Introduction to Phase Transitions in Statistical Physics and Field Theory

Motivation

Basic Concepts and Facts about Phase Transitions:

- Phase Transitions in Fluids and Magnets
- Thermodynamics and Statistical Mechanics
- Order Parameter and Symmetry Breaking
- Susceptibilities and Correlation Functions
- Characteristics of Phase Transitions
- Universality Classes
- Important Models
Motivation

- Phase Transitions are of relevance everywhere in physics - starting with the theory of the early universe, in particle physics, in condensed matter physics, and of course they are relevant in daily life.

- In the critical region of continuous transitions cooperative phenomena generate a large scale, the correlation length, although the fundamental interactions are short range - most details of the microscopic structure become unimportant and a power law behaviour emerges. Such scaling laws occur also in the clustering of galaxies, the distribution of earthquakes, the turbulence in fluids and plasmas, etc.

- Most interesting for particle physicists is: the connection between statistical field theory of critical phenomena and quantum field theory, both from a theoretical point of view:
  for the understanding of the role of renormalization and renormalizable field theories in particle physics, for the calculation of critical parameters, and in a practical sense:
  methods from statistical mechanics allow simulations of lattice gauge field theories and lead to non-perturbative results.
Phase Transitions in Fluids and Magnets

At a phase transition (PT) one observes a sudden change in the properties of a system, e.g.

- liquid $\leftrightarrow$ gas
- paramagnet $\leftrightarrow$ ferromagnet
- normal conductor $\leftrightarrow$ superconductor

Phase diagram of a typical fluid:

Solid lines are phase boundaries separating regions of stable phases. In crossing these lines:

- Mass density $\rho$ and energy density $\epsilon$ change discontinuously, they jump

  $\rightarrow$ 1. order phase transition

- $C$ is the critical end point

  $\rightarrow$ 2. order phase transition
The vapour pressure curve is the liquid-gas coexistence line:

"Order parameter" (OP): \( \rho_l - \rho_g \)

\[ T = T_c : \rho_c = \rho_l(T_c) = \rho_g(T_c) \]

\( T > T_c : \) no distinction between gas and liquid

At \( T_c \): continuous transition, but \( C_v \) diverges for \( \rho = \rho_c \)
Magnetic Phase Transitions

The phase diagram:

- $H > 0$
- $H = 0$
- $H < 0$

Analog to fluids:

- $T < T_c$: 1. order PT
- $T = T_c$: 2. order critical point

The order parameter:

- $M(T < T_c, H = 0) \neq 0$
- $M(T, H \to 0^+) = -M(T, H \to 0^-)$
- $M(T \geq T_c, H = 0) = 0$

Spontaneous Magnetization:
Thermodynamics and Statistical Mechanics

We consider only the equilibrium case. Thermodynamics assumes no particular model for a system, the variables are macroscopic quantities - for the system as a whole. There are

**Extensive Variables:**

- $U =$ internal energy,
- $V =$ volume,
- $S =$ entropy,
- $N_i =$ particle number of type $i$,
- $\mathbf{M} =$ total magnetic moment.

**Intensive Variables:**

- $T =$ temperature,
- $p =$ pressure,
- $\rho =$ density $= \text{mass}/\text{volume},$
- $\mu_i =$ chemical potential of type $i$,
- $\mathbf{H} =$ external magnetic field.

The magnetic field and the magnetic moment are in general vectors in $N$ dimensions

$$\mathbf{H} = H \mathbf{n}, \quad \mathbf{n} = (n_1, \ldots, n_N), \quad \mathbf{n}^2 = 1$$

A change in $U$ (here without the internal energy of the field) is described by

$$dU = TdS - pdV - \mathbf{M}d\mathbf{H}$$

Mostly one treats either **Fluids**, where $\mathbf{M} = 0$ or **Magnetic systems** with $dV = 0$. 
Thermodynamic Potentials

Instead of using the internal energy \( U = U(S, V, \bar{H}) \) one may use other potentials. Potential changes are achieved by Legendre transformations.

<table>
<thead>
<tr>
<th>Thermodynamic Potentials</th>
<th>Fluids</th>
<th>Magnets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Internal energy</td>
<td>( U(S, V) )</td>
<td>( U(S, \bar{H}) )</td>
</tr>
<tr>
<td></td>
<td>( dU = TdS - pdV )</td>
<td>( dU = TdS - \bar{M}d\bar{H} )</td>
</tr>
<tr>
<td>Helmholtz free energy</td>
<td>( F(T, V) = U - TS )</td>
<td>( F(T, \bar{H}) = U - TS )</td>
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<tr>
<td></td>
<td>( dF = -SdT - pdV )</td>
<td>( dF = -SdT - \bar{M}d\bar{H} )</td>
</tr>
<tr>
<td>Gibbs free energy</td>
<td>( G(T, p) = F + pV )</td>
<td>( G(T, \bar{M}) = F + \bar{M}\bar{H} )</td>
</tr>
<tr>
<td></td>
<td>( dG = -SdT - Vdp )</td>
<td>( dG = -SdT + \bar{H}d\bar{M} )</td>
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</tbody>
</table>

A closed system has in equilibrium: at fixed \( S \) minimal \( U \), at fixed \( U \) maximal \( S \) and at fixed \( T \) minimal \( F \).

In finite volumes: ensemble average = time average
Thermodynamic Limit

We would like to work in the thermodynamic limit, i. e. in the limit where

\[ V \to \infty , \quad N_i \to \infty , \quad \text{at fixed intensive variables} \]

Reasons :

i) For finite systems the thermodynamic quantities are always analytic functions of their variables. Singular behaviour as required for phase transitions does not occur. Because of the finite volume the correlation length is finite.

ii) Spontaneous symmetry breaking exists only in the thermodynamic limit. In finite systems the breaking is explicit by an external field or by the boundary conditions.

iii) In the thermodynamic limit the different ensembles are equivalent.

iv) In the thermodynamic limit there is no boundary (surface) dependence.

\[ \Rightarrow \] The usual extensive variables become infinite in the thermodynamic limit. We therefore use densities :

\[ \epsilon = \frac{U}{V} \quad \text{energy density} \]
\[ \vec{M} = \frac{\vec{M}}{V} \quad \text{magnetization} \]
\[ s = \frac{S}{V} \quad \text{entropy density} \]
\[ f = \frac{F}{V} = \epsilon - Ts \quad \text{free energy density} \]
### Thermodynamic Relations

<table>
<thead>
<tr>
<th>Fluids ((T, V))</th>
<th>Magnets ((T, H), N = 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(df = -s dT + \left( \frac{\partial f}{\partial V} \right)_T \ dV)</td>
<td>(df = -s dT - M dH)</td>
</tr>
<tr>
<td>(s = - \left( \frac{\partial f}{\partial T} \right)_V)</td>
<td>(s = - \left( \frac{\partial f}{\partial T} \right)_H)</td>
</tr>
<tr>
<td>(\epsilon = f + Ts = \left( \frac{\partial f}{\partial \beta} \right)_V)</td>
<td>(\epsilon = f + Ts = \left( \frac{\partial f}{\partial \beta} \right)_H)</td>
</tr>
<tr>
<td>(p = -f - V \left( \frac{\partial f}{\partial V} \right)_T = - \left( \frac{\partial V f}{\partial V} \right)_T)</td>
<td>(M = - \left( \frac{\partial f}{\partial H} \right)_T)</td>
</tr>
<tr>
<td>(C_V = \left( \frac{\partial e}{\partial T} \right)_V)</td>
<td>(C_H = \left( \frac{\partial e}{\partial T} \right)_H)</td>
</tr>
<tr>
<td>(\kappa_T = - \frac{1}{V} \left( \frac{\partial V}{\partial p} \right)_T)</td>
<td>(\chi = \left( \frac{\partial M}{\partial H} \right)_T)</td>
</tr>
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We use \(k = 1\) as unit, so that \(\beta = 1/T\)
In contrast to thermodynamics it provides a link between microscopic and macroscopic physics, because it starts with a Hamiltonian $\mathcal{H}$, which describes the microscopic interaction.

Canonical ensemble, the partition function:

$$Z = \text{Tr} \ e^{-\beta \mathcal{H}} = \sum_{\alpha} e^{-\beta E_{\alpha}}$$

$E_{\alpha} = $ Eigenvalue of $\mathcal{H}$ in state $\alpha$

$$U = \langle E \rangle = \frac{1}{Z} \sum_{\alpha} E_{\alpha} e^{-\beta E_{\alpha}} = -\frac{\partial \ln Z}{\partial \beta} = \langle \mathcal{H} \rangle$$

$$F = -T \ln Z \ , \quad f = -\frac{T}{V} \ln Z$$

$$Z = e^{-\beta F}$$
The fluctuation in the energy is given by

\[ (\Delta U)^2 = \langle (E - \langle E \rangle)^2 \rangle = \langle E^2 \rangle - \langle E \rangle^2 = \frac{\partial^2 \ln Z}{\partial \beta^2} \]

More general, the average of a quantity \( X \) can be calculated by introducing a source term \( XY \) in the Hamiltonian:

\[ \mathcal{H} = \mathcal{H}_0 - XY \]

and taking the derivative with respect to \( Y \):

\[ \frac{1}{\beta} \left. \frac{\partial \ln Z}{\partial Y} \right|_{Y=0} = \frac{1}{\beta Z} \frac{\partial}{\partial Y} \left( \sum_{\alpha} e^{-\beta(E_{\alpha} - X_{\alpha} Y)} \right) \bigg|_{Y=0} = \langle X \rangle = - \frac{\partial F}{\partial Y} \bigg|_{Y=0} \]

and

\[ \frac{1}{\beta} \left. \frac{\partial \langle X \rangle}{\partial Y} \right|_{Y=0} = \langle X^2 \rangle - \langle X \rangle^2 \]
Symmetry breaking for a ferromagnet, \( \vec{H} = H \vec{n} \), \( \vec{H}, \vec{n} \) \( N \)-dim vectors

**\( T < T_c \):** ferromagnetic phase, the spins are aligned, \( M \) measures the degree of ordering – the ordered state is not symmetric under rotation

- \( H \neq 0 \) \( \implies \vec{M} \) aligns in \( \vec{H} \)-direction, for \( \vec{H} \rightarrow -\vec{H} \), \( \vec{M} \) jumps to \( -\vec{M} \)
  - explicit symmetry breaking

- \( H = 0 \) \( \implies \) direction of \( \vec{M} \) is spontaneously chosen by the system
  - spontaneous symmetry breaking

**\( T > T_c \):** paramagnetic phase,

- \( H \neq 0 \) \( \implies \vec{M} \sim \vec{H} \), for \( \vec{H} \rightarrow -\vec{H} \),
  - \( \vec{M} \) changes continuously to \( -\vec{M} \)
  - explicit symmetry breaking

- \( H = 0 \) \( \implies \vec{M} = 0 \)
  - symmetry of state

**\( T = T_c \):** critical line, like paramagnetic phase

- \( H = 0 \) \( \implies \partial M / \partial H \) is infinite
The Order Parameter

 Definition of the order parameter:

An (ideal) order parameter for a phase transition is a variable \( \phi \) with

\[
\langle \phi \rangle \equiv 0 \quad \text{in Phase I}
\]
\[
\langle \phi \rangle \neq 0 \quad \text{in Phase II}
\]

Here, \( \langle \cdot \rangle \) means the "thermal average" over a long period in equilibrium at constant temperature = the time average. Further remarks:

- \( \phi = \phi(\vec{x}) \) is usually a local, fluctuating variable, which is not always an observable.
- \( \phi \) can be a scalar, a vector etc., real or complex.
- The name "order parameter" is confusing, because it is also used for \( \langle \phi \rangle \), e.g. when the "order parameter" is shown in a plot. Better: \( \phi(\vec{x}) \) is the order parameter field or local order parameter, \( \langle \phi(\vec{x}) \rangle \) is the order parameter.
- In a homogeneous system \( \langle \phi(\vec{x}) \rangle \) is independent of \( \vec{x} \)

\[
\langle \phi(\vec{x}) \rangle = \langle \phi \rangle
\]
The Order Parameter

The Hamiltonian is a function of $\phi$:

$$H = H(\phi(\vec{x}))$$

In many cases: $H$ is invariant under certain transformations of $\phi$, if $\bar{H} = 0$, e.g.

- reversal: $\phi(\vec{x}) \rightarrow -\phi(\vec{x})$  Ising model
- change of phase: $\phi(\vec{x}) \rightarrow e^{i\alpha} \phi(\vec{x})$  $He^4$, fluid to superfluid
- rotation: $\vec{\phi}(\vec{x}) \rightarrow A\vec{\phi}(\vec{x})$  Ferromagnet

Magnet: $\vec{\phi}(\vec{x})$ corresponds to the local magnetization and

$$\vec{M} = \langle \vec{\phi}(\vec{x}) \rangle = \langle \vec{\varphi} \rangle \quad \text{where} \quad \vec{\varphi} = \frac{1}{V} \int d^d x \vec{\phi}(\vec{x})$$

$\vec{\varphi}$ is the volume average of the local magnetization and $\vec{M}$ its thermal average, a global variable, the total magnetization of the magnet.
The Order of a Phase Transition

A phase transition occurs, if in the thermodynamic limit the free energy density $f$ (or other thermodynamic potentials) is non-analytic as a function of its parameters $(T, H, \text{etc.})$.

A definition of order: A phase transition is of $n'$ th order, if the thermodynamic potential has $(n - 1)$ continuous derivatives, but the $n'$ th derivative is discontinuous or divergent.

Modern Classification (M.E. Fisher)

- Discontinuous or first order phase transition: one or more of the first derivatives of $f$ has a finite discontinuity; the order parameter is the first derivative of its conjugate variable $\implies$ it jumps at $T_c$.

- Continuous (or second order) phase transition: the first derivatives of $f$ are continuous, but one higher derivative is singular - includes all higher order transitions $\implies$ the order parameter is continuous at $T_c$.

- Crossover: is no real phase transition, but there is a rapid change in the behaviour of the system $\implies$ in (finite volume) simulations difficult to distinguish from true phase transitions.
Susceptibilities and Correlation Functions

General Magnet:
\[ \mathcal{H} = \mathcal{H}_0 - \vec{H} \int d^d x \vec{\phi}(\vec{x}) \]

\(N\)-dimensional vectors:
\[ \vec{H} = H\vec{n}, \quad \vec{\phi}(\vec{x}), \quad \vec{\varphi} \]

On a lattice:
\[ \vec{\varphi} = \frac{1}{N_x} \sum_{\vec{x}} \vec{\phi}(\vec{x}) \]

\(d\)-dimensional vector:
\[ \vec{x} \]

\[ \mathcal{H} = \mathcal{H}_0 - V \vec{\varphi} \vec{H} \]
\[ X_a = V \varphi_a, \quad Y_a = H_a \]

\[ \langle V \varphi_a \rangle = -\frac{\partial F}{\partial H_a} \quad \text{or} \quad M_a = -\frac{\partial f}{\partial H_a} = \langle \varphi_a \rangle \]

Tensor of susceptibilities:
\[ \chi_{ab} = -\frac{\partial^2 f}{\partial H_a \partial H_b} = \frac{\partial M_a}{\partial H_b} = \frac{\partial \langle \varphi_a \rangle}{\partial H_b} \]
The Susceptibilities

\[ \chi_{ab} = \frac{\partial \langle \varphi_a \rangle}{\partial H_b} = \frac{\partial}{\partial H_b} \left( \frac{1}{Z} \sum_{\alpha} \varphi_\alpha e^{-\beta(E_\alpha - V \varphi_\alpha H)} \right) \]

\[ = \frac{1}{Z} \sum_{\alpha} \varphi_\alpha^a \beta V \varphi_\alpha^b e^{-\beta(\ldots)} - \frac{1}{Z^2} \left( \sum_{\alpha} \varphi_\alpha^a e^{-\beta(\ldots)} \right) \left( \sum_{\alpha} \beta V \varphi_\alpha^a e^{-\beta(\ldots)} \right) \]

\[ \chi_{ab} = \beta V \left[ \langle \varphi_a \varphi_b \rangle - \langle \varphi_a \rangle \langle \varphi_b \rangle \right] \]

Total magnetic susceptibility ~ fluctuation of the length of \( \varphi \):

\[ \chi_{tot} = \sum_a \chi_{aa} = \text{Tr} \chi \]

\[ \chi_{tot} = \beta V \left[ \langle \varphi^2 \rangle - \langle \varphi \rangle^2 \right] \]
The Susceptibilities

Assumption: \( H_0 \) is invariant under \( O(N) \)-rotations of \( \vec{\phi}(\vec{x}) \)

Then \( \vec{M} \) aligns with \( \vec{H} \):
\[
\vec{M} = M\vec{n}
\]

\[
\vec{\varphi} = \varphi_\parallel \vec{n} + \varphi_\perp \quad \varphi_\perp \vec{n} = 0 \quad \Rightarrow \quad M = \langle \varphi_\parallel \rangle \quad \vec{M}_\perp = \langle \varphi_\perp \rangle = 0
\]

\[
\chi_{tot} = \chi_L + (N - 1)\chi_T
\]

with
\[
\chi_L = \frac{\partial M}{\partial H} \quad \chi_T = \frac{M}{H}
\]

Longitudinal susceptibility:
\[
\chi_L = \beta V \left( \langle \varphi_\parallel^2 \rangle - \langle \varphi_\parallel \rangle^2 \right)
\]

\( \leftrightarrow \) Fluctuation of the longitudinal field component = fluctuation of \( M \)

Transverse susceptibility:
\[
\chi_T = \beta V \langle \varphi_\perp^2 \rangle / (N - 1)
\]

\( \leftrightarrow \) Fluctuation of a transverse field component

\[
\chi_{ab} = \chi_L n_a n_b + \chi_T \left( \delta_{ab} - n_a n_b \right)
\]
The Correlation Functions

Correlation functions (2-point connected) of the components of $\vec{\phi}(\vec{x})$:

$$G_{ab}(\vec{x}_1, \vec{x}_2) = \langle \phi_a(\vec{x}_1)\phi_b(\vec{x}_2) \rangle - \langle \phi_a(\vec{x}_1) \rangle \langle \phi_b(\vec{x}_2) \rangle$$

$$= \langle (\phi_a(\vec{x}_1) - \langle \phi_a(\vec{x}_1) \rangle)(\phi_b(\vec{x}_2) - \langle \phi_b(\vec{x}_2) \rangle) \rangle$$

Translation invariance:

$$G_{ab}(\vec{x}_1, \vec{x}_2) = G_{ab}(\vec{x}) \; , \; \vec{x} = \vec{x}_2 - \vec{x}_1$$

Rotation invariance:

$$G_{ab}(\vec{x}) = G_{ab}(r) \; , \; r = |\vec{x}|$$

$$\Rightarrow \quad G_{ab}(\vec{x}) = \langle \phi_a(0)\phi_b(\vec{x}) \rangle - \langle \phi_a(0) \rangle \langle \phi_b(\vec{x}) \rangle$$

$$\int d^dx G_{ab}(\vec{x}) = V \left[ \langle \phi_a(0)\varphi_b \rangle - \langle \phi_a(0) \rangle \langle \varphi_b \rangle \right]$$

$$= V \left[ \langle \varphi_a\varphi_b \rangle - \langle \varphi_a \rangle \langle \varphi_b \rangle \right] = \chi_{ab}/\beta$$

FD-Theorem:

$$\chi_{ab} = \beta \int d^dx G_{ab}(\vec{x})$$
The Correlation Functions

Longitudinal and transverse correlation functions:

\[ G_L(\vec{x}) = \langle \phi(0)\phi(\vec{x}) \rangle - \langle \phi \rangle^2 \]

\[ G_T(\vec{x}) = \frac{1}{N-1} \langle \phi_{\perp}(0)\phi_{\perp}(\vec{x}) \rangle \]

Like for \( \chi \):

\[ G_{ab} = G_L \ n_a n_b + G_T \ (\delta_{ab} - n_a n_b) \]

\[ G_{tot} = \sum_a G_{aa} = G_L + (N-1)G_T = \langle \phi(0)\bar{\phi}(\vec{x}) \rangle - \langle \bar{\phi} \rangle^2 \]

In general: For larger distances \( r = |\vec{x}| \) the fields become more and more uncorrelated and

\[ G(\vec{x}) \sim \frac{e^{-r/\xi}}{r^\tau} \]

\( \xi \) or \( \xi_{L,T} \) is the correlation length.

If \( \xi \to \infty \) a long range order develops and \( G(\vec{x}) \) has a power behaviour.
Characteristics of First Order Transitions

There is a latent heat $\ell \neq 0$:

$$\ell = \lim_{T \to T_c^+} \epsilon(T) - \lim_{T \to T_c^-} \epsilon(T) \neq 0, \quad \ell = \int_{T_c^-}^{T_c^+} C_v(T) dT, \quad C_v = C_H$$

The order parameter $M$ jumps at $T_c$ because:

$$\lim_{T \to T_c^-} M \neq 0, \quad \lim_{T \to T_c^+} M = 0$$

The correlation length $\xi$ stays finite at $T_c$ and:

$$\lim_{T \to T_c^+} \xi - \lim_{T \to T_c^-} \xi \neq 0,$$

The fluctuations / susceptibilities at $T_c$ grow $\sim V$:

$$\chi = \beta V \left( \langle \varphi^2 \rangle - M^2 \right) = \beta V \langle (\varphi - M)^2 \rangle \sim V \beta (\Delta M)^2 \sim V$$

$$C_v = \frac{\beta^2}{V} \langle (E^2) - \langle E \rangle^2 \rangle = \beta^2 V \langle (\frac{E}{V} - \epsilon)^2 \rangle \sim V \beta^2 (\epsilon_+ - \epsilon_-)^2 \sim V$$
Characteristics of Continuous Transitions

At $T = T_c$:

- There is no latent heat

- The order parameter vanishes

- The correlation length diverges, i.e. the fluctuations of the order parameter field grow infinitely, a long range order is established

$$M(T_c) = 0 \quad \lim_{T \to T_c} \xi \to \infty$$

Close to $T_c$:

- The critical behaviour is described by power laws with critical exponents.

The exponents are the same for $T \to T_c^+$ and $T \to T_c^-$. 
Characteristics of Continuous Transitions

Reduced temperature: \( t = (T - T_c)/T_c \)

The following laws are valid in the thermodynamic limit

Singular behaviour of an observable \( F(t) \) near \( T_c \) or \( t = 0 \) for \( H = 0 \)

\[ F(t) = A|t|^\lambda(1 + b|t|^\lambda_1 + \ldots), \quad \lambda_1 > 0 \]

\( \lambda = \text{critical exponent} \)

\[ \lambda = \lim_{t \to 0} \frac{\ln |F(t)|}{\ln |t|} \]

\( \lambda \) can, in principle, be different for \( t \to 0^+ \) and \( t \to 0^- \), but is not!

\( A = \text{critical amplitude} \), is different for \( t \to 0^+ \) and \( t \to 0^- \)!

\( (1 + b|t|^\lambda_1 + \ldots) = \text{corrections to scaling} \), \( b \) is different for \( t \to 0^\pm \)
The Scaling Laws

Leading terms of the scaling laws, thermodynamic limit

\( t > 0, H = 0, \text{ high } T \text{ phase} \)

\[
M = 0 \\
C_v = C_{ns} + \frac{A^+}{\alpha} t^{-\alpha}
\]

\( N = 1 : \quad N > 1 : \)

\[
\chi = C^+ t^{-\gamma} \quad \chi_L = C^+ t^{-\gamma} = \chi_T \\
\xi = \xi^+ t^{-\nu} \quad \xi_L = \xi^+ t^{-\nu} = \xi_T
\]

• \( t = 0, H = 0, \text{ critical point} \)

\[
r \text{ large : } \quad G(r) \sim \frac{1}{r^{d-2+\eta}} \quad \eta \text{ measures deviations from pure Gaussian behaviour}
\]
The Scaling Laws

$t < 0, H = 0$, low $T$ phase

\[ M = B(-t)^\beta \]
\[ C_v = C_{ns} + \frac{A^-}{\alpha}(-t)^{-\alpha} \]

$N = 1 :$
\[ \chi = C^-(-t)^{-\gamma} \]
\[ \xi = \xi^-(t)^{-\nu} \]

$N > 1 :$
\[ \chi, \xi_L, T \text{ diverge} \]

Goldstone effect

$t = 0, H > 0$, critical isotherm

\[ M = B_c H^{1/\delta} \Leftrightarrow H = D_c M^\delta \]
\[ C_v = C_{ns} + \frac{A_c}{\alpha_c} H^{-\alpha_c}, \quad \alpha_c = \frac{\alpha}{\beta \delta} \]

$N = 1 :$
\[ \chi = C^c H^{1/\delta-1}, \quad C^c = \frac{B^c}{\delta} \]
\[ \xi = \xi^c H^{-\nu_c}, \quad \nu_c = \frac{\nu}{\beta \delta} \]

$N > 1 :$
\[ \chi_L = C^c H^{1/\delta-1}, \quad \chi_T = B^c H^{1/\delta-1} \]
\[ \xi_{L,T} = \xi_{L,T}^c H^{-\nu_c} \]
The Scaling Laws

Further facts and comments

- **Specific heat:**

  \( C_{ns} \) is a background term coming from the non-singular part of \( f \);
  the amplitudes \( A^\pm \) are positive; \( \alpha < 1 \) because \( \ell = 0 \):

  \[
  \lim_{\Delta T \to 0} \int_{T_c}^{T_c+\Delta T} t^{-\alpha} dT = T_c \lim_{\Delta T \to 0} \frac{t^{1-\alpha}}{1-\alpha} \left| \frac{\Delta T}{T} \right| = 0 \quad \text{only if} \quad \alpha < 1
  \]

  \( \alpha < 0, \alpha = 0 \) (special case) are possible.

- **\( \eta < 2 \)**, because of the FD-theorem, at \( T_c \):

  \[
  \chi = \beta \int d^d x G(\vec{x}) \sim \int_{R_0}^R drr^{d-1} \frac{1}{r^{d-2+\eta}} \sim \frac{R^{2-\eta}}{2 - \eta}
  \]

  For \( R \to \infty \) the expression must diverge \( \Rightarrow \) \( 2 - \eta > 0 \)
The Scaling Laws

- **Hyperscaling laws:**
  The six main critical exponents $\alpha, \beta, \gamma, \delta, \eta, \nu$ are not independent, because of 4 (hyper)scaling laws

\[
\alpha + 2\beta + \gamma = 2 \quad 2 - \alpha = \beta(\delta - 1) \\
(2 - \eta)\nu = \gamma \quad d\nu = 2 - \alpha
\]

The laws were first derived as inequalities from thermodynamics.

- For $d > 4$ the critical exponents assume the mean field values (for the Landau-Ginzburg model also at $d = 4$)

\[
\gamma = 1, \ \nu = \frac{1}{2}, \ \eta = 0, \ \alpha = 0, \ \beta = \frac{1}{2}, \ \delta = 3
\]

the hyperscaling relation $2 - \alpha = d\nu$ does not hold for $d \geq 5$. 

IGS-Block course, LI, Oct 2006, J. Engels – p.27/??
The power laws are called scaling laws, because they are scale-free.

Homogeneous functions of one variable satisfy

\[ f(bx) = g(b)f(x) \]

\[ \implies g(b) = \lambda^p, \quad p = \text{the degree of homogeneity} \]

Power law functions are homogeneous:

\[ f(x) = Ax^\theta, \quad f(bx) = A(bx)^\theta = b^\theta f(x) \]

\[ \implies \text{a change in the units of } x \text{ can be compensated by a change in the units of } f(x) - \text{the function looks on all scales alike} - \text{that is scale freedom.} \]
Remarkable: Systems with physically different kinds of continuous transitions can be assigned to one Universality Class (UC).

Systems with short range, isotropic couplings with the same space dimension $d$, and same symmetry and dimension $N$ of the order parameter belong usually to the same UC.

Members of the same UC have:

- different values of $T_c$, but
- the same critical exponents
- different critical amplitudes, but universal amplitude ratios / combinations, as e.g.

$$C^+/C^-, \ A^+/A^-, \ R_X = C^+ D_c B^{\delta-1}$$

- the same scaling functions, e.g. the coexistence curve $\rho/\rho_c(T/T_c)$ for fluids:

Guggenheim 1945
Universality Classes

Historically:

The equations between the critical exponents, and the scaling laws are satisfied automatically, if
\[ \hat{f} = \beta f(t, H) \quad \text{and} \quad G(r, t) \]
are generalized homogeneous functions of their arguments

\[ \hat{f}(b^p x, b^q y) = b^\hat{f}(x, y) \]

Equation of state \( H = M^\delta \psi_H(t/M^{1/\beta}) \) Widom(1965)

Correl. function \( G(r, t) = \psi_G(r/\xi)/r^{d-2+\eta} \) Kadanoff(1966)

Here, \( \psi_H \) and \( \psi_G \) are universal apart from metric factors.

\[ \Rightarrow \quad \text{A theory, which explains the fact that all different members of a UC share its properties, must identify the important or relevant parameters of a system.} \]

Idea of Renormalization Group Theory: \textit{Systematic reduction of the number of degrees of freedom by successive scale transformations}
Important Models

The universality classes of these models cover many important systems of Statistical Physics and Particle Physics. The knowledge of the universal parameters of these simple systems serves to classify phase transitions.

In the rest of the lecture we include a factor $\beta = 1/T$ in $f$, $\mathcal{H}$ and therefore in the couplings, e.g. in the magnetic field $\vec{H}$, that is we use reduced variables. However, we use the old names

$$\beta \mathcal{H} \rightarrow \mathcal{H}, \quad \hat{f} = \beta f \rightarrow f, \quad \beta \vec{H} \rightarrow \vec{H}$$

Due to $\vec{H}$-rescaling the susceptibility looses a factor, whereas the magnetization remains the same

$$M_a = -\frac{\partial f}{\partial H_a}, \quad \chi_{ab} = \frac{\partial M_a}{\partial H_b}$$

$$M_a^{\text{new}} = M_a^{\text{old}}, \quad \chi_{ab}^{\text{new}} = \chi_{ab}^{\text{old}} / \beta = \int d^d x G_{ab}(\vec{x})$$
Lattice Models

Lattice models: Spin variables $\vec{s}_i = \vec{\phi}(\vec{x}_i)$ are defined on the sites $i$ of a regular $d$–dimensional lattice (usually hypercubic).

A. $O(N)$–invariant spin models (non-linear $\sigma$–models)

Heisenberg Hamiltonian

$$\mathcal{H} = -J \sum_{<ij>} \vec{s}_i \cdot \vec{s}_j - \bar{H} \sum_i \vec{s}_i$$

$<ij>$ – summation over nearest neighbour sites,

$\vec{s}_i = N$–component unit vector at site $i$, $\vec{s}_i^2 = 1$,

like $\bar{H}$, $J$ is $\beta$–rescaled and therefore $J \sim 1/T$,

Partition Function

$$Z = \int \prod_i d^N s_i \delta(\vec{s}_i^2 - 1) e^{-\mathcal{H}}$$
Lattice Models

\( N = 1 \): the Ising model, \( s_i = \pm 1 \), scalar spin variables

\( N = 2 \): the XY model, \( \vec{s}_i = (\cos \theta_i, \sin \theta_i) \)

\( N = 3 \): the Heisenberg model (of a real 3d ferromagnet)

\( N = 4 \): the \( O(4) \) model

\( d = 1 \): the Ising model has no phase transition

\( d = 2 \): the Ising model has a 2. order phase transition at 
\( J_c = \ln(1 + \sqrt{2})/2 \), the \( Z(2) \)-symmetry, which breaks at 
\( J_c \) is discrete (model solved by Onsager 1944);

\( N \geq 2 \): no real symmetry breaking phase transition 
(Mermin- Wagner- Colemann theorem): instead a 
Kosterlitz-Thouless transition

\( d = 3 \): 2. order phase transition

\( d \geq 4 \): 2. order phase transition with mean field exponents
Lattice Models

B. The \( q \)-state Potts model

**Hamiltonian**

\[
\mathcal{H} = -J \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j}
\]

\( s_i \) is a scalar with \( q \) states \( \sigma_i = 1, 2, \ldots, q \);
\( \delta \) the Kronecker symbol

**Partition Function**

\[
Z = \prod_i \sum_{\sigma_i = 1}^{q} e^{-\mathcal{H}}
\]

\( d = 2 \): 2. order phase transitions for \( q \leq 4 \), 1. order for \( q \geq 5 \)

\( d = 3 \): 2. order phase transitions for \( q = 2 \), 1. order for \( q \geq 3 \)
Lattice Models

C. More general Hamiltonians with spins $\vec{\phi}_i$ of unbound length

$$\mathcal{H} = -J \sum_{<ij>} \vec{\phi}_i \vec{\phi}_j + \sum_i V(\vec{\phi}_i^2) - \bar{H} \sum_i \vec{\phi}_i$$

where $\int_{-\infty}^{\infty} e^{bx^2-V(x)} dx < \infty$ for all real $b$

Special case: the $\phi^4$–Hamiltonian

$$\mathcal{H} = -J \sum_{<ij>} \vec{\phi}_i \vec{\phi}_j + \sum_i [\lambda(\vec{\phi}_i^2 - 1)^2 + \vec{\phi}_i^2] - \bar{H} \sum_i \vec{\phi}_i$$

It is the lattice discretization of the Landau-Ginzburg-Wilson Hamiltonian of continuum theory.
C. The Landau-Ginzburg-Wilson Hamiltonian

\[ \mathcal{H} = \int d^d x \left\{ \frac{1}{2} |\nabla \vec{\phi}(x)|^2 + \frac{r}{2} \vec{\phi}(x)^2 + \frac{u}{4!} (\vec{\phi}(x)^2)^2 - H \vec{\phi}(x) \right\} \]

Translation from lattice

\[ \vec{\phi}_i \rightarrow \vec{\phi}(x) J^{-1/2} , \quad r = \frac{2 - 4d}{J} - 2d , \quad u = \frac{4! \lambda}{J^2} , \quad \bar{H} \rightarrow \bar{H} J^{1/2} \]

Partition Function: \[ Z = \int [d^N \phi(x)] e^{-\mathcal{H}} \]

The Gaussian Model is a special case of the LGW-Hamiltonian, where \( u = 0 \) \( \implies \) the components of \( \vec{\phi}(x) \) do not interact, it is sufficient to consider a single component, the theory corresponds to a free, Euclidean theory of a particle with mass \( m = r^{1/2} \).


