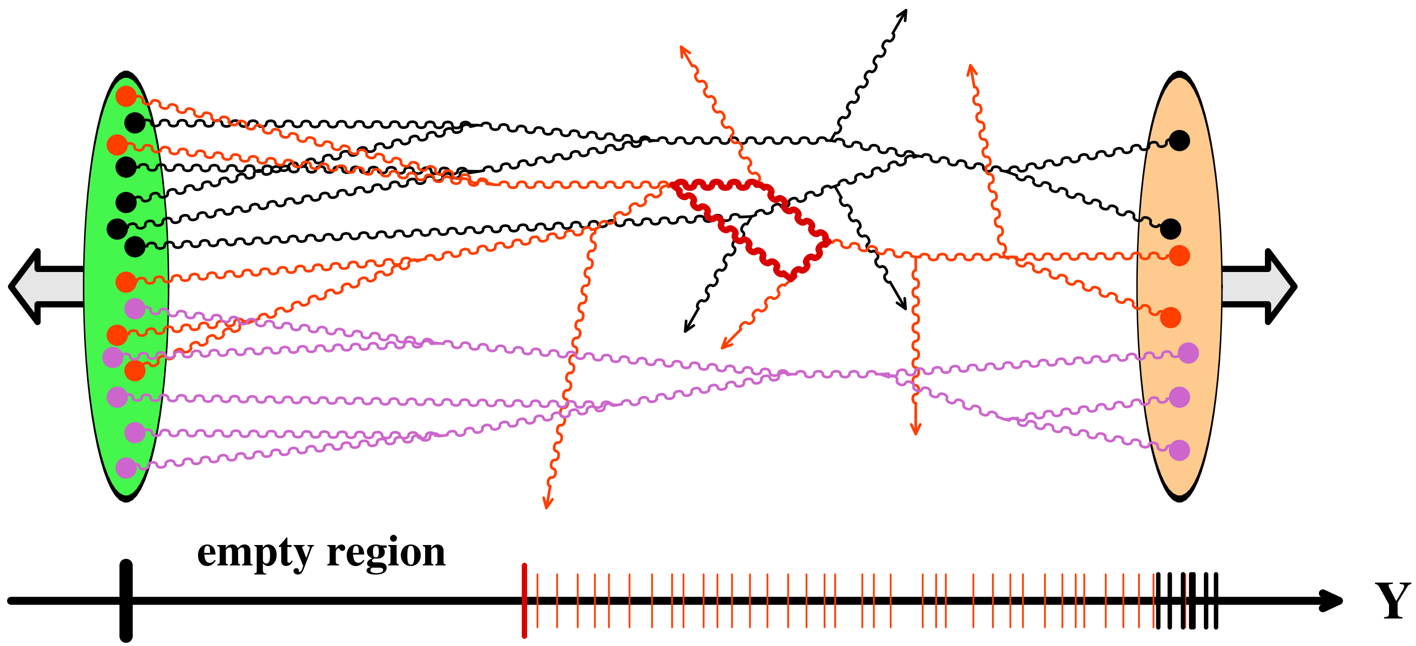
- There are boundary conditions both at $x_0 = -\infty$ and $x_0 = +\infty$ $\triangleright$ not an initial value problem $\triangleright$ hard...

Remarks on factorization

- As we have seen, the fact that the calculation of the first moment $\overline{N}$ can be formulated as an initial value problem seems quite helpful for proving factorization
- If the retarded nature of the fields is crucial, then factorization does not hold for the generating function $F(z)$, or equivalently for the individual probabilities P_n
- Note : by differentiating the result for $F(z)$ with respect to z, and then setting $z = 1$, we can obtain formulas for higher moments of the distribution

Exclusive processes

- So far, we have considered only inclusive quantities – i.e. the P_n are defined as probabilities of producing particles anywhere in phase-space
- What about events where a part of the phase-space remains unoccupied ? e.g. rapidity gaps



Main issues

1. How do we calculate the probabilities $P_n^{\rm excl}$ with an excluded region in the phase-space ?
   Can one calculate the total gap probability $P_{\rm gap} = \sum_n P_n^{\rm excl}$?
2. What is the appropriate distribution of sources $W_Y^{\rm excl}[\rho]$ to describe a projectile that has not broken up ?
3. How does it evolve with rapidity ?

   See : Hentschinski, Weigert, Schafer (2005)
4. Are there some factorization results, and for which quantities do they hold ?

Exclusive probabilities

- The probabilities $P_n^{\rm excl}[\mathbf{\Omega}]$, for producing n particles – only in the region $\mathbf{\Omega}$ – can also be constructed from the vacuum diagrams, as follows :

$$P_n^{\rm excl}[\mathbf{\Omega}] = \frac{1}{n!}\,\mathcal{D}_{\mathbf{\Omega}}^n \;\; e^{iV}\; e^{-iV^*}$$

where $\mathcal{D}_{\mathbf{\Omega}}$ is an operator that removes two sources and links the corresponding points by a cut (on-shell) line, for which the integration is performed only in the region $\mathbf{\Omega}$

- One can define a generating function,

$$F_{\mathbf{\Omega}}(z) \equiv \sum_n P_n^{\rm excl}[\mathbf{\Omega}]\; z^n \quad ,$$

whose derivative is given by the same diagram topologies as the derivative of the generating function for inclusive probabilities

Exclusive probabilities

- Differences with the inclusive case :
  - In the diagrams that contribute to $F'_\Omega(z)/F_\Omega(z)$, the cut propagators are restricted to the region $\boldsymbol{\Omega}$ of the phase-space

    ▷ at leading order, this only affects the boundary conditions for the classical fields in terms of which one can write $F'_\Omega(z)/F_\Omega(z)$

    ▷ not more difficult than the inclusive case
  - Contrary to the inclusive case – where we know that $F(1) = 1$ – the integration constant needed to go from $F'_\Omega(z)/F_\Omega(z)$ to $F_\Omega(z)$ is non-trivial. This is due to the fact that the sum of all the exclusive probabilities is smaller than unity

    ▷ $F_\Omega(1)$ is in fact the probability of not having particles in the complement of $\boldsymbol{\Omega}$ – i.e. the gap probability

Survival probability

- We can write :

$$F_{\Omega}(z) = F_{\Omega}(1) \quad \exp\left\{ \int_1^z d\tau \frac{F'_{\Omega}(\tau)}{F_{\Omega}(\tau)} \right\}$$

▷ the prefactor $F_{\Omega}(1)$ will appear in all the exclusive probabilities

- This prefactor is nothing but the famous "survival probability" for a rapidity gap

▷ One can in principle calculate it by the general techniques developed for calculating inclusive probabilities :

$$F_{\Omega}(1) \; = \; F^{\text{incl}}_{1-\Omega}(0)$$

▷ Note : it is incorrect to say that a certain process with a gap can be calculated by multiplying the probability of this process without the gap by the survival probability

Factorization ?

- In order to discuss factorization for exclusive quantities, one must calculate their 1-loop corrections, and study the structure of the divergences...

- Except for the case of Deep Inelastic Scattering, nothing is known regarding factorization for exclusive processes in a high density environment

- For the overall framework to be consistent, one should have factorization between the gap probability, $F_\Omega(1)$, and the source density studied in Hentschinski, Weigert, Schafer (2005) (and the ordinary $W_Y[\rho]$ on the other side)

- The total gap probability is the "most inclusive" among the exclusive quantities one may think of. For what quantities – if any – does factorization work ?