Uniqueness of distributional solutions to the 2D vorticity Navier–Stokes equation and its associated nonlinear Markov process

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Abstract

In this work we prove uniqueness of distributional solutions to $2D$ Navier–Stokes equations in vorticity form $u_t - \nu\Delta u + \mathrm{div}(K(u)u) = 0$ on $(0,\infty) \times \mathbb{R}^2$ with Radon measures as initial data, where K is the Biot–Savart operator in 2-D. As a consequence, one gets the uniqueness of probabilistically weak solutions to the corresponding McKean–Vlasov stochastic differential equations. It is also proved that for initial conditions with density in L^4 these solutions are strong, so can be written as a functional of the Wiener process, and that pathwise uniqueness holds in the class of weak solutions, whose time marginal law densities are in $L^{\frac{4}{3}}$ in space-time. In particular, one derives a stochastic representation of the vorticity u of the fluid flow in terms of a solution to the McKean–Vlasov SDE. Finally, it is proved that the family $\mathbb{P}_{s,\zeta}$, $s \geq 0$, ζ =probability measure on $\mathbb{R}^d$, of path laws of the solutions to the McKean–Vlasov SDE, started with ζ at s, form a nonlinear Markov process in the sense of McKean.

MSC: 60H15, 47H05, 47J05.
Keywords: Navier–Stokes equation, vorticity, McKean–Vlasov equation, distributional solution.

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1 Introduction

Consider here the 2-D incompressible Navier–Stokes equation

$$\begin{aligned} &y_t - \nu\Delta y + (y\cdot\nabla)y = \nabla p \quad \text{in } (0,\infty)\times\mathbb{R}^2,\\ &\nabla\cdot y = 0 \qquad\qquad\qquad\quad\ \ \text{in } (0,\infty)\times\mathbb{R}^2,\\ &y(0,x) = y_0(x),\ x\in\mathbb{R}^2. \end{aligned} \tag{1.1}$$

Let $u = u(t,x)$ denote the vorticity of the velocity field $y = \{y_1, y_2\}$, that is,

$$u(t,x) = \operatorname{curl} y(t,x) = D_1y_2(t,x) - D_2y_1(t,x),\ (t,x)\in(0,\infty)\times\mathbb{R}^2,$$

where $D_j = \frac{\partial}{\partial x_j}$, $j = 1,2$, and the symbol $\nabla, \operatorname{div}$ refer to spatial derivatives.

Equation (1.1) can then be rewritten as the *vorticity equation*

$$\begin{aligned} &u_t - \nu\Delta u + \operatorname{div}(yu) = 0 \qquad \text{in } (0,\infty)\times\mathbb{R}^2,\\ &u(0,x) = u_0(x) = \operatorname{curl} y_0(x),\quad x\in\mathbb{R}^2. \end{aligned} \tag{1.2}$$

Here, the velocity field $y(t,x)$ is given by the Biot–Savart formula

$$y(t,x) = (\nabla^\perp E * u(t))(x),\ \forall (t,x)\in(0,\infty)\times\mathbb{R}^2, \tag{1.3}$$

where

$$E(x) = \frac{1}{2\pi}\ \ln|x|,\ x\in\mathbb{R}^2,$$

hence

$$\nabla^\perp E(x) = \frac{(-x_2, x_1)}{2\pi|x|^2},\ x = (x_1,x_2)\in\mathbb{R}^2\setminus\{0\},$$

which is the Biot–Savart kernel. We set

$$K(z) = \nabla^\perp E * z,\ z\in L^p(\mathbb{R}^2),\ p\in(1,2) \tag{1.4}$$

and note (see, e.g., [21], Lemma 2.2) that by the generalized Young inequality

$$|K(z)|_{L^q(\mathbb{R}^2)} \le C|z|_{L^p(\mathbb{R}^2)},\ \forall z\in L^p(\mathbb{R}^d), \tag{1.5}$$

and, if $y\in L^p(\mathbb{R}^2,\mathbb{R}^2)$ with $\operatorname{div} y = 0$ and $u = \operatorname{curl} y\in L^q$, that we have

$$y = K(u), \tag{1.6}$$

where $p\in(1,2)$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$. Then, we may write (1.2) as

$$\begin{aligned}&u_t - \nu\Delta u + \operatorname{div}(uK(u)) = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^2,\\ &u(0,x) = u_0(x), \qquad\qquad\qquad x \in \mathbb{R}^2.\end{aligned} \tag{1.7}$$

This is a special case of a so called *generalized mean-field Fokker–Planck equation* with locally integrable singular kernel K usually derived from Riesz potentials. Besides this kernel, other singular potentials of this form arise in chemotaxis mathematical models and in some classes of mean field equations (see [18]–[23], [27]).

There is an extensive literature on the well-posedness of the vorticity equation (1.7) and, implicitly, on the Navier–Stokes equation (1.1) in the spaces $L^p((0,\infty)\times\mathbb{R}^2)$ (see, e.g., [6], [13], [21], [24]) and in $(BMO)^{-1}(\mathbb{R}^2)$, (see [25]).

Our main objective here is the relationship of (1.7) with the McKean–Vlasov stochastic differential equation

$$\begin{aligned}&dX(t) = K(u(t,\cdot))(X(t))dt + \sqrt{2\nu}\, dW(t),\ t \geq 0,\\ &X(0) = X_0,\end{aligned} \tag{1.8}$$

on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with the normal filtration $(\mathcal{F}_t)_{t\geq 0}$ and 2-D $(\mathcal{F}_t)$-Brownian motion $W(t)$, where

$$u(t,x)dx = \mathbb{P}\circ X(t)^{-1}(dx);\ t>0,\ u_0(dx) = \mathbb{P}\circ X_0^{-1}(dx). \tag{1.9}$$

SDE (1.8) describes the microscopic dynamics of the vorticity flow $u = u(t,x)$ and, implicitly, of the Navier–Stokes velocity field $y(t) = K(u(t))$. Whereas weak existence of solutions to (1.8) is a consequence by the above mentioned existence results for (1.7) (see, e.g., Theorem 2.1 below) and the general technique from Section 2.2 in [2] (see Theorem 4.1 below), weak uniqueness results for (1.8) appear to be less known.

Another open question is whether the path laws of the solutions to (1.8) form a nonlinear Markov process in the sense of McKean (see [26] and the recent paper [29]). It turns out that to solve both problems, weak uniqueness for (1.8) and the question whether the laws of its solutions lead to a Markov process, require a new uniqueness result for (1.7), namely uniqueness in the most general class of solutions for (1.7), the so-called distributional solutions (see (2.2) and (2.3) below).

So, the first main result (which is purely analytic) of this paper is such a uniqueness result in the class of distributional solutions to (1.7) with measure initial data, formulated as Theorem 2.4 (under the restrictions (2.19)–(2.21)) and as Corollary 3.1, where the latter is devoted to the case of probability measures as initial data. Furthermore, we also prove distributional uniqueness for the linearized equation corresponding to (1.7) (see Theorem 4.2 and Corollary 4.3, which is crucial for the subsequent probabilistic applications). As a consequence of both, we prove the existence of probabilistically weak solutions to (1.8) (see Theorem 4.4). For initial conditions which are probability measures with densities in L^4 we prove that these solutions are in fact strong and that pathwise uniqueness holds in the class of all solutions whose path laws have time marginals densities in $L^{\frac{4}{3}}$ in space-time (see Theorem 4.5). Our last main result, Theorem 4.6, guarantees that the family of the path laws $\mathbb{P}_{s,\zeta}$ of the solutions to (1.8), started at time $s \geq 0$ with probability measure ζ on $\mathbb{R}^d$, form a nonlinear Markov process in the sense of McKean [26] (see Definition 4.7 and also [29]). We would like to stress that all these results are heavily depending on our uniqueness result in Theorem 2.4, and the fact that it gives uniqueness in the class of distributional solutions. Uniqueness in smaller classes as, e.g., mild solutions (see [21] or the more general results in [22]) does not suffice. We refer, e.g., to the proof of Theorem 4.4, where this becomes obvious.

As a by-product of our existence result, Theorem 4.1, for (1.8), we get a a probabilistic representation of the solutions to (1.7) as time-marginal law densities of the nonlinear Markov process gotten from the paths laws of the solutions to (1.8). Thus, McKean's general programme, already envisioned in [26], is completed in this paper for the $2D$ vorticity Navier–Stokes equation (1.7).

For the existence theory for nonlinear Fokker–Planck equations with Nemytski-type drift term and their implications to McKean–Vlasov SDEs, we refer to [2]–[4]. As regards the literature on the stochastic representation of solutions to Navier–Stokes equations, the works [10], [11], [15], [18] should be primarily cited. In particular, in [10] one gets the probabilistic representation of solutions to the vorticity equation (1.7) as

$$u(t,x) = \mathbb{E}[u_0(X^{t,x}(t))],\ (t,x) \in (0,\infty) \times \mathbb{R}^d,$$

where $X^{t,x}$ is the stochastic flow map defined by the equation

$$\begin{aligned} dX^{t,x}(s) &= -v(t-s, X^{t,x}(s))ds + \sqrt{2\nu}\, dW_s, \\ X^{t,x}(0) &= x, \\ v(t,x) &= -\frac{1}{2}\int_0^\infty \frac{1}{s}\,\mathbb{E}[u(t,x+W(s))W_s^\perp]ds, \end{aligned}$$

and $W(t)=(W^1(t),W^2(t)))$ is a $2-D$ Brownian motion and $W^\perp(t)=(W^2(t),W^1(t))$. This representation formula is extended later on in [11] to $3-D$ equations of the form (1.1). In the present paper, however, we use a different approach which takes advantage of the interpretation of the vorticity equation (1.7) as a nonlinear Fokker–Planck equation, which is associated with a McKean–Vlasov SDE by virtue of the superposition principle (see [2] and see also [31] for the case of usual SDEs).

Finally, let us comment on probabilistic approaches to (1.8) and also (1.7). We start with pointing out the recent paper [23] which contains a substantial discussion on the related literature, and we also refer to the references therein. Furthermore, we would like to mention reference [32] where also weak existence of solutions to (1.8) for every probability measure on $\mathbb{R}^2$ as initial data was proved, however, by a completely different method, not employing the nonlinear superposition principle. Furthermore, in [23] (see Theorem 6.3) it is proved that there exists a unique strong solution to (1.8) with initial data in certain Besov spaces (which include $L^{1+\varepsilon}$ data). So, this result is quite different from ours and again the methods are completely different from those in our present paper. In addition, uniqueness of smooth solutions to (1.7) under additional restrictions on the behaviour at $t=0$ is proved (see [23, Theorem 6.1]). A further related paper should be mentioned, namely [33], in which a very nice theory for existence and uniqueness of soloutions to ordinary SDEs with singular drifts beyond the Ladyzhenskaya–Prodi–Serrin condition is developed, which applies to (1.8) after determining and fixing u. However, the general weak uniqueness results in [23] only give uniqueness in a class of martingale solutions obtained by a certain limiting procedure, and the Markov property is only proved for Lebesgue almost all times. So, our result in the present paper on (1.8) giving rise to a nonlinear Markov process (in the sense of McKean, i.e., Theorem 4.6) is much more general.

Notations. $L^p(\mathbb{R}^2)$, $1\le p\le\infty$ (denoted L^p) is the space of all Lebesgue measurable and p-integrable functions on $\mathbb{R}^2$, with the standard norm $|\cdot|_p$. $(\cdot,\cdot)_2$ denotes the inner product in L^2. By $L^p_{\rm loc}$ we denote the corresponding local space. For any open set $\mathcal{O}\subset\mathbb{R}^2$ let $W^{k,p}(\mathcal{O})$, $k\ge1$, denote the standard Sobolev space on $\mathcal{O}$ and by $W^{k,p}_{\rm loc}(\mathcal{O})$ the corresponding local space. We set $W^{1,2}(\mathcal{O})=H^1(\mathcal{O})$, $W^{2,2}(\mathcal{O})=H^2(\mathcal{O})$, $H^1_0(\mathcal{O})=\{u\in H^1(\mathcal{O}),$ $u=0$ on $\partial\mathcal{O}\}$, where $\partial\mathcal{O}$ is the boundary of $\mathcal{O}$. By $H^{-1}=H^{-1}(\mathbb{R}^2)$ we denote the dual space of $H^1(\mathbb{R}^d)$. $C^\infty_0(\mathcal{O})$ is the space of infinitely differentiable real-valued functions with compact support in $\mathcal{O}$ and $\mathcal{D}'(\mathcal{O})$ is the dual of $C_0^\infty(\mathcal{O})$, that is, the space of Schwartz distributions on $\mathcal{O}$. By $C_b(\mathbb{R})$, we denote the space of continuous and bounded functions on $\mathbb{R}^2$. We shall denote by $\mathcal{M}(\mathbb{R}^2)$ the space of all finite Radon measures on $\mathbb{R}^2$. Given a Banach space $\mathcal{X}$ and $0 < T \le \infty$, we denote by $C([0,T];\mathcal{X})$ the space of all continuous $\mathcal{X}$-valued functions on $[0,T]$. For $1 \le p \le \infty$, we shall denote by $L^p(0,T;\mathcal{X})$ the space of $\mathcal{X}$-valued, L^p-Bochner integrable functions on $(0,T)$. By $C_0^\infty([0,\infty)\times\mathbb{R}^2)$ we denote the space $\{y \in C^\infty([0,\infty);\mathbb{R}^2)$; y with compact support in $[0,\infty)\}$. For $1 < p < \infty$, let $L^{p,\infty}(\mathbb{R}^2)$ denote the Lorentz space of all measurable functions $y : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$|y|_{L^{p,\infty}} := \sup_{\lambda>0}\{\lambda^p \text{ meas}(x \in \mathbb{R}^2;\ |y(x)| > \lambda)\}^{\frac{1}{p}} < \infty.$$

Throughout this work, $\nabla = \text{grad}$ refers only to the spatial derivatives, i.e., in the x-variables and

$$\nabla \cdot y = \operatorname{div} y,\ \forall y \in (L^p(\mathbb{R}^2))^2,\ 1 \le p \le \infty.$$

If X_1, X_2 are two Banach spaces, we shall denote by $L(X_1, X_2)$ the space of linear continuous operators from X_1 to X_2. By $\mathcal{P}$ we denote the set of all probability measures on $\mathbb{R}^2$ and set

$$\mathcal{P}^a = \left\{ y \in L^1(\mathbb{R}^2); y \ge 0, \text{ a.e. in } \mathbb{R}^2; \int_{\mathbb{R}^2} y(x)dx = 1 \right\}. \tag{1.10}$$

2 The existence and uniqueness for the vorticity equation (1.7)

A function $u \in L^{r_1}_{\text{loc}}(0,\infty;L^{r_2}),\ r_1, r_2 \ge 1$, is called a *mild solution to* (1.7) if it is a solution to the integral equation

$$u(t) = e^{\nu t\Delta}u_0 - \operatorname{div}\int_0^t e^{\nu(t-s)\Delta}(K(u(s))u(s))dx,\ t > 0, \tag{2.1}$$

where $e^{t\Delta}$ is the heat semigroup in $\mathbb{R}^2$, which is well defined on all L^p, $1 \le p \le \infty$. (If $u_0 \in \mathcal{M}(\mathbb{R}^2)$, then $\|e^{t\Delta}u_0\|_{L^p} \le Ct^{-1+\frac{1}{p}}\|u_0\|_{\mathcal{M}}$, $\forall t > 0$, for all $p > 1$.)

This definition extends to mild solutions y to (1.1) via the Biot–Savart formula (1.6).

Given $u_0 \in \mathcal{M}(\mathbb{R}^2)$, a function $u \in L^1_{\rm loc}(0,\infty;L^p)$ for some $p \in (1,2)$ is said to be a distributional solution to (1.7) if

$$K(u)u \in L^1_{\rm loc}((0,\infty)\times\mathbb{R}^2), \tag{2.2}$$

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}^2} u(t,x)(\varphi_t(t,x)+\nu\Delta\varphi(t,x)+K(u(t,x))\cdot\nabla\varphi(t,x))dtdx\\ +\int_{\mathbb{R}^2}\varphi(0,x)u_0(dx)=0,\ \forall\varphi\in C_0^\infty([0,\infty)\times\mathbb{R}^2).\end{aligned} \tag{2.3}$$

The next existence theorem is due to Y. Giga, T. Miyakawa & H. Osada (see [21], Theorem 4.2).

Theorem 2.1 *Assume that $u_0 = \operatorname{curl} y_0 \in \mathcal{M}(\mathbb{R}^2)$, where $y_0 \in L^{2,\infty}(\mathbb{R}^2)$, $\nabla\cdot y_0 = 0$ on $\mathbb{R}^2$. Then, equation* (1.7) *has a solution $u : [0,\infty) \to \mathcal{M}(\mathbb{R}^2)$, which is bounded and continuous in the weak topology. Moreover, one has the estimate*

$$|u(t)|_r \le C_r\, t^{-1+\frac{1}{r}},\quad \forall\, t>0,\ 1<r<\infty, \tag{2.4}$$

and $u(t) = \operatorname{curl} y(t)$, where y is a mild solution to the Navier–Stokes equation (1.1) *and*

$$|y(t)|_r \le C_r\, t^{\frac{1}{r}-\frac{1}{2}},\quad \forall\, t>0, 2<r\le\infty, \tag{2.5}$$

$$\sup\{|D^k_x D^j_t u(t)|_\infty;\ t\in[\varepsilon,T]\} \le C^T_{\varepsilon,k,j}, \tag{2.6}$$

for all $0<\varepsilon<T<\infty$ and all $k,j = 0,1,\dots$

Remark 2.2 According to Theorem 1.2 in [20], Theorem 2.1 extends to all initial conditions $u_0 \in \mathcal{M}(\mathbb{R}^2)$.

We set

$$\begin{aligned}k(x) = (k^1(x),\ k^2(x)) := \nabla^\perp E(x) = \frac{(-x_2,x_1)}{2\pi|x|^2},\\ x\in(x_1,x_2)\in\mathbb{R}^2\setminus\{0\},\end{aligned} \tag{2.7}$$

and

$$K^i(u) := k^i * u,\ u\in L^p,\ p\in(1,2),\ i=1,2.$$

We have

$$\nabla k(x) = (\partial_j k^i)_{1\le i,j\le 2} = \frac{1}{2\pi|x|^2}\begin{pmatrix} \dfrac{2x_1x_2}{|x|^2} & -\left(1-\dfrac{2x_2^2}{|x|^2}\right)\\ \left(1-\dfrac{2x_1^2}{|x|^2}\right) & -\dfrac{2x_1x_2}{|x|^2}\end{pmatrix},$$

and hence, all four components $\partial_j k^i$ of $\nabla k(x)$ define kernels of degree (-2) satisfying all assumptions of Theorem 1 in [12], which implies that, for all $p \in (1,\infty)$; $i,j \in \{1,2\}$,

$$\partial_j K^i(u)(x) := \lim_{\varepsilon\to 0+} \int_{|x-y|\geq\varepsilon} \partial_j k^i(x-y)u(y)dy,\ x\in\mathbb{R}^2,\ u\in L^p, \tag{2.8}$$

defines bounded, linear operators $\partial_i K^j : L^p \to L^p$. The limit in (2.8) is meant in L^p as well as a.e. Moreover, the p-norm of the right hand side of (2.8) with $\sup\limits_{\varepsilon\in(0,1]}$ replacing $\lim\limits_{\varepsilon\to 0+}$ is up to a constant (only depending on p) dominated by $|u|_p$. On the other hand, it is elementary to check and well known that for the distributional derivatives $\partial_j(K^i(u))$ of $K^i(u)$, $i=1,2$, $u\in L^p$, $p\in(1,2)$,

$$\partial_j(K^i(u)) = \partial_j K^i(u) + \frac{1}{2}\,\mathrm{sign}(i-j)u \tag{2.9}$$

for some numerical constant $c>0$. In particular, we have

$$\mathrm{div}(K(u)) = 0 \text{ and } \mathrm{curl}(K(u)) = u,\ \forall u\in L^p,\ p\in(1,2), \tag{2.10}$$

where div and curl are taken in the sense of Schwartz distributions. Furthermore, together with the fact that the operators in (2.8) are bounded on every L^p, $p\in(1,\infty)$, (2.9) implies that

$$|\nabla K(u)|_p \leq c_p |u|_p,\ \forall u\in L^p\cap\left(\bigcup_{q\in(1,2)} L^q\right),\ p\in(1,\infty). \tag{2.11}$$

Theorem 2.1 can be complemented as follows.

Theorem 2.3 *Let u be the solution of* (1.7) *from Theorem* 2.1 *and let* $T>0$. *Then u is a distributional solution to equation* (1.7) *and*

(i)
$$\int_0^T \int_{\mathbb{R}^d} |u(t,x)|\,|K(u(t,\cdot))(x)|dx\,dt < \infty. \tag{2.12}$$

(ii) *If $u_0\in\mathcal{P}$, then*
$$u(t)\in\mathcal{P}^a,\ \forall t>0. \tag{2.13}$$

(iii) *Assume that* $u_0 \in L^q$ *for some* $q \in (1,2)$. *Then, for all* $p \in [1,\infty]$ *we have* $u \in L^\infty(0,T;L^p)$ *and*

$$|u|_{L^\infty(0,T;L^p)} \le c|u_0|_p, \tag{2.14}$$

where c *only depends on* p.

Therefore, $u \in L^4(0,T;L^{\frac{4}{3}} \cap L^4) \cap L^\infty(0,T;H^{-1})$ *for* $u_0 \in L^{\frac{4}{3}} \cap L^4$.

Proof. (i): Let $p \in (1,2)$, $q := \frac{2p}{2-p}$ and $q' := \frac{q}{q-1}$. Then, by (1.5),

$$\int_0^T \int_{\mathbb{R}^d} |u(t,x)|\,|K(u(t,x))|dx\,dt \le C \int_0^T |u(t)|_{q'} |u(t)|_p dt.$$

But by (2.4) the last integral is finite. In particular, (2.2) holds, hence clearly u is a distributional solution to (1.7).

(ii): As seen in the proof of Theorem 4.2 in [21], the solution u to (1.7) given by Theorem 2.1 can be obtained by

$$u = \lim_{u\to\infty} u_n \text{ uniformly on compact sets of } (0,\infty)\times\mathbb{R}^2 \tag{2.15}$$

where u_n are the unique global smooth solutions to (1.7) with $u_n(\cdot,0) = u_0^n$ and $\{u_0^n\}$ is a smooth approximation of the initial data $u_0 \in \mathcal{M}(\mathbb{R}^2)$ in the narrow topology. Moreover, u_n is expressed as (see, e.g., (2.1) in [21])

$$u_n(t,x) = \int_{\mathbb{R}^2} \Gamma_{K(u_n)}(t,x;0,\xi) u_0^n(\xi) d\xi, \tag{2.16}$$

where $\Gamma_v \equiv \Gamma_v(t,x;s,\xi)$ is the fundamental solution to the linear parabolic operator

$$L_v(u) = u_t - \nu\Delta u + (u\cdot\nabla)v,\ (t,x) \in (0,\infty)\times\mathbb{R}^2.$$

We have

$$\begin{aligned} &\Gamma_v \ge 0;\ \int_{\mathbb{R}^2} \Gamma_v(t,x;s,\xi)d\xi = 1,\ 0 \le s \le t < \infty,\ x \in \mathbb{R}^2, \\ &\lim_{t\downarrow s} \int_{\mathbb{R}^2} \Gamma_v(t,x;s,\xi) f(\xi) d\xi = f(x),\ \forall f \in C_b(\mathbb{R}^2). \end{aligned} \tag{2.17}$$

If $u_0 \in \mathcal{P}$, then the sequence $\{u_0^n\}$ can be chosen in such a way that $u_0^n \geq 0$ and $u_0^n \to u_0$ in $\mathcal{M}(\mathbb{R}^2)$ narrowly as $n \to \infty$. Then, by (2.15)–(2.17) it follows that $u \geq 0$. Furthermore, by Theorem 1.2 in [20] we know that

$$\int_{\mathbb{R}^2} u(t,x)dx = \int_{\mathbb{R}^2} u_0(x)dx = 1,\ \forall t \in (0,T),$$

as claimed.

(iii) Since $q \in (1,2)$, by (2.10) we know that $u = \mathrm{curl}(K(u))$. Hence, we may apply Theorem 4.3 in [21] to obtain the following representation

$$u(x,t) = \int_{\mathbb{R}^2} \Gamma(t,x;0,\xi)u_0(x)d\xi,\ t \geq 0,\ x \in \mathbb{R}^d, \tag{2.18}$$

where $\Gamma(t,x;s,\xi),\ x,\xi \in \mathbb{R}^2,\ t > s \geq 0$, is a positive continuous function, which satisfies

$$\begin{aligned}
&\int_{\mathbb{R}^2} \Gamma(t,x;s,\xi)d\xi = \int_{\mathbb{R}^2} \Gamma(t,x;s,\xi)dx = 1,\\
&\Gamma(t,x;s,\xi) \simeq \frac{1}{(t-s)}\ e^{-\frac{C|x-\xi|^2}{(t-s)}},\ t > s \geq 0.
\end{aligned}$$

By the Young inequality $\forall 1 \leq p \leq \infty$, if $u_0 \in L^p$, this yields

$$\begin{aligned}
|u(t)|_p &= \left| \int \Gamma(t,x;0,\xi)u_0(\xi)d\xi \right|_p\\
&\leq C \left| \int \frac{1}{t-s}\ e^{-\frac{C|x-\xi|^2}{t-s}}\ u_0(\xi)d\xi \right|_p\\
&\leq C \left| \int \frac{1}{t}\ e^{-\frac{C|x|^2}{t}}\ dx \right| |u_0|_p\\
&\leq C|u_0|_p.
\end{aligned}$$

Thus, $u \in L^\infty(0,T;L^p)$, and $|u|_{L^\infty(0,T;L^p)} \leq c|u_0|_p$, as claimed. □

Theorem 2.1 is completed in [21] by a uniqueness theorem for (1.1) and implicitly for (1.7), in the class of mild solutions with sufficiently small atomic part $(u_0)_{pp}$ of the Radon measure u_0. Such a uniqueness result was extended in [20] to $u_0 \in \mathcal{M}(\mathbb{R}^2)$ in the class of mild solutions $u \in C((0,T];L^1 \cap L^\infty) \cap L^\infty(0,T;L^1)$.

We shall prove here for the purposes of the McKean–Vlasov equation (1.8) a sharper uniqueness result, namely within the much larger class of all distributional solutions to (1.7), in the sense of (2.2), (2.3), which belong to the class $\{u\in L^4(0,T;\, L^4\cap L^{\frac43});\ \forall T>0\}$. More precisely, we prove

Theorem 2.4 *Let u_1, u_2 be two distributional solutions to* (1.7) *such that*

$$u_1, u_2 \in L^4(0,T; L^4\cap L^{\frac43}),\ \forall T>0, \tag{2.19}$$

$$u_1-u_2\in L^\infty(0,T;H^{-1}), \tag{2.20}$$

$$u_1(t)-u_2(t)\in L^1 \text{ for a.e. } t\in(0,T), \tag{2.21}$$

$$\lim_{t\downarrow 0}\ \operatorname*{ess\,sup}_{s\in(0,t)}\int_{\mathbb{R}^2}(u_1(s,x)-u_2(s,x))\varphi(x)dx=0,\ \forall\varphi\in C_0^\infty(\mathbb{R}^2). \tag{2.22}$$

Then, $u_1\equiv u_2$.

We note that the uniqueness class considered here is larger than that covered by [20] and the method of proof is different. In fact, in the case of nonlinear Fokker–Planck equations with Nemytski-type drift term, a result of this type was proved in [5] in the class of $L^2((0,T)\times\mathbb{R}^2)$ distributional solutions u_1,u_2 such that $u_1-u_2\in L^\infty(0,T;H^{-1})$, $\forall T>0$, but there is not a large overlap. Though the idea of the proof is borrowed from [5], the argument used here requires sharp estimates specific to the drift term $\operatorname{div}(uK(u))$. It should be also mentioned that the uniqueness condition (2.22) does not exclude the class of solutions with measure initial data $u_0\in\mathcal{M}(\mathbb{R}^2)$. In fact, such a condition agrees with Theorem 2.1 which provides a distributional solution $u:[0,\infty)\to\mathcal{M}(\mathbb{R}^2)$, which is weakly continuous. We also note that, as in the proof of (2.12), by (2.19) it follows that $u_iK(u_i)\in L^1_{\rm loc}((0,\infty)\times\mathbb{R}^d)$, $i=1,2$, which is a condition required by (2.2).

3 Proof of Theorem 2.4

We set $z=u_1-u_2$. Then, we have by (1.7)

$$z_t-\nu\Delta z+\operatorname{div}(K(z)u_1-zK(u_2))=0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^2), \tag{3.1}$$

where K is the Biot–Savart operator (1.4).

We also recall that in 2-D we have

$$|w|_4^2\le 2|w|_2|\nabla w|_2,\ \forall w\in H^1. \tag{3.2}$$

It follows by (1.5) that, for $u_1, u_2 \in L^4 \cap L^{\frac{4}{3}}$,

$$|u_1 K(z)|_2 \le |K(z)|_4 |u_1|_4 \le C|z|_{\frac{4}{3}} |u_1|_4. \tag{3.3}$$

Similarly, we get the estimate

$$|zK(u_2)|_2 \le C|z|_4 |u_2|_{\frac{4}{3}}, \tag{3.4}$$

for some constant C independent of u_1, u_2. Since for $u \in L^{\frac{4}{3}} \cap L^4$, by interpolating between $L^{\frac{4}{3}}$ and L^4, it follows also that $|u|_2 \le |u|_{\frac{4}{3}}^{\frac{1}{2}} |u|_4^{\frac{1}{2}}$. Therefore, since $u_1, u_2 \in L^2(0,T; L^{\frac{3}{4}} \cap L^4)$,

$$z \in L^2(0,T;L^2) \cap L^\infty(0,T;H^{-1}). \tag{3.5}$$

Consider the operator $\Phi_\varepsilon : L^2 \to L^2$,

$$\Phi_\varepsilon(y) = (\varepsilon I - \Delta)^{-1} y,\ \forall y \in L^2,\ \varepsilon > 0. \tag{3.6}$$

and we note that

$$\Phi_\varepsilon \in L(L^2, H^2) \cap L(H^{-1}, H^1) \cap L(L^2, L^2). \tag{3.7}$$

Then, applying Φ_ε in (3.1), we get

$$\begin{aligned}(\Phi_\varepsilon(z(t)))_t - \nu\Delta\Phi_\varepsilon(z(t)) + \Phi_\varepsilon(\mathrm{div}(K(z(t))u_1(t) - z(t)K(u_2(t)))) = 0 \\ \text{in } \mathcal{D}'((0,T) \times \mathbb{R}^2).\end{aligned} \tag{3.8}$$

Taking into account that, by (3.3), (3.4), and (2.19),

$$K(z)u_1 - zK(u_2) \in L^2(0,T;L^2),$$

it follows by (3.7), (3.8) and (3.5) that

$$\frac{d}{dt}\,\Phi_\varepsilon(z) \in L^2(0,T;L^2) \tag{3.9}$$

and since, by (3.5) and (3.7), $\Phi_\varepsilon(z) \in L^2(0,T;H^2)$, we infer that $\Phi_\varepsilon(z) \in C([0,T];H^1)$. In particular, this implies that there is

$$\lim_{t\to 0} \Phi_\varepsilon(z(t)) = f_\varepsilon \text{ in } H^1. \tag{3.10}$$

Now, we set

$$h_\varepsilon(t) = (\Phi_\varepsilon(z(t)), z(t))_2,\ t \in (0,T), \tag{3.11}$$

and

$$K_\varepsilon(y) = \nabla^\perp \Phi_\varepsilon(y),\ \forall y \in L^2,\ \varepsilon > 0. \tag{3.12}$$

We note that $K_\varepsilon(z) = \nabla^\perp \Phi_\varepsilon(z) \in L^2(0,T;H^1) \cap C([0,T];L^2)$. Taking into account that, by (3.6),

$$\varepsilon\Phi_\varepsilon(z) - \Delta\Phi_\varepsilon(z) = z \text{ on } \mathbb{R}^2, \tag{3.13}$$

it follows by (3.11), (3.12) that

$$h_\varepsilon(t) = |K_\varepsilon(z(t))|_2^2 + \varepsilon|\Phi_\varepsilon(z(t))|_2^2,\ t \in (0,T). \tag{3.14}$$

Since $\Phi_\varepsilon(z) \in C([0,T];H^1)$, by (3.9) we see that h_ε is absolutely continuous on $[0,T]$ and, by (3.8) and (3.13) we have, for $\varepsilon \in (0,1)$,

$$\begin{aligned}
h'_\varepsilon(t) &= 2\left(\frac{d}{dt}\,\Phi_\varepsilon(z(t)), z(t)\right)_2 \\
&= 2(\nu\Delta\Phi_\varepsilon(z(t)) - \Phi_\varepsilon(\mathrm{div}(K(z(t))u_1(t) - z(t)K(u_2(t)))), z(t))_2 \\
&= -2\nu|z(t)|_2^2 + 2\varepsilon\nu(\Phi_\varepsilon(z(t)), z(t))_2 \\
&\quad +2(K(z(t))u_1(t) - z(t)K(u_2(t)), \nabla\Phi_\varepsilon(z(t)))_2 \\
&\le -2\nu|z(t)|_2^2 + 2\varepsilon\nu h_\varepsilon(t) + 2(K(z(t)u_1(t), \nabla\Phi_\varepsilon(z(t))))_2 \\
&\quad +2|z(t)K(u_2(t)) \cdot \nabla\Phi_\varepsilon(z(t))|_1,\ \text{a.e. } t \in (0,T).
\end{aligned} \tag{3.15}$$

On the other hand, we have

$$h_\varepsilon(0+) = \lim_{t\downarrow 0} h_\varepsilon(t) = 0. \tag{3.16}$$

Indeed, we have, for a.e. $t > 0$,

$$\begin{aligned}
0 \le h_\varepsilon(t) \le |\Phi_\varepsilon(z(t)) - f_\varepsilon|_{H^1}\, |z|_{L^\infty(0,T;H^{-1})} + (f_\varepsilon - \varphi, z(t))_2 + (\varphi, z(t))_2, \\
\forall \varphi \in C_0^\infty(\mathbb{R}^2),
\end{aligned}$$

and so, by (3.10) and (2.22) we have

$$h_\varepsilon(0+) = \lim_{t\downarrow 0}\ \operatorname*{ess\,sup}_{s\in(0,t)} h_\varepsilon(s) \le |f_\varepsilon - \varphi|_{H^1} |z|_{L^\infty(0,T;H^{-1})},\ \forall \varphi \in C_0^\infty(\mathbb{R}^2).$$

Since $|f_\varepsilon - \varphi|_{H^1}$ can be chosen sufficiently small for a suitable $\varphi \in C_0^\infty(\mathbb{R}^2)$, by (2.20) we get (3.16), as desired.

Now, taking into account that $|\nabla\Phi_\varepsilon(z(t))|_2^2 = |K_\varepsilon(z(t))|_2^2$, we see by (3.15) that

$$\begin{aligned} h'_\varepsilon(t) + 2\nu|z(t)|_2^2 \le 2\nu\varepsilon h_\varepsilon(t) + 2|K_\varepsilon(z(t))|_2\,|K(z(t))u_1(t)|_2 \\ +2|z(t)K(u_2(t))\cdot\nabla\Phi_\varepsilon(z(t))|_1,\ \text{a.e. } t\in(0,T). \end{aligned} \tag{3.17}$$

Note that

$$\begin{aligned} |K_\varepsilon(z(t))|_2|K(z(t))u_1(t)|_2 \le |K(z(t))|_4|u_1(t)|_4|K_\varepsilon(z(t))|_2, \\ \le |K(z(t))|_4^2 + |u_1(t)|_4^2|K_\varepsilon(z(t))|_2^2, \quad \text{a.e. } t\in(0,T). \end{aligned} \tag{3.18}$$

We also have, by (1.5) and (3.2),

$$\begin{aligned} &|z(t)K(u_2(t))\cdot\nabla\Phi_\varepsilon(z(t))|_1 \le |z(t)|_2|K(u_2(t))|_4|K_\varepsilon(z(t))|_4 \\ &\quad\le C|z(t)|_2|u_2(t)|_{\frac43}|K_\varepsilon(z(t))|_2^{\frac12}|\nabla K_\varepsilon(z(t))|_2^{\frac12} \\ &\quad\le C|z(t)|_2|u_2(t)|_{\frac43}||K_\varepsilon(z(t))|_2^{\frac12}|z(t)|_2^{\frac12} \\ &\quad\le C|z(t)|_2^{\frac32}|u_2(t)|_{\frac43}||K_\varepsilon(z(t))|_2^{\frac12} \\ &\quad\le \frac{\nu}{4}|z(t)|_2^2 + C_\nu|u_2(t)|_{\frac43}^4|K_\varepsilon(z(t))|_2^2,\ \text{a.e. } t\in(0,T). \end{aligned} \tag{3.19}$$

Here, we have used the inequality

$$|\nabla K_\varepsilon(z)|_2 \le C|z|_2,\ \forall z\in L^2\cap L^{\frac43}, \tag{3.20}$$

which follows, since by Lemma A.1 in the Appendix we know that

$$K_\varepsilon(z) = -K(z) + \varepsilon K(\Phi_\varepsilon(z)),$$

and, therefore, by (2.11),

$$\begin{aligned} |\nabla K_\varepsilon(z)|_2 &\le |\nabla K(z)|_2 + \varepsilon|\nabla K(\Phi_\varepsilon(z))|_2 \\ &\le C(|z|_2 + \varepsilon|\Phi_\varepsilon(z)|_2) \\ &\le 2C|z|_2. \end{aligned}$$

(Here and everywhere in the following we have denoted by C several positive constants independent of ε and u_1, u_2.)

Then, substituting (3.18)–(3.19) into (3.17) and recalling that, by (3.14), $|K_\varepsilon(z)|_2^2 \le h_\varepsilon$, yields

$$
\begin{aligned}
&h'_\varepsilon(t)+\nu|z(t)|_2^2 \\
&\quad\le C(\varepsilon h_\varepsilon(t)+|u_1(t)|_4^2|K_\varepsilon(z(t))|_2^2+|K(z(t))|_4^2+|u_2(t)|_{\frac43}^4|K_\varepsilon(z(t))|_2^2) \\
&\quad\le C\left(\left(\varepsilon+|u_1(t)|_4^2+|u_2(t)|_{\frac43}^4\right)h_\varepsilon(t)\right)+C|K(z(t))|_4^2,\ \text{a.e. } t\in(0,T),
\end{aligned}
\tag{3.21}
$$

we get by (3.21) that

$$
\begin{aligned}
&\frac{d}{dt}\left(h_\varepsilon(t)\exp\left(-C\left(\varepsilon t+\int_0^t\left(|u_1(s)|_4^2+|u_2(s)|_{\frac43}^4\right)ds\right)\right)\right) \\
&\quad\le C|K(z(t))|_4^2\exp\left(-C\left(\varepsilon t+\int_0^t\left(|u_1(s)|_4^2+|u_2(s)|_{\frac43}^4\right)ds\right)\right), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{a.e. } t\in(0,T).
\end{aligned}
\tag{3.22}
$$

Since, by (1.5),

$$
|K(z(t))|_4^2\le C(|u_1(t)|_{\frac43}^2+|u_2(t)|_{\frac43}^2),\ \text{a.e. } t\in(0,T),
$$

and so $|K(z)|_4^2\in L^1(0,T)$, we see by (3.16) and (3.22) that

$$
\begin{aligned}
&0\le h_\varepsilon(t) \\
&\le C\int_0^t|K(z(s))|_4^2\exp\left(C\left(\varepsilon(t-s)+\int_s^t\left(|u_1(\tau)|_4^2+|u_2(\tau)|_{\frac43}^4\right)d\tau\right)\right)ds, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\forall\varepsilon>0,\ t\in[0,T].
\end{aligned}
\tag{3.23}
$$

Taking into account that, by (2.19),

$$
|u_1|_4^2+|u_2|_{\frac43}^4\in L^1(0,T),
$$

by (3.14) and (3.23) we have

$$
\sup_{\varepsilon\in(0,1]}\|K_\varepsilon(z)\|_{C([0,T];L^2)}=\sup_{\varepsilon\in(0,1]}\|\nabla^\perp(\varepsilon I-\Delta)^{-1}z\|_{C([0,T];L^2)}<\infty. \tag{3.24}
$$

We set $\theta_\varepsilon(t)=\Phi_\varepsilon(z(t))$ and note that

$$
\varepsilon\theta_\varepsilon(t)-\Delta\theta_\varepsilon(t)=z(t),\ \text{a.e. } t\in(0,T) \tag{3.25}
$$

and, by (3.14), (3.23), it follows that, for all $\varepsilon\in(0,1),\ t\in[0,T]$,

$$\varepsilon|\theta_\varepsilon(t)|_2^2 + |\nabla\theta_\varepsilon(t)|_2^2 = h_\varepsilon(t) \le \sup_{\substack{\varepsilon\in(0,1)\\ t\in[0,T]}} h_\varepsilon(t) \le C < \infty. \tag{3.26}$$

We introduce the space

$$\mathcal{G} := \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^2) : |\nabla u| \in L^2(\mathbb{R}^2)\}$$

equipped with the inner product $(\nabla\cdot, \nabla\cdot)_2$ (see p. 11 in [19]). Then, by [17], we have

($\mathcal{G}$.1) $\mathcal{G} = \left\{T \in \mathcal{D}'(\mathbb{R}^2) : \frac{\partial T}{\partial x_i} \in L^2(\mathbb{R}^2),\ 1 \le i \le 2\right\}.$

($\mathcal{G}$.2) The quotient space

$$\dot{\mathcal{G}} := \mathcal{G}/\{constants\}$$

is a Hilbert space. Furthermore, for every Cauchy sequence $u_n \in \mathcal{G}$, $n \in \mathbb{N}$, there exist $u \in \mathcal{G}$ and $c_n \in \mathbb{R}$ such that $\lim\limits_{n\to\infty} u_n = u$ in $\mathcal{G}$ and $\lim\limits_{n\to\infty}(u_n + c_n) = u$ in L^2_{loc}.

By (3.26) there exist subsequences $\varepsilon_k \in (0, 1]$, $k \in \mathbb{N}$, and $\ell_n \in \mathbb{N}$, $n \in \mathbb{N}$, such that $\varepsilon_k \to 0$, as $k \to \infty$ and for $v_n := \frac{1}{\ell_n}\sum\limits_{k=1}^{\ell_n} \theta_{\varepsilon_k}$, $n \in \mathbb{N}$, we have

$$\nabla v_n \to F \text{ in } L^2((0,T)\times\mathbb{R}^2), \tag{3.27}$$

and, for some Lebesgue zero set $N \subseteq (0, T)$,

$$\nabla v_n(t) \to F(t) \text{ in } L^2(\mathbb{R}^2), \text{ for every } t \in (0,T)\setminus N. \tag{3.28}$$

Below we fix $t \in (0, T) \setminus N$. By ($\mathcal{G}$.2) we know that there exist $\theta(t) \in \mathcal{G}$ and $c_n(t) \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \nabla v_n(t) = \nabla\theta(t) \text{ in } L^2 \tag{3.29}$$

and

$$\lim_{n\to\infty}(v_n(t) + c_n(t)) = \theta(t) \text{ in } L^2_{\mathrm{loc}}. \tag{3.30}$$

Furthermore, by (3.25) we have, for every $n \in \mathbb{N}$,

$$\frac{1}{\ell_n}\sum_{k=1}^{\ell_n} \varepsilon_k\theta_{\varepsilon_k}(t) - \Delta v_n(t) = z(t). \tag{3.31}$$

Hence, taking the limit $n \to \infty$ in $\mathcal{D}'(\mathbb{R}^2)$, by (3.26) we conclude that, for all $t \in (0,T) \setminus N$,

$$-\Delta\theta(t) = z(t), \quad \text{in } \mathcal{D}'(\mathbb{R}^2). \tag{3.32}$$

Moreover, by (3.29)–(3.30) it follows that $\theta(t) \subset W^{1,2}_{\rm loc}(\mathbb{R}^2)$, for all $t \in (0,T) \setminus N$, and by (3.27), (3.28) that $\nabla\theta \in L^2((0,T) \times \mathbb{R}^2)$. Furthermore, by (3.26)

$$|\nabla\theta(t)|_2 \le C < \infty, \quad \text{a.e.} t \in (0,T).$$

This yields

$$\begin{aligned}\lim_{n\to\infty} \int_{[1\le|x|\le2]} |\nabla\theta(t,nx)|dx &= \lim_{n\to\infty} \frac{1}{n^2} \int_{[n\le|y|\le2n]} |\nabla\theta(t,y)|dy \\ &\le \sqrt{3\pi} \lim_{n\to\infty} \frac{1}{n} \left(\int |\nabla\theta(t,y)|^2 dy \right)^{\frac12} = 0, \text{ a.e. } t \in (0,T).\end{aligned}$$

We recall also that $z(t) \in L^1$, a.e. $t \in (0,T)$ by (2.21). Then, by Lemma A.11 in [7], we have

$$\nabla\theta(t) = -\nabla E * z(t), \text{ a.e. } t \in (0,T), \tag{3.33}$$

and this yields

$$\nabla^\perp\theta(t) = -\nabla^\perp E * z(t) = -K(z(t)), \text{ a.e. } t \in (0,T), \tag{3.34}$$

where $E(x) \equiv \frac{1}{2\pi} \ln|x|$. It follows, therefore, by (3.26) that

$$\begin{aligned}K_\varepsilon(z(t)) = \nabla^\perp\theta_\varepsilon(t) \to -\nabla^\perp E * z(t) = -K(z(t)) \text{ weakly in } L^2 \\ \text{for a.e. } t \in (0,T).\end{aligned} \tag{3.35}$$

Now, by (3.24), it follows by the lower semicontinuity of the L^2-norm that

$$|K(z(t))|_2 \le C, \quad \text{a.e. } t \in (0,T),$$

and so $|K(z)|_2 \in L^\infty(0,T)$.

For $0 < \varepsilon' < \varepsilon \le 1$ by the resolvent equation for $(\varepsilon I - \Delta)^{-1}$ and (3.35), we have for a.e. $t \in (0,T)$ and $h \in L^2$, $|h|_2 \le 1$,

$$(h, K_\varepsilon(z(t)))_2 = (h, \nabla^\perp\Phi_{\varepsilon'}(z(t)))_2 + \frac{(\varepsilon'-\varepsilon)}{\varepsilon} (h, \varepsilon(\varepsilon I - \Delta)^{-1}\nabla^\perp\Phi_{\varepsilon'}(z(t)))_2.$$

Hence,

$$
\begin{aligned}
|(h, K_\varepsilon(z(t)))_2| &\le \limsup_{\varepsilon'\to 0} |(h, K_{\varepsilon'}(z(t)))_2| \\
&\quad + \limsup_{\varepsilon'\to 0} |(\varepsilon(\varepsilon I - \Delta)^{-1} h, K_{\varepsilon'}(z(t)))_2| \\
&= |(h, K(z(t)))_2| + |(\varepsilon(\varepsilon I - \Delta)^{-1} h, K(z(t)))_2| \\
&\le 2|K(z(t))|_2 \ \text{ a.e. } t \in (0,T).
\end{aligned}
$$

Therefore,

$$
|K_\varepsilon(z(t))|_2 \le 2|K(z(t))|_2, \quad \text{a.e. } t \in (0,T),\ \forall \varepsilon > 0. \tag{3.36}
$$

We come back to (3.17) and, taking into account (3.2) and (2.11), we obtain that

$$
\begin{aligned}
2|K(z(t))u_1(t)|_2|K_\varepsilon(z(t))|_2 &\le |K(z(t))u_1(t)|_2^2 + |K_\varepsilon(z(t))|_2^2 \\
&\le C|K(z(t))|_2|\nabla K(z(t))|_2|u_1(t)|_4^2 + |K_\varepsilon(z(t))|_2^2 \\
&\le C|K(z(t))|_2|z(t)|_2|u_1(t)|_4^2 + |K_\varepsilon(z(t))|_2^2 \\
&\le \frac{\nu}{4}|z(t)|_2^2 + \frac{C}{\nu}|K(z(t))|_2^2|u_1(t)|_4^4 + |K_\varepsilon(z(t))|_2^2, \text{ a.e. } t \in (0,T).
\end{aligned} \tag{3.37}
$$

Then, substituting (3.19), (3.37), (3.36) in (3.17), we get

$$
\begin{aligned}
h_\varepsilon'(t) + |z(t)|_2^2 \le C(h_\varepsilon(t) + (1 + |u_1(t)|_4^4 + |u_2(t)|_{\frac{4}{3}}^{4})|K(z(t))|_2^2),& \\
\text{a.e. } t \in (0,T),&
\end{aligned} \tag{3.38}
$$

Then, recalling that $|K(z)|_2 \in L^\infty(0,T)$ and that $z, u_1, u_2 \in L^4(0,T; L^{\frac{4}{3}} \cap L^4)$, it follows that the right-hand side of (3.38) is in $L^1(0,T)$. Then, integrating over $(0,t)$ and taking into account (3.16), it follows via Gronwall's lemma that

$$
h_\varepsilon(t) \le C\int_0^t \left(1 + |u_1(s)|_4^4 + |u_2(s)|_{\frac{4}{3}}^4\right) |K(z(s))|_2^2 ds,\ \forall t \in (0,T),
$$

and, therefore, once again by (3.14) we have

$$
|K_\varepsilon(z(t))|_2^2 \le C\int_0^t \left(1 + |u_1(s)|_4^4 + |u_2(s)|_{\frac{4}{3}}^4\right) |K(z(s))|_2^2 ds,\ \text{a.e. } t \in (0,T).
$$

Then, by (3.35) and the weak lower semicontinuity of the L^2-norm, it follows that

$$|K(z(t))|_2^2 \le C \int_0^t \left(1+|u_1(s)|_4^4+|u_2(s)|_{\frac43}^{4}\right)|K(z(s))|_2^2 ds,\ \text{ a.e. } t\in(0,T).$$

The latter implies via Gronwall's lemma that $K(z(t))=0$, for a.e. $t\in(0,T)$. Hence, by (3.34) and (3.32), we have therefore $z\equiv 0$, a.e. on $[0,T]$, as claimed. □

Theorem 2.4 implies the uniqueness of distributional solutions y to (1.7) in the sense of (2.2)–(2.3) satisfying (2.19)–(2.21) with initial data $u_0\in\mathcal{M}(\mathbb{R}^2)$, $u_0\ge 0$. Namely, we have

Corollary 3.1 *Let $u_0\in\mathcal{P}^a$ and let $u_1,u_2\in L^4(0,T;L^4)$ be two nonnegative distributional solutions to* (1.7) *in the sense of* (2.2), (2.3)*, such that $u_1-u_2\in L^\infty(0,T;H^{-1})$. Then $u_1\equiv u_2$.*

Proof. We note first that, if u is such a solution to (1.7), then we have

$$\int_{\mathbb{R}^2} u(t,x)dx=\int_{\mathbb{R}^2} u_0(dx)=1,\quad \text{a.e. } t\in(0,T). \tag{3.39}$$

Indeed, if $u^\varepsilon=u*\rho_\varepsilon$, where $\rho_\varepsilon=\frac{1}{\varepsilon^3}\,\rho\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)$ is a mollifier in $\mathbb{R}^3$, then we have

$$u_t^\varepsilon-\nu\Delta u^\varepsilon+\operatorname{div}((K(u)u)*\rho_\varepsilon)=0 \text{ on } (0,T)\times\mathbb{R}^2,$$

and, integrating over $\mathbb{R}^2$, we get

$$\int_{\mathbb{R}^2} u^\varepsilon(t,x)dx=\int_{\mathbb{R}^2} u_0^\varepsilon dx,\ \forall t\in(0,T),\ \forall\varepsilon>0,$$

which for $\varepsilon\to 0$ yields (3.39), as claimed.

In particular, $u_1,u_2\in L^\infty(0,T;L^1)$, hence by interpolation $u_1,u_2\in L^4(0,T;L^4\cap L^{\frac43})$.

Then, it follows from Lemma 2.3 in [28] that there is a $dt\otimes dx$ version $\widetilde{u}$ of u such that, for $\widetilde{u}(t,dx):=\widetilde{u}(t,x)dx$, $t>0$, and $\widetilde{u}(0,dx):=u_0(dx)$ we have that $t\mapsto\int_{\mathbb{R}^d}\varphi(x)\widetilde{u}(t,dx)$ is continuous on $[0,T]$, for every $\varphi\in C_b(\mathbb{R}^2)$.

If $\widetilde{u}_1,\widetilde{u}_2$ are two such $dt\otimes dx$ versions of u_1,u_2, respectively, we have

$$\begin{aligned}&\lim_{t\to 0}\ \operatorname*{ess\,sup}_{s\in(0,t)}|(u_1(s)-u_2(s),\varphi)_2|\\ &\qquad=\lim_{t\to 0}\ \operatorname*{ess\,sup}_{s\in(0,t)}\left|\int_{\mathbb{R}^2}(\widetilde{u}_1(s,x)-\widetilde{u}_2(s,x))\varphi(x)dx\right|=0,\quad \forall\varphi\in C_0^\infty(\mathbb{R}^2).\end{aligned}$$

Then, by Theorem 2.4 it follows that $u_1\equiv u_2\,dt\otimes dx$ a.e., as claimed. □

Remark 3.2 Following the previous proof, one can get the uniqueness in Theorem 2.4 in the class of solutions $u \in L^{q_1}(0,T;L^{p_1}) \cap L^{q_2}(0,T;L^{p_2})$, where $p_1 \in (2,\infty)$, $p_2 \in (1,2)$ and $\frac{1}{q_1} + \frac{1}{p_1} \leq 1$, $i = 1,2$. This class of solutions seems to be more appropriate if one takes into account estimate (2.4).

4 Existence and weak uniqueness for the McKean–Vlasov equation (1.8)

It is well known (see [2], [3]) that the existence of a distributional narrowly continuous solution u for a Fokker–Planck equation with Nemytskii drift terms implies the existence of a probabilistically weak solution $X(t)$ to the corresponding McKean–Vlasov equation such that

$$u(t,x) = \mathcal{L}_{X(t)}(x),\ \forall\, t > 0 \text{ and } u_0(dx) = \mathbb{P} \circ X(0)^{-1}(dx), \tag{4.1}$$

where $\mathcal{L}_{X(t)}$ is the density of $\mathbb{P} \circ X(t)^{-1}$ w.r.t. Lebesgue measure. This result follows by the nonlinear superposition principle in [2, Section 2] (which in turn is derived from the linear superposition principle in [31]). Applying this to equation (1.8) and, respectively, to the vorticity equation (1.7), which as seen earlier can be viewed as a Fokker–Planck equation with the drift $K(u)$ satisfying $uK(u) \in L^1$, the following existence result for equation (1.8) follows by Theorem 2.3 and [2, Section 2].

Theorem 4.1 *Let $u_0 \in \mathcal{P}$. Then, there is a probabilistically weak solution X to* (1.8) *such that* (4.1) *holds, where u is the mild solution to equation* (1.7)*, from Theorem* 2.1*.* (See also Remark 2.2.)

We recall that the process $X = X(t)$ is called a *probabilistically weak solution* to (1.8) if there is a 2-dimensional $(\mathcal{F}_t)$-Brownian motion $W(t), \geq 0$, on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ such that $X : [0,\infty) \times \Omega \to \mathbb{R}^2$ is progressively measurable, $\mathbb{P}$-a.s. continuous in t and satisfies (1.8), i.e.,

$$dX(t) = K(u(t,\cdot))(X(t))dt + \sqrt{2\nu}\, dW(t),\ t \geq 0, \tag{4.2}$$

with one dimensional time marginal laws $\mathcal{L}_{X(t)} = \mathbb{P} \circ X(t)^{-1} = u(t),\ t \geq 0$.

The process $X(t)$ is called the *probabilistic representation* of the solution u to the vorticity equation (1.7).

In particular, we have by the Biot–Savart formula (1.6) the probabilistic representation

$$y(t) = K(\mathcal{L}_{X(t)}),\ \forall t \geq 0, \tag{4.3}$$

of the solution y to the Navier–Stokes equation (1.1).

We shall discuss now the weak uniqueness of probabilistically weak solutions X to the McKean–Vlasov equation (1.8). To this purpose, we shall prove first the *linearized uniqueness* for equation (1.7). Namely, we have

Theorem 4.2 *Let $u \in L^4(0,T;L^{\frac{4}{3}})$ and let*

$$\begin{gathered} u_1, u_2 \in L^4(0,T;L^4 \cap L^{\frac{4}{3}}),\ u_1 - u_2 \in L^\infty(0,T;H^{-1}), \\ u_1(t) - u_2(t) \in L^1 \textit{ for a.e. } t \in (0,T), \end{gathered} \tag{4.4}$$

such that

$$\lim_{t \downarrow 0} \operatorname*{ess\,sup}_{s \in (0,T)} \int_{\mathbb{R}^2} (u_1(s,x) - u_2(s,x))\varphi(x)dx = 0,\ \forall \varphi \in C_0^\infty(\mathbb{R}^2) \tag{4.5}$$

be two solutions to the equation

$$\begin{gathered} \int_0^\infty \int_{\mathbb{R}^2} (\varphi_t + \nu\Delta) + K(u) \cdot \nabla\varphi)v\,dxdt + \int_{\mathbb{R}^2} \varphi(0,x)u_0(dx) = 0, \\ \forall \varphi \in C_0^2([0,T];\mathbb{R}^2), \end{gathered} \tag{4.6}$$

where $u_0 \in \mathcal{M}(\mathbb{R}^2)$. Then, $u_1 \equiv u_2$.

Proof. The proof is the same as that of Theorem 2.4, but with some simplifications. Namely, we set $z = u_1 - u_2$ and get, by (4.1),

$$\begin{aligned} &z_t - \nu\Delta z + \operatorname{div}(K(u)z) = 0 \text{ in } \mathcal{D}'((0,T) \times \mathbb{R}^2), \\ &z(0) = u_0. \end{aligned} \tag{4.7}$$

If $z_\varepsilon = \Phi_\varepsilon(z)$, we obtain for z_ε the equation

$$(z_\varepsilon)_t - \nu\Delta z_\varepsilon + \Phi_\varepsilon(\operatorname{div}(K(u)z)) = 0 \text{ in } (0,T) \times \mathbb{R}^2$$

and so, arguing as in the proof of Theorem 2.4, we obtain (see (3.14)–(3.15))

$$\begin{aligned} |K_\varepsilon(z)(t)|_2^2 + \int_0^t |z(s))|_2^2 ds &\leq C \int_0^t |K(u(s))|_4^2 |K_\varepsilon(z)|_2^2 ds \\ &\leq C \int_0^t |u(s)|_{\frac{4}{3}}^2 |K_\varepsilon(z)|_2^2 ds,\ \forall t \in [0,T], \end{aligned} \tag{4.8}$$

from which, taking into account (3.36), we get for $\varepsilon \to 0$

$$|K(z(t))|_2^2 \le C \int_0^t (1 + |u(s)|_4^4 + |u(s)|_{\frac{4}{3}}^4) |K(z(s))|_2^2 ds, \text{ a.e. } t \in (0,T),$$

which implies as above that $z \equiv 0$. $\square$

Arguing as in the proof of Corollary 3.1, it follows, by Theorem 4.2, the following uniqueness result.

Corollary 4.3 *Let $u_0 \in \mathcal{P}^a$ and let $u_1, u_2 \in L^4(0,T;L^4)$ be two nonnegative solutions to (4.6) such that $u_1 - u_2 \in L^\infty(0,T;H^{-1})$. Then, $u_1 \equiv u_2$.*

Now, let us prove weak uniqueness for the McKean–Vlasov equation (1.8). (For the definition of weak solutions we refer to Definition 3.1 (a) part (i) in [29].)

Theorem 4.4 *Let $T > 0$ and let $X(t)$, $\widetilde{X}(t)$, $t \ge 0$, on stochastic bases $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathbb{P})$, $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\ge0}, \widetilde{\mathbb{P}})$ respectively, be two probabilistically weak solutions to (1.8) such that, for*

$$u(t,\cdot) = \mathcal{L}_{X(t)}, \quad \widetilde{u}(t,\cdot) = \mathcal{L}_{\widetilde{X}(t)}, \ t > 0,$$

we have

$$u, \widetilde{u} \in L^4(0,T;L^4) \cap L^\infty(0,T;H^{-1}). \tag{4.9}$$

Then X and $\widetilde{X}$ have the same laws, that is,

$$\mathbb{P} \circ X^{-1} = \widetilde{\mathbb{P}} \circ \widetilde{X}^{-1}. \tag{4.10}$$

Proof. By Itô's formula, both u and $\widetilde{u}$ satisfy (2.3) and by our definition of weak solution also (2.2). Hence, by Corollary 3.1, $u \equiv \widetilde{u}$. Furthermore, again by Itô's formula, both $\mathbb{P} \circ X^{-1}$ and $\widetilde{\mathbb{P}} \circ \widetilde{X}^{-1}$ satisfy the martingale problem with the initial condition u_0 for the linearized Kolmogorov operator

$$L_u := \Delta + K(u) \cdot \nabla. \tag{4.11}$$

Note that Theorem 4.2 above remains true if we replace the role of the starting time 0 by any $s \ge 0$. Therefore, the assertion follows by Lemma 2.12 in [31] applied to the linear Kolmogorov operator in (4.11) and the family $\mathcal{R}_{[s,T]}$, $0 \le s \le T$, of $\mathcal{R}$-regular narrowly continuous solutions are defined as follows. $\mathcal{R}_{[s,T]}$ is the set of all narrowly continuous solutions $(u(t))_{t\geq s}$ of (2.2), (2.3) starting at (s,ζ) with $\zeta \in \mathcal{P}(\mathbb{R}^d)$ such that

$$u \in L^4(s,T;L^4) \cap L^\infty(s,T;H^{-1}).$$

Obviously, this family fulfills conditions (2.9) and (2.14) in [31], as is required for Lemma 2.12 in [31]. □

Now, let us turn to the probabilistically strong solutions to (1.8).

Theorem 4.5 *Let $u_0 \in \mathcal{P}^a \cap L^4$. Then, the solution to* (1.8) *from Theorem* 4.1 *is, in fact, a probabilistically strong solution, i.e., is a functional of the Brownian motion $W(t)$, $t \geq 0$. Furthermore, pathwise uniqueness holds in the class of all probabilistically weak solutions to* (1.8) *with the same Brownian motion, having path laws with one dimensional time marginal law densities in $L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}})$.*

Proof. Let u be as in Theorem 4.1, with initial u_0 and let (X,W) be the corresponding weak solution to (1.8). Then, fixing u in (1.8), we are in the case of a usual SDE and may apply the results in Sections 1.3 and 2.1.1 in [22]. To this end, we aim to prove

$$K(u) \in L^\infty(0,T;W^{1,4}). \tag{4.12}$$

Since $u_0 \in \mathcal{P}^a \cap L^4 \subset L^{\frac{4}{3}}$, by Theorem 2.3 (iii) we know that $u \in L^\infty(0,T;L^{\frac{4}{3}})$, hence by (1.5) and (2.14) we have $K(u) \in L^\infty(0,T;L^4)$. Furthermore, by (2.11) and (2.14), $\nabla K(u) \in L^\infty(0,T;L^4)$ and thus (4.12) is proved. Hence, the assertion follows by Theorems 1.3.1 and 2.1.3 in [22] and Lemmas A.2 and A.3 in [16]. □

Finally, we prove that the path laws of the probabilistically weak solution to (1.8) from Theorem 4.1 form a Markov process. To this end, we first note that clearly both Theorems 2.1 and 2.4 hold if we consider (1.7) on $[s,\infty)\times\mathbb{R}^2$ for any $s \geq 0$. Then, renaming the initial condition in (2.3) by $\zeta \in \mathcal{P}$, due to Theorem 2.1 and Remark 2.2 we have, for each $(s,\zeta) \in [0,\infty)\times\mathcal{P}$, a solution to (2.2), (2.3) with the initial condition ζ at time s, which according to Lemma 2.3 in [28] has a narrowly continuous version on $[0,\infty)$. Let us denote this narrowly continuous solution by $\mu^{s,\zeta} = (\mu_t^{s,\zeta})_{t\geq s}$. Below we identify a measure which is absolutely continuous w.r.t. Lebesgue measure dx with its density. Furthermore, for $(s,\zeta) \in [0,\infty) \times \mathcal{P}$, we denote the corresponding probabilistically weak solution by $X(t,s,\zeta)_{t\geq s}$, defined on a stochastic basis $(\Omega^{s,\zeta}, \mathcal{F}^{s,\zeta}, (\mathcal{F}^{s,\zeta})_{t\geq s}, \mathbb{P}^{s,\zeta})$ and define

$$\mathbb{P}_{s,\zeta} := \mathbb{P}^{s,\zeta} \circ X(\cdot, s, \zeta)^{-1}.$$

Then, $\mathbb{P}_{s,\zeta}$ is a probability measure on $\Omega_s := C([s,\infty);\mathbb{R}^d)$, i.e., the set of all continuous paths in $\mathbb{R}^d$ starting at time s equipped with the topology of locally uniform convergence and corresponding Borel σ-algebra $\mathcal{B}(\Omega_s)$.

Define, for $\tau \geq s$,

$$\pi^s_\tau : \Omega_s \to \mathbb{R}^d, \quad \pi^s_\tau(w) := w(\tau), \ w \in \Omega_s,$$

and, for $r \geq s$,

$$\mathcal{F}_{s,r} := \sigma(\pi^s_\tau : s \leq \tau \leq r).$$

Theorem 4.6 *The family* $\mathbb{P}_{s,\zeta}, \ (s,\zeta) \in [0,\infty)\times\mathcal{P}$*, forms a nonlinear Markov process in the sense of Definition* 4.7 *below with* $\mathcal{P}_0 := \mathcal{P}$*.*

The following is a moderately concretized version of the one by McKean from [26].

Definition 4.7 Let $\mathcal{P}_0 \subseteq \mathcal{P}$. A *nonlinear Markov process* is a family $(\mathbb{P}_{s,\zeta})_{(s,\zeta)\in\mathbb{R}_+\times\mathcal{P}_0}$ of probability measures $\mathbb{P}_{s,\zeta}$ on $\mathcal{B}(\Omega_s)$ such that

(i) The marginals $\mathbb{P}_{s,\zeta} \circ (\pi^s_t)^{-1} =: \mu^{s,\zeta}_t$ belong to $\mathcal{P}_0$ for all $0 \leq s \leq r \leq t$ and $\zeta \in \mathcal{P}_0$.

(ii) The *nonlinear Markov property* holds, i.e. for all $0 \leq s \leq r \leq t$, $\zeta \in \mathcal{P}_0$

$$\mathbb{P}_{s,\zeta}(\pi^s_t \in A|\mathcal{F}_{s,r})(\cdot) = p_{(s,\zeta),(r,\pi^s_r(\cdot))}(\pi^r_t \in A) \quad \mathbb{P}_{s,\zeta} - \text{a.s.} \quad \text{for all } A \in \mathcal{B}(\mathbb{R}^d), \tag{MP}$$

where $p_{(s,\zeta),(r,y)}, y \in \mathbb{R}^d$, is a regular conditional probability kernel from $\mathbb{R}^d$ to $\mathcal{B}(\Omega_r)$ of $\mathbb{P}_{r,\mu^{s,\zeta}_r}[\,\cdot\,|\pi^r_r = y]$, $y \in \mathbb{R}^d$ (i.e., in particular, $p_{(s,\zeta),(r,y)} \in \mathcal{P}(\Omega_r)$ and $p_{(s,\zeta),(r,y)}(\pi^r_r = y) = 1$).

The term *nonlinear* Markov property originates from the fact that in the situation of Definition 4.7 the map $\mathcal{P}_0 \ni \zeta \mapsto \mu^{s,\zeta}_t$ is, in general, not convex.

Remark 4.8 The one-dimensional time marginals $\mu_t^{s,\zeta} = \mathbb{P}_{s,\zeta} \circ (\pi_t^s)^{-1}$ of a nonlinear Markov process satisfy the *flow property*, i.e.,

$$\mu_t^{s,\zeta} = \mu_t^{r,\mu_r^{s,\zeta}}, \quad \forall\, 0 \le s \le r \le t, \zeta \in \mathcal{P}_0. \tag{4.13}$$

Proof of Theorem 4.6. For $(s,\zeta) \in [0,\infty) \times \mathcal{P}$, consider the narrowly continuous solution $\mu^{s,\zeta} = (\mu_t^{s,\zeta})_{t \ge s}$, to (2.2), (2.3) introduced in front of the formulation of Theorem 4.6 and define

$$\mathfrak{P}_0 := \{u_0 \in \mathcal{P}^a : u_0 \in L^4\}.$$

Then, by Theorem 2.3 (iii) we have for every $(s, u_0) \in [0,\infty) \times \mathfrak{P}_0$,

$$\mu^{s,u_0} \in L^\infty(s,T;L^4) \subset L^4(s,T;L^4) \cap L^\infty(s,T;H^{-1}),$$

and (2.6), (2.13) imply that $\mu_t^{s,u_0} \in \mathfrak{P}_0$ for every $0 \le s \le t$, $u_0 \in \mathfrak{P}_0$ and $\mu^{t,\zeta} \in \mathfrak{P}_0$ for every $0 \le s < t$, $\zeta \in \mathcal{P}$. Furthermore, Corollary 3.1 implies that μ^{s,u_0}, $(s,u_0) \in [0,\infty) \times \mathfrak{P}_0$, satisfy the flow property (4.13). Then, by Corollary 4.3 and [29, Lemma 3.4] we see that Corollary 3.9 in [29] (with $\mathcal{P}_0 := \mathcal{P}$) applies to prove the assertion. $\square$

Appendix

Proof of (3.20)

Lemma A.1 *Let $z \in L^{\frac{4}{3}}, \varepsilon > 0$. Then*

$$K_\varepsilon(z) = -K(z) + \varepsilon K(\Phi_\varepsilon(z)). \tag{A1}$$

Proof. We first recall that $\varepsilon\Phi_\varepsilon$ is a contraction on every L^p, $p \in [1,\infty]$, and that

$$\Phi_\varepsilon(z) = (g_\varepsilon * z),$$

where

$$g_\varepsilon(x) := \int_0^\infty e^{-\varepsilon t}\ \frac{1}{4\pi t}\ e^{-\frac{1}{4t}|x|^2} dt,\ x \in \mathbb{R}^2. \tag{A2}$$

(See, e.g., [30, p. 132, formula (26)].) Then, $\varepsilon g_\varepsilon \in \mathcal{P}^a$ and an elementary computation yields

$$\nabla^\perp g_\varepsilon(x) = -k(x) \int_0^\infty e^{-\varepsilon |x|^2 t}\, \gamma(dt),\ x \in \mathbb{R}^2 \setminus \{0\}, \tag{A3}$$

where γ is a probability measure on $[0,\infty)$ with density

$$t \mapsto \frac{1}{4t^2}\, e^{-\frac{1}{4t}},\ t \in [0,\infty),$$

and k is as in (2.7). Hence, if B_1 denotes the unit ball in $\mathbb{R}^2$ with centre zero, we have

$$\sup_{\varepsilon>0}\left(\mathbb{1}_{B_1}|\nabla^\perp g_\varepsilon|\right) \le \mathbb{1}_{B_1}|k| \in L^1 \quad\text{and}\quad \sup_{\varepsilon>0}\left(\mathbb{1}_{B_1^c}|\nabla^\perp g_\varepsilon|\right) \le \mathbb{1}_{B_1^c}|k| \in L^\infty \tag{A4}$$

and

$$|\nabla^\perp g_\varepsilon(x)| \nearrow |k(x)| = \frac{1}{2\pi|x|}, \quad \forall x \in \mathbb{R}^2 \setminus \{0\}. \tag{A5}$$

We first prove (A1) for $\varphi \in C_0^\infty =: \mathcal{D}$. Let $\mathcal{D}'$ denote its dual and $\mathcal{S}$ the space of all Schwartz test functions. Then, we have by the resolvent equation of $\Phi_{\varepsilon'}$, $\varepsilon' > 0$, that, for all $\varepsilon' \in (0,\varepsilon)$,

$$\Phi_\varepsilon(\varphi) = \Phi_{\varepsilon'}(\varphi) + (\varepsilon' - \varepsilon)\Phi_{\varepsilon'}(\Phi_\varepsilon(\varphi)). \tag{A6}$$

By (A4), (A5) and Lebesgue's dominated convergence theorem, we have

$$\nabla^\perp\Phi_{\varepsilon'}(\varphi) = (\nabla^\perp g_{\varepsilon'}) * \varphi \underset{\varepsilon'\to 0}{\longrightarrow} -k*\varphi = -K(\varphi) \text{ in } L^1,\ \text{hence in } \mathcal{D}',$$

and, for all $\widetilde{\varphi} \in \mathcal{D}$, since $\Phi_\varepsilon(\varphi) \in \mathcal{S}$, hence $\Phi_\varepsilon(\varphi) * \widetilde{\varphi} \in \mathcal{S}$,

$$\begin{aligned}
{}_{\mathcal{D}'}\big\langle \nabla^\perp\Phi_{\varepsilon'}(\Phi_\varepsilon(\varphi)), \widetilde{\varphi}\big\rangle_{\mathcal{D}} &= \int_{\mathbb{R}^2} \nabla^\perp g_{\varepsilon'}\Phi_\varepsilon(\varphi) * \widetilde{\varphi}\, dx \underset{\varepsilon'\to 0}{\longrightarrow} -\int_{\mathbb{R}^2} k\, \Phi_\varepsilon(\varphi) * \widetilde{\varphi}\, dx \\
&= -\int_{\mathbb{R}^2} k * \Phi_\varepsilon(\varphi)\widetilde{\varphi}\, dx \\
&= -\int_{\mathbb{R}^2} K(\Phi_\varepsilon(\varphi))\widetilde{\varphi}\, dx.
\end{aligned}$$

Therefore,

$$\nabla^\perp\Phi_{\varepsilon'}(\Phi_\varepsilon(\varphi)) \to -K(\Phi_\varepsilon(\varphi)) \ \text{ in } \mathcal{D}',$$

and, consequently, applying $\nabla^\perp$ to (A6), and passing to the limit in $\mathcal{D}'$ with $\varepsilon' \to 0$, we obtain that (A1) holds for $\varphi \in C_0^\infty$. Now, we approximate z in $L^{\frac{4}{3}}$ by $\varphi_n \in C_0^\infty$, $n \in \mathbb{N}$, and since then also $\Phi_\varepsilon(\varphi_n) \to \Phi_\varepsilon(z)$ in $L^{\frac{4}{3}}$ as $n \to \infty$, using the generalized Young inequality and the fact that k, $\nabla^\perp g_\varepsilon \in L^{2,\infty}$, we can pass to the limit with $n \to \infty$ in L^4 to obtain (A1), for $z \in L^{\frac{4}{3}}$. $\square$

Acknowledgement. M. Röckner was supported by the DFG through SFB 1283/2 2021-317210226, V. Barbu was supported by the CNCS–UEFISCDI project PN-III-P4-PCE-2021-0006, within PNCDI III and D. Zhang by the NSFC (No. 12271352, 12322108) and the Shanghai Rising Star Program 21QA1404500.

References

[1] Barbu, V., *Semigroup Approach to Nonlinear Diffusion Equations*, World Scientific, Singapore, 2021.

[2] Barbu, V., Röckner, M., From Fokker–Planck equations to solutions of distribution dependent SDE, *Annals of Probability*, 48 (2020), 1902-1920.

[3] Barbu, V., Röckner, M., Solutions for nonlinear Fokker–Planck equations with measures as initial data and McKean-Vlasov equations, *J. Functional Anal.*, 280 (7) (2021), 108925.

[4] Barbu, V., Röckner, M., *The evolution to equilibrium of solutions to nonlinear Fokker-Planck equations*, *Indiana University Math. Journal*, 72 (1) (2023), 89-131.

[5] Barbu, V., Röckner, M., Uniqueness for nonlinear Fokker-Planck equations and for McKean–Vlasov SDEs: the degenerate case, *J. Functional Analysis*, 285 (3) (2023), 109980.

[6] Ben-Artizi, M., Global solutions of two dimensional Navier–Stokes equations and Euler equations, *Archives Rational Mech. Anal.*, 128 (1994), 329-358.

[7] Benilan, Ph., Brezis, H., Crandall, M.G., A semilinear elliptic equation in $L^1(\mathbb{R}^N)$, *Ann. Scuola Norm. Sup. Pisa*, Serie IV, vol. II (1975), 523-555.

[8] Brezis, H., *Functional Analysis Sobolev Spaces and Partial Differential Equations*, Springer 2011, New York, Dordrecht, Heidelberg, London.

[9] Brezis, H., Crandall, M.G., Uniqueness of solutions of the initial-value problem for $u_t - \Delta\beta(u) = 0$, *J. Math. Pures et Appl.*, 58 (1979), 153-163.

[10] Busnello, B., A probabilistic approach to the two dimensional Navier–Stokes equations, *Annals of Probab.*, 27 (1999), 1750-1780.

[11] Busnello, B., Flandoli, F., Romito, M., A probabiloistic representation for the vorticity of a three-dimensional viscous fluid and for general systems of parabolic equations, *Proc. Edinburgh Math. Soc.*, 48 (2) (2005), 295-336.

[12] Calderon, A.P., Zygmund, A., On singular integrals, *American J. Math.*, 78 (1956), 289-309.

[13] Carillo, S., Shaftingen, J.V., Yung, P.L., The incompressible Navier–Stokes flow in two dimensions with prescribed vorticity, *Harmonic Analysis Partial Differential Equations and Applications*, S. Chanillo et al. (eds.), Springer, 2017.

[14] Chavanis, P.H., Generalized stochastic Fokker–Planck equations, *Entropy*, 17 (2015), 3205-3252.

[15] Constantin, P., Iyer, G., A stochastic Lagrangian representation of the three-dimensional incompressible Navier–Stokes equations, *Ann. Appl. Probability*, 21 (2011), 1466-1492.

[16] Crippa, G., De Lellis, C., Estimates and regularity results for the Di Perna–Lions flow. *J. Reine Angew. Math.*, 616 (2008), 15-46.

[17] Deny, J., Lions, J.L., Les espaces du type de Beppo Levi, *Annales de l'Institut Fourier,* t. 5 (1954), 305-370.

[18] Flandoli, F., Leocata, M., Ricci, C., On the macroscopic limit of Brownian particles with local interaction, *Stoch. Dyn.*, 20 (6) (2020), 2040007, 24 pp.

[19] Fukushima, Masatoshi; Oshima, Yoichi; Takeda, Masayoshi, *Dirichlet Forms and Symmetric Markov Processes*, Studies in Mathematics, 19 Walter de Gruyter & Co., Berlin, 2011, x+489 pp.

[20] Gallagher, I., Gallay, T., Uniqueness for the two-dimensional Navier–Stokes equations with a measure as initial vorticity, *Math. Annalen*, 332 (2005), 287-327.

[21] Giga, Y., Miyakawa, T., Osada, H., Two dimensional Navier–Stokes flow with measures as initial velocity, *Archives Rat. Mech. Anal.*, 104 (1988), 223-250.

[22] Grube, S., Strong solutions to McKean-–Vlasov stochastic differential equations with coefficients of Nemytskii-type, PhD thesis. Bielefeld University, 2023, https://pub.uni-bielefeld.de/record/2979897.

[23] Hao, Z., Röckner, M., Zhang, X., Second order fractional meanfield SDEs with singular kernels and measure initial data, arXiv: 2302.04392v2.

[24] Kato, T., The Navier–Stokes equations for an incompressible fluid in $\mathbb{R}^2$, with a measure as the initial velocity, *Different. Integral Eqns.*, 7 (1994), 949-966.

[25] Koch, H., Tataru, D., Well posedness for the Navier–Stokes equations, *Advances in Mathematics*, 157 (2001), 22-35.

[26] McKean, Jr., H.P., A class of Markov processes associated with nonlinear parabolic equations, *Proceedings of the National Academy of Sciences of the United States of America*, 56 (6) (1966), 1907–1911.

[27] Olivera, C., Richard, A., Tomasevic, M., Quantitative particle approximation of nonlinear Fokker–Planck equations with singular kernel, 2020. arXiv:2011.00537.

[28] Rehmeier, M., Flow selection for (nonlinear) Fokker–Planck–Kolmogorov equations, *J. Diff. Eqn.*, 328 (2022), 105-133.

[29] Rehmeier, M., Röckner, M., On nonlinear Markov processes in the sense of McKean, arXiv:2212.12424v2.

[30] Stein, E.M., *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[31] Trevisan, D., Well-posedness of multidimensional diffusion processes with weakly differentiable coefficients, *Electron J. Probab.,* 21 (2016), 22-41.

[32] Zhang, X., Weak solutions of McKean–Vlasov SDEs with supercritical drifts, to appear in *Commun. Math. Stat.*, 2023+.

[33] Zhang, X., Zhao, G., Stochastic Lagrangian path for Leray solutions of 3D Navier–Stokes equations, *Commun. Math. Phys.*, 381 (2) (2021), 491-525.