THE THREE DIMENSIONAL STOCHASTIC ZAKHAROV SYSTEM

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Abstract. We study the three dimensional stochastic Zakharov system in the energy space, where the Schrödinger equation is driven by linear multiplicative noise and the wave equation is driven by additive noise. We prove the well-posedness of the system up to the maximal existence time and provide a blow-up alternative. We further show that the solution exists at least as long as it remains below the ground state. Two main ingredients of our proof are refined rescaling transformations and the normal form method. Moreover, in contrast to the deterministic setting, our functional framework also incorporates the local smoothing estimate for the Schrödinger equation in order to control lower order perturbations arising from the noise. Finally, we prove a regularization by noise result which states that finite time blowup before any given time can be prevented with high probability by adding sufficiently large non-conservative noise. The key point of its proof is a Strichartz estimate for the Schrödinger equation with a potential solving the free wave equation.

1. Introduction and main results

1.1. The Zakharov system. We consider the three dimensional stochastic Zakharov system

$$\begin{cases} \mathrm{i}\,\mathrm{d}X + \Delta X\,\mathrm{d}t = \mathrm{Re}(Y)X\,\mathrm{d}t - \mathrm{i}\mu X\,\mathrm{d}t + \mathrm{i}X\,\mathrm{d}W_1(t), \\ \frac{1}{\alpha}\mathrm{i}\,\mathrm{d}Y + |\nabla|Y\,\mathrm{d}t = -|\nabla||X|^2\,\mathrm{d}t + \mathrm{d}W_2(t), \\ (X(0),Y(0)) = (X_0,Y_0) \in H^1 \times L^2. \end{cases} \tag{1.1}$$

Here, the wave speed $\alpha > 0$ is a fixed constant, $\{W_j(t), t \geq 0\}$ are independent Wiener processes white in time and colored in space,

$$W_j(t,x) = \sum_{k=1}^{\infty} \mathrm{i}\phi_k^{(j)}(x)\beta_k^{(j)}(t), \quad x \in \mathbb{R}^3, \quad t \geq 0$$

for $j = 1, 2$, $\{\phi_k^{(1)}\} \subseteq H^3(\mathbb{R}^3)$ and $\{\phi_k^{(2)}\} \subseteq H^1(\mathbb{R}^3)$ are real-valued and satisfy the summability conditions in (1.10), and $\{\beta_k^{(j)}\}$ are real-valued independent Brownian motions on a stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$. The stochastic term $X\,\mathrm{d}W_1(t)$ is taken in the sense of Itô, and

$$\mu = \frac{1}{2}\sum_{k=1}^{\infty} |\phi_k^{(1)}|^2 < \infty.$$

In particular, $-\mathrm{i}\mu X\,\mathrm{d}t + \mathrm{i}X\,\mathrm{d}W_1(t)$ is the usual Stratonovich differential $\mathrm{i}X \circ \mathrm{d}W_1(t)$.

In 1972, Zakharov introduced the quadratically coupled system of a Schrödinger and a wave equation (2.1) in order to describe rapid oscillations of the electric field (Langmuir waves) in a non- or weakly magnetized plasma, see [58]. We provide more information on the model in Subsection 2.1 below.

The deterministic Zakharov system has attracted a lot of attention in the literature as its coupling of a Schrödinger and a wave equation leads to interesting phenomena and presents various mathematical challenges. Moreover, the Zakharov system is closely connected to the focusing cubic Schrödinger equation which arises in the subsonic limit $\alpha \to \infty$, see [41, 44, 49]. We refer to [9, 15, 25, 48] and the references therein for the local well-posedness theory and to [16, 27, 30, 32] and the references therein for the long-time behavior of the Zakharov system.

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In contrast, very little is known about the stochastic Zakharov system. The available results [8, 29, 56] are all restricted to the one dimensional case and consider the Zakharov system with additive noise.

In this article, we propose to consider the Zakharov system with linear multiplicative noise in the Schrödinger equation and additive noise in the wave equation. We provide heuristic arguments for this type of noise from a modeling perspective in Subsection 2.1 below. Linear multiplicative noise in the Schrödinger equation also preserves the conservation of mass of the Schrödinger component

$$\|X(t)\|_{L^2}^2 = \|X_0\|_{L^2}^2 \tag{1.2}$$

from the deterministic setting. Moreover, adding additive noise in the wave equation, at least formally we obtain the cubic focusing Schrödinger equation with linear multiplicative noise as the subsonic limit $\alpha \to \infty$ of the Zakharov system.

The aim of this work is to develop the well-posedness theory for the three dimensional stochastic Zakharov system (1.1) in the energy space $H^1 \times L^2$. We prove the well-posedness up to the maximal existence time and provide a blow-up alternative. Using the variational characterization of the ground state, we also show the well-posedness of the stochastic Zakharov system below the ground state.

One major difficulty in solving the Zakharov system in the energy space is that the Schrödinger component has to be controlled in H^1 while the wave component in the nonlinearity $\mathrm{Re}(Y)X$ only belongs to L^2. Moreover, as pointed out in [56], since the path of Brownian motions has merely $C^{1/2-}$ regularity, one cannot expect that in the stochastic case solutions belong to the usual Bourgain space $X^{s,b}$ with $b > 1/2$ which was used in the deterministic setting.

Our proof utilizes a refined rescaling approach and a normal form method, which is new in the stochastic literature. The rescaling approach allows a pathwise treatment of the system so that we can apply a normal form transform which exploits the resonance structure of the nonlinearity in the Schrödinger equation and reveals that the loss of derivatives indicated above can be avoided. On the other hand, the rescaling leads to first order perturbation terms originating from the noise. Our functional framework thus also incorporates local smoothing estimates in order to control these first order perturbations, which do not appear in the deterministic setting. More precisely, in order to exploit the sharp local smoothing effect, we adapt an approach developed in [11] for the Schrödinger map problem. Furthermore, compared with previous work on stochastic nonlinear Schrödinger equations (SNLS for short) [2, 3, 60, 61], we remark that we can deal with noise of much lower regularity here.

Another delicate fact is that the control of the first order perturbation requires a small amplitude of the path of the Brownian motions, which, however, is only possible up to a small enough stopping time. Consequently, we encounter another difficulty in extending the solution up to the maximal existence time, as well as in proving a blow-up alternative, which are the foundation for the long-time analysis. Our strategy utilizes the refined rescaling transformations which have been recently developed in [59, 61] to tackle the global well-posedness problem for SNLS, particularly in the critical regime. Rather than performing one rescaling transformation, we apply a series of rescaling transformations which incorporate stopping times constructed iteratively to keep track of the trajectory of the noise. We can then construct solutions on every small random interval. Finally, probabilistic arguments involving the Hölder continuity of the noise are used to glue these solutions together up to the maximal existence time.

Our method also permits to investigate the regularization by noise effect on well-posedness. To that purpose, we first note that finite time blowup might occur for the Zakharov system. In fact, finite time blow-up was proved in [26] in dimension $d = 2$ and conjectured in [42] for $d = 3$.

We prove that, with high probability, non-conservative noise can prevent blow-up on bounded time intervals for any $H^1 \times H^1$ initial data, hence even for initial data above the ground state constraint which may lead to finite time blow-up in the deterministic case. Its proof uses a new type of rescaling transformation, which reveals a geometrical Brownian motion which decays exponentially fast at infinity and dampens the nonlinearity. The key point is then to prove Strichartz estimates for the Schrödinger operator with a potential solving the free wave equation.

Before formulating our main results, we first review the literature for both the deterministic and stochastic Zakharov system.

The deterministic Zakharov system. The deterministic Zakharov system (2.3) is a Hamiltonian system and conserves the mass

$$M(u) := \frac{1}{2} \int |u|^2 \, \mathrm{d}x, \tag{1.3}$$

and the Hamiltonian

$$E_Z(u,v) := \int \frac{|\nabla u|^2}{2} + \frac{|v|^2}{4} + \frac{\operatorname{Re} v |u|^2}{2} \, \mathrm{d}x. \tag{1.4}$$

The conservation laws suggest the energy space $H^1 \times L^2$ as a natural state space for the well-posedness of the Zakharov system (2.3). In dimensions $d = 2, 3$ local well-posedness in the energy space and global well-posedness under a smallness condition were proved by Bourgain-Colliander [9], see also Ginibre-Tsutsumi-Velo [25], which covers the case $d = 1$ as well. There has been extensive research on the question of determining the optimal range of regularity parameters (s, l) such that the Zakharov system is locally well-posed in $H^s \times H^l$. A comprehensive answer has recently be given by Candy-Herr-Nakanishi [15] in dimensions $d \geq 4$ and by Sanwal [48] in dimensions $d \leq 3$. We refer to [15, 48] for previous work on and the historic development of that problem.

As mentioned above, the Zakharov system (2.3) is closely related to the focusing cubic nonlinear Schrödinger equation (NLS)

$$\mathrm{i}\partial_t u + \Delta u = -|u|^2 u, \tag{1.5}$$

since the latter arises as the subsonic limit $\alpha \to \infty$ of the Zakharov system, see [41, 44, 49]. The close connection between the Zakharov system and the cubic NLS also appears in the Hamiltonian

$$E_Z(u,v) = E_S(u) + \frac{1}{4} \| v + |u|^2 \|_{L^2}^2, \tag{1.6}$$

where $E_S(u)$ is the Hamiltonian for the focusing cubic NLS, i.e.,

$$E_S(u) := \int \frac{|\nabla u|^2}{2} - \frac{|u|^4}{4} \, \mathrm{d}x. \tag{1.7}$$

This close relationship indicates the difficulty in studying the global behavior of solutions to the Zakharov system. In fact, in the 1D case the cubic nonlinearity in the NLS is L^2-subcritical with respect to scaling and hence (1.5) is globally well-posed in L^2. But in the 2D case the cubic nonlinearity becomes L^2-critical and there exist solutions (e.g., the pseudo-conformal blow-up solutions) which form a singularity in finite time. For higher dimensions, the cubic nonlinearity is even more singular and becomes H^1-critical when $d = 4$.

In the seminal work [36], Kenig-Merle proved that the ground state is the sharp threshold for global well-posedness, scattering and blowup for the radial energy-critical focusing NLS in dimensions $3 \leq d \leq 5$. In the 3D case, the corresponding ground state Q is the unique positive radial solution of the nonlinear elliptic equation

$$-\Delta Q + Q = Q^3. \tag{1.8}$$

The Kenig-Merle approach can also be used for the Zakharov system. In dimension $d = 3$, the scattering behavior for radial solutions with small energy was proved by Guo-Nakanishi [31] and for radial solutions below the ground state by Guo-Nakanishi-Wang [30]. For the non-radial case with additional conditions we refer to [27, 28, 33, 51].

In the energy-critical dimension $d = 4$, global well-posedness and scattering for radial initial data below the ground state was proved by Guo-Nakanishi [32]. For the non-radial case, the small-data global well-posedness and scattering were proved in [10]. Recently, the global well-posedness below the ground state was shown by Candy, Nakanishi and the first author [16]. The scattering for general non-radial data below the ground state remains a challenging open problem. We refer to the recent progress by Candy [14] on the existence of a minimal energy almost periodic solution if the scattering fails.

The stochastic case. Much less is known in the stochastic case. The Zakharov system with additive noise on a bounded interval with zero Dirichlet boundary condition was studied in [29] based on the Galerkin method. Very recently, Tsutsumi [56] proved the global well-posedness for the 1D Zakharov system driven by additive noise, using the Fourier restriction method. Finally, the subsonic limit of the 1D Zakharov system with additive noise in the wave equation was studied by Barrué [8].

The stochastic Zakharov system with multiplicative noise, the well-posedness in the energy space and the stochastic Zakharov system in higher dimensions have not been studied before.

At least formally, in the subsonic limit $\alpha \to \infty$ the stochastic Zakharov system (1.1) reduces to the focusing SNLS driven by linear multiplicative noise

$$\mathrm{i}\,\mathrm{d}X + \Delta X\,\mathrm{d}t = -|X|^2 X\,\mathrm{d}t - \mathrm{i}\mu X\,\mathrm{d}t + \mathrm{i}X\,\mathrm{d}W(t), \tag{1.9}$$

where $W = W_1 - \mathrm{i}|\nabla|^{-1}W_2$. In the case where $W_2 \equiv 0$, the H^1 local well-posedness of (1.9) is well known, see, e.g., [3, 12, 18].

The global existence and large time behavior of solutions to SNLS are quite delicate. In the case $d = 2$, the cubic nonlinearity is L^2-critical. On the one hand, the stochastic solutions exist globally below the ground state [43]. Furthermore, critical mass blow-up solutions have recently been constructed in [53] using the modulation method. Hence, the ground state is still the sharp threshold for the global well-posedness and blowup when $d = 2$.

We also refer to [22] for the construction of stochastic log-log blow-up solutions, and to [46, 47, 54] for the construction and conditional uniqueness of multi-bubble Bourgain-Wang type blow-up solutions and (non-pure) multi-solitons, which provide new examples for the mass quantization conjecture and the soliton resolution conjecture. For the refined uniqueness of pure multi-solitons in the low asymptotical regime, related to an open problem raised by Martel [40], we refer to [13].

1.2. **Main results.** We first present the hypothesis for the spatial functions in the noise, which also relates to the lateral spaces we introduce in Subsection 5.1 in order to capture the local smoothing estimates.

Hypothesis (H). The spatial functions $\{\phi_k^{(j)}\}$, $j = 1, 2$, satisfy the following summability conditions:

$$\sum_{k=1}^{\infty} \|\phi_k^{(1)}\|_{H^3}^2 + \sum_{j=1}^{3}\sum_{k=1}^{\infty} \int \sup_{y\in\mathbb{R}^2} |\phi_k^{(1)}(r\mathbf{e}_j + y)|\,\mathrm{d}r < \infty, \quad \sum_{k=1}^{\infty} \|\phi_k^{(2)}\|_{H^1}^2 < \infty, \tag{1.10}$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ denotes the natural orthonormal basis of $\mathbb{R}^3$.

It is standard that under the summability (1.10), W_1 and W_2 are Wiener processes in H^3 and H^1, respectively. See, e.g., [39].

We present the definition of solutions to (1.1), which are taken in the probabilistic strong and analytically weak sense. Since the wave speed α is a fixed constant in the well-posedness theory of (1.1), we set $\alpha = 1$ for convenience.

Definition 1.1. *Fix $T \in (0, \infty)$. We say that (X, Y) is a probabilistic strong solution to* (1.1) *on $[0, \tau]$, where $\tau \in (0, T]$ is an $\{\mathscr{F}_t\}$-stopping time, if (X, Y) is an $H^1 \times L^2$-valued $\{\mathscr{F}_t\}$-adapted process which belongs to $C([0, \tau], H^1 \times L^2)$ and satisfies $\mathbb{P}$-a.s. for any $t \in [0, \tau]$,*

$$\begin{cases} X(t) = \displaystyle\int_0^t \mathrm{i}\Delta X\,\mathrm{d}s - \int_0^t \mathrm{i}\,\mathrm{Re}(Y)X\,\mathrm{d}s - \int_0^t \mu X\,\mathrm{d}s + \int_0^t X\,\mathrm{d}W_1(s), \\ Y(t) = \displaystyle\int_0^t \mathrm{i}|\nabla|Y\,\mathrm{d}s + \int_0^t \mathrm{i}|\nabla||X|^2\,\mathrm{d}s - \mathrm{i}W_2(t), \end{cases} \tag{1.11}$$

as equations in $H^{-1} \times H^{-1}$.

Given an $\{\mathscr{F}_t\}$-stopping time τ^*, we also call (X, Y) a probabilistic strong solution to (1.1) *on $[0, \tau^*)$ if (X, Y) is an $\{\mathscr{F}_t\}$-adapted process belonging to $C([0, \tau^*), H^1 \times L^2)$ such that for any $T \in (0, \infty)$ and any $\{\mathscr{F}_t\}$-stopping time $\tau < \tau^*$, (X, Y) is a probabilistic strong solution to* (1.1) *on $[0, \tau \wedge T]$.*

The first main result of this paper provides the local well-posedness of (1.1) and a blow-up alternative.

Theorem 1.2 (Well-posedness up to maximal existence time)**.** *Assume (H). Let $(X_0, Y_0) \in H^1 \times L^2$. Then, $\mathbb{P}$-a.s. there exist an $\{\mathscr{F}_t\}$-stopping time τ^* and a unique $\{\mathscr{F}_t\}$-adapted solution $(X, Y) \in C([0, \tau^*); H^1 \times L^2)$ to the stochastic Zakharov system* (1.1) *in the sense of Definition 1.1.*

Moreover, $\mathbb{P}$-a.s. we have either $\tau^* = \infty$, or

$$\limsup_{t\to\tau^*}(\|X(t)\|_{H^1} + \|Y(t)\|_{L^2}) = \infty \quad \text{if } \tau^* < \infty. \tag{1.12}$$

Remark 1.3. In the statement of Theorem 1.2 uniqueness means that for any $T \in (0,\infty)$ and any $\{\mathscr{F}_t\}$-stopping time $\tau < \tau^*$, the process (X,Y) is the unique solution of (1.1) in

$$(C([0,\tau\wedge T]; H^1) \cap \mathbb{X}(0,\tau\wedge T)) \times C([0,\tau\wedge T]; L^2)$$

in the sense of Definition 1.1, where the function space $\mathbb{X}$ is introduced in (5.6).

Remark 1.4.
(i) To the best of our knowledge, Theorem 1.2 provides the first well-posedness result for the stochastic Zakharov system with multiplicative noise, in the physical dimension three, and in the energy space.
(ii) The proof makes use of the refined rescaling approach, recently developed for the critical stochastic nonlinear Schrödinger equations, and the normal form method, employing both Strichartz and local smoothing estimates. We explain the key ideas of the proof in Subsection 2.2 below. In particular, our strategy also applies in the context of stochastic nonlinear Schrödinger equations, allowing to treat noise with less regularity than in the previous papers [2, 3, 59, 61], as well as the case of infinite modes of noise.

Next, we consider the well-posedness below the ground state, which is known as the sharp threshold for global well-posedness in the deterministic case.

We recall from [30, (2.9)] that the ground state minimizes the action functional

$$J(Q) = \inf\{J(\varphi) : \varphi \neq 0, K(\varphi) = 0\}, \tag{1.13}$$

where

$$J(Q) := E_S(Q) + M(Q) \tag{1.14}$$

with E_S being the Hamiltonian for the cubic focusing NLS in (1.7), M the mass functional given by (1.3), and K stands for the scaling derivative of the action J, defined by

$$K(\varphi) := \partial_\lambda|_{\lambda=1} J(\lambda^{\frac{3}{2}}\varphi(\lambda x)) = \int |\nabla\varphi|^2 - \frac{3}{4}|\varphi|^4 \, \mathrm{d}x. \tag{1.15}$$

Theorem 1.5 (Well-posedness below the ground state)**.** *Assume (H). Let $(X_0, Y_0) \in H^1 \times L^2$ satisfy $E_Z(X_0,Y_0)M(X_0) < E_S(Q)M(Q)$. Let (X,Y) be the corresponding unique solution to (1.1) on $[0,\tau^*)$ from Theorem 1.2, where τ^* is the maximal existence time. Define the $\{\mathscr{F}_t\}$-stopping time*

$$\sigma_* := \inf\{t > 0 : E_Z(X(t), Y(t))M(X(t)) \geq E_S(Q)M(Q)\}. \tag{1.16}$$

Then, if $K(X_0) \geq 0$, we have $\tau^* \geq \sigma_*$, $\mathbb{P}$-a.s., that is, (X,Y) exists at least up to the stopping time σ_*.

Remark 1.6. In the deterministic case the conservation of mass and energy shows that $\sigma_* = \infty$ if $E_Z(u_0,v_0)M(u_0) < E_S(Q)M(Q)$, implying global existence below the ground state (cf. [30]). However, the conservation of the energy breaks down in the stochastic case due to the presence of the Itô type stochastic integrals. In view of the possible blow-up phenomena above the ground state in the deterministic case, we would expect $\sigma_* < \infty$. This motivates the study of the regularization by noise effect on well-posedness below.

It is widely expected in the probability community that suitable noise may improve the well-posedness and long-time behavior of deterministic systems, which is called the *regularization by noise* effect. We refer to [23] where regularization by noise was proved for uniqueness of transport equations, and to [24] for the regularization effect of transport noise on vorticity blow-up in 3D Navier-Stokes equations.

For stochastic Schrödinger equations, the numerical experiments in [20, 21] show that conservative smooth multiplicative noise can delay blowup, while white noise can even prevent blowup. In the case of non-conservative noise, the noise is able to prevent blow-up with high probability [4], and it can also improve the scattering behavior of solutions [34].

Regarding the Zakharov system, the existence of finite time blow-up solutions was proven in [26] in dimension $d = 2$ and conjectured in [42] in dimension $d = 3$. Numerical evidence for finite time blow-up solutions was found in [37, 45]. In Theorem 1.7 below, we show the regularization effect of non-conservative noise on the blow-up behavior of solutions of the stochastic Zakharov system.

Theorem 1.7 (Noise regularization on well-posedness)**.** *Consider the stochastic Zakharov system* (1.1)*, where the driving noise W_1 is a one-dimensional Brownian motion with non-zero imaginary part, i.e., $\operatorname{Im}\phi_1^{(1)} \neq 0$, $\phi_k^{(1)} = 0$ for $2 \leq k < \infty$, and $W_2 \equiv 0$. Then, for any deterministic initial data $X_0, Y_0 \in H^1$ and any $0 < T < \infty$, we have*

$$\mathbb{P}\left(X(t) \text{ does not blow up on } [0,T]\right) \longrightarrow 1, \quad \text{as } \phi_1^{(1)} \to \infty. \tag{1.17}$$

Remark 1.8. (i) Unlike in Theorems 1.2 and 1.5, the noise considered in Theorem 1.7 is of *non-conservative type*, i.e., $\operatorname{Im} W^{(1)} \neq 0$. The non-conservative noise permits to define the so-called "physical probability law" which has important applications in quantum measurements, see, e.g., the monograph [7] for the physical context. Theorem 1.7 shows that even for deterministic data which leads to a singularity in finite time, the one-dimensional non-conservative noise can prevent blow-up on bounded time intervals with high probability, as long as the noise is strong enough.

(ii) This regularization by noise effect relies on the observation that the non-conservative noise gives rise to a geometrical Brownian motion in the nonlinearity which has exponential decay at infinity and thus is able to weaken the nonlinearity. The key step in order to explore this effect is to prove Strichartz estimates for the Schrödinger equation with a potential

$$(\partial_t - \mathrm{i}\Delta + \mathrm{i}v_1)z = g, \tag{1.18}$$

where the potential $v_1 := e^{\mathrm{i}t|\nabla|}Y_0$ solves the free wave equation.

The global uniform Strichartz estimate for (1.18) is the key ingredient in [16] in proving the global well-posedness below the ground state for the deterministic problem in dimension $d = 4$. Here, we prove Strichartz estimates on bounded time intervals using perturbation arguments, which suffices for the finite time analysis.

Organization of the paper. In Section 2, a heuristic derivation of the stochastic Zakharov system and the strategy of the proof is discussed. Section 3 is mainly concerned with the refined rescaling transformations. We prove the equivalence of solving the stochastic and random Zakharov systems, and also show how solutions can be glued together. Section 4 is devoted to the normal form transform. In Section 5, we provide the functional framework for the proof of Theorem 1.2 and provide corresponding multi-linear estimates. In Section 6 and Section 7 we give the proofs of Theorems 1.2 and 1.5, respectively. In Section 8 we prove Theorem 1.7 concerning the noise regularization effect on well-posedness. Finally, we prove the equivalence of different solution concepts we use in this article, i.e. weak solutions, mild solutions, and normal form solutions, in Appendix A and Appendix B.

2. Motivation

2.1. Heuristic derivation of the stochastic Zakharov system.

We recall that the Zakharov system was introduced in [58] as a model in plasma physics to describe rapid oscillations of the electric field in a non- or weakly magnetized plasma. A more formal derivation using multiple-scale modulational analysis is presented in [52, Chapter 13], see also [17, 55].

The starting point for the model is to consider the plasma as two interpenetrating fluids, the ion fluid and the electron fluid, each described by a set of hydrodynamic equations. Since the fluids consist of charged particles, they interact with the electromagnetic fields in the plasma, which evolve via Maxwell's equations. Consequently, the two-fluid model is described by an Euler-Maxwell system, where Euler's and Maxwell's equations are coupled via the current density (given as the sum of the products of charge, number density, and velocity field of the fluids) in the Maxwell equations, while the gradient of the pressure term in the Euler equations is complemented by the Lorentz force, see [52]. Considering long wavelength small-amplitude Langmuir waves, one can reduce the rather complex Euler-Maxwell system to the Zakharov system, which reads in its scalar version

$$\begin{cases} \mathrm{i}\partial_t u + \Delta u = Vu, \\ \dfrac{1}{\alpha^2}\partial_t^2 V - \Delta V = \Delta|u|^2. \end{cases} \tag{2.1}$$

We refer again to [52, Chapter 13] for this derivation, see also [55]. In (2.1), $u\colon \mathbb{R}\times\mathbb{R}^3\to\mathbb{C}$ describes the complex envelope of the electric field, $V\colon \mathbb{R}\times\mathbb{R}^3\to\mathbb{R}$ is the ion density fluctuation, and the fixed constant $\alpha>0$ is called the ion sound speed.

To be more precise, making the modulational ansatz

$$E=\frac{\varepsilon}{2}(\mathcal{E}(\hat{t},\tilde{x})e^{-\mathrm{i}\omega_e t}+\text{c.c.})+\varepsilon^2\overline{\mathcal{E}}(\hat{t},\tilde{x})+\dots,$$
$$n_i=n_0+\frac{\varepsilon^2}{2}(\tilde{n}_i(\tilde{t},\tilde{x})e^{-\mathrm{i}\omega_e t}+\text{c.c.})+\varepsilon^2\overline{n}_i(\tilde{t},\tilde{x})+\dots$$

with $\tilde{x}=\varepsilon x$, $\tilde{t}=\varepsilon t$, $\hat{t}=\varepsilon^2 t$, and ω_e being the plasma frequency as in [52], u represents (one component of) the amplitude $\mathcal{E}$ and V the mean ion density fluctuation $\overline{n}_i$. Note that in the above ansatz it is assumed that the unperturbed plasma density n_0 is known. This leads to the question how the model is affected if there is insufficient information on or internal random fluctuations in n_0. Taking into account this uncertainty and replacing n_0 by $n_0+\varepsilon^2\dot{W}_1(\tilde{t},\tilde{x})$, it turns out that we have to replace V by $V+\dot{W}_1$ on the right-hand side of the Schrödinger equation in (2.1). We obtain

$$\mathrm{i}\partial_t u+\Delta u=(V+\dot{W}_1)u=Vu+u\dot{W}_1,$$

suggesting to consider the Schrödinger part of the Zakharov system with linear multiplicative noise.

Another source of uncertainty is the plasma temperature. In the long-wavelength limit it is assumed that the compression of the wave is adiabatic, so that we obtain for the pressure on the right-hand side of the Euler equation for the ion fluid $\nabla p_i=\gamma_i T_i\nabla n_i$, where γ_i denotes the specific heat ratio of the ions and T_i the ion temperature. Modeling thermal fluctuations by $T_i+\varepsilon^2 c(\tilde{x})\dot{\beta}(\tilde{t})$, we obtain for the right-hand side of the wave equation in (2.1) $\Delta|u|^2+\Delta c\dot{\beta}$. We refer to [52] for the details.

Summing up, we propose to model thermal fluctuations in the plasma by additive noise in the wave equation, leading to

$$\begin{cases}\mathrm{i}\partial_t u+\Delta u=Vu+\dot{W}_1u,\\ \dfrac{1}{\alpha^2}\partial_t^2V-\Delta V=\Delta|u|^2+\dot{W}_2,\end{cases}\tag{2.2}$$

with space-time noise $\dot{W}_1,\dot{W}_2$.

Finally, we note that the Zakharov system has an equivalent first-order formulation obtained by setting $v:=V-\mathrm{i}\frac{1}{\alpha}|\nabla|^{-1}\partial_t V$. In the deterministic case this transformation leads to

$$\begin{cases}\mathrm{i}\partial_t u+\Delta u=\mathrm{Re}(v)u,\\ \dfrac{1}{\alpha}\mathrm{i}\partial_t v+|\nabla|v=-|\nabla||u|^2,\end{cases}\tag{2.3}$$

while we get

$$\begin{cases}\mathrm{i}\partial_t u+\Delta u=\mathrm{Re}(v)u+\dot{W}_1u,\\ \dfrac{1}{\alpha}\mathrm{i}\partial_t v+|\nabla|v=-|\nabla||u|^2+|\nabla|^{-1}\dot{W}_2,\end{cases}\tag{2.4}$$

in the stochastic case. Writing $\dot{W}_2$ for the noise in the wave equation again and (X,Y) instead of (u,v), we arrive at (1.1) as a stochastic model for the Zakharov system.

2.2. **Key ideas of the proof.** In the following we present three main ingredients of the proofs of our main theorems.

(i) Refined rescaling transformations. The rescaling transformation is a Doss-Sussman type transformation first developed for finite dimensional stochastic differential equations, and now has been applied successfully to various infinite dimensional stochastic equations. We refer to [1] for the stochastic Camassa-Holm equation and to [5, 6, 19, 59, 60] for the SNLS. One main feature of the rescaling transformation is that it permits to transform the original stochastic equation to a random equation, and thus allows sharp analysis in a *pathwise* way, which is difficult for the usual stochastic analysis. In particular, the resulting solutions depend continuously on the initial data pathwisely and satisfy the strict cocycle property. This leads to stochastic dynamical systems which are favorable for the analysis of large time dynamics, including the scattering and blow-up behavior and the study of solitons. We refer to [34, 46, 47, 53, 54, 61] for corresponding references.

Unlike in [2, 3] for the Schrödinger equation, we perform two different rescaling transforms in order to respect the different types of noise in the stochastic Zakharov system (1.1). More precisely, for the Schrödinger component, we use the transform

$$u = e^{-W_1} X, \tag{2.5}$$

and for the wave component we use the transform

$$v(t) := Y(t) - \mathcal{T}_t(W_2) \;\; \text{with } \mathcal{T}_t(W_2) := -\mathrm{i} \int_0^t e^{\mathrm{i}(t-s)|\nabla|} \, \mathrm{d}W_2(s). \tag{2.6}$$

Then, the original stochastic Zakharov system (1.1) can be reduced to the random system

$$\begin{cases} \mathrm{i}\partial_t u + e^{-W_1}\Delta(e^{W_1}u) = \mathrm{Re}(v)u + \mathrm{Re}(\mathcal{T}_t(W_2))u, \\ \mathrm{i}\partial_t v + |\nabla| v = -|\nabla||u|^2, \\ (u(0), v(0)) = (X_0, Y_0), \end{cases} \tag{2.7}$$

or, equivalently, the system with random perturbations

$$\begin{cases} \mathrm{i}\partial_t u + \Delta u = \mathrm{Re}(v)u - b\cdot\nabla u - cu + \mathrm{Re}(\mathcal{T}_t(W_2))u, \\ \mathrm{i}\partial_t v + |\nabla| v = -|\nabla||u|^2, \\ (u(0), v(0)) = (X_0, Y_0), \end{cases} \tag{2.8}$$

where the random coefficients of the lower order perturbations are defined by

$$b = 2\nabla W_1 = 2\mathrm{i}\sum_{k=1}^{\infty} \nabla\phi_k^{(1)}\beta_k^{(1)}, \tag{2.9}$$

$$c = |\nabla W_1|^2 + \Delta W_1 = -\sum_{j=1}^{d}\left(\sum_{k=1}^{\infty}\partial_j\phi_k^{(1)}\beta_k^{(1)}\right)^2 + \mathrm{i}\sum_{k=1}^{\infty}\Delta\phi_k^{(1)}\beta_k^{(1)}. \tag{2.10}$$

The solutions to the random Zakharov system (2.7) (or, (2.8)) are taken in the analytically weak sense, analogous to the stochastic case (1.11) in Definition 1.1 above.

The relation between the two systems (1.1) and (2.8) can be seen by a formal application of Itô's formula. However, the rigorous justification of the application of Itô's formula is non-trivial in the infinite dimensional case, due to the failure of the C_b^2-regularity of the nonlinearity (see also [2] for detailed explanations in the context of SNLS). The rigorous proof for the equivalence of (1.1) and (2.8) is given in Section 3.

In order to deal with the first order perturbation, we will employ local smoothing estimates for the Schrödinger equation (see (ii) below). The resulting functional framework only allows us to control this first order term if the variation of the noise is small. This requires the refined rescaling transformations, recently developed in the context of critical stochastic nonlinear Schrödinger equations [61], in order to extend solutions up to the maximal existence time.

More precisely, for any stopping time σ we define the increments of the noise by

$$W_{1,\sigma}(t) := W_1(\sigma + t) - W_1(\sigma),$$
$$\mathcal{T}_{\sigma+t,\sigma}(W_2) := -\mathrm{i}\int_{\sigma}^{\sigma+t} e^{\mathrm{i}(\sigma+t-s)|\nabla|}\,\mathrm{d}W_2(s).$$

Using the transformations

$$u_\sigma(t) := e^{W_1(\sigma)}u(\sigma+t),$$
$$v_\sigma(t) := v(\sigma+t) + e^{\mathrm{i}t|\nabla|}\mathcal{T}_\sigma(W_2),$$

where $0 \le t \le \tau$ with τ being an $\{\mathscr{F}_{\sigma+t}\}$-stopping time, we transform the random Zakharov system (2.8) to the new system

$$\begin{cases} \partial_t u_\sigma(t) = \mathrm{i}e^{-W_{1,\sigma}(t)}\Delta(e^{W_{1,\sigma}(t)}u_\sigma(t)) - \mathrm{i}\,\mathrm{Re}(v_\sigma(t))u_\sigma(t) - \mathrm{i}\,\mathrm{Re}(\mathcal{T}_{\sigma+t,\sigma}(W_2))u_\sigma(t), \\ \partial_t v_\sigma(t) = \mathrm{i}|\nabla|v_\sigma(t) + \mathrm{i}|\nabla||u_\sigma(t)|^2. \end{cases} \tag{2.11}$$

Note that (2.11) reduces to (2.8) if $\sigma \equiv 0$.

The advantage of this transform is that it enables us to control the problematic first order perturbation by controlling the increment of the noise via stopping time techniques. In the proof we will iteratively construct sequences $(\sigma_n)_{n\in\mathbb{N}}$ and $(\tau_n)_{n\in\mathbb{N}}$ of $\{\mathscr{F}_t\}$ and $\{\mathscr{F}_{\sigma_n+t}\}$ stopping times, respectively, in order to split the trajectory of the noise. On each small random interval $[\sigma_n, \sigma_n + \tau_{n+1}]$ we can then construct the local solution.

Probabilistic arguments involving the Hölder continuity of the noise will then be applied to glue together all these local solutions, providing the solution up to the maximal existence time and the blow-up alternative. The detailed arguments are contained in Section 3.

(ii) Normal form method and local smoothing. Solving the random system (2.8) in the energy space $H^1 \times L^2$, one of the main difficulties is the regularity of the nonlinearity. At first sight, there seems to be a derivative loss as we have to control the Schrödinger component in H^1, but in the nonlinearity $\mathrm{Re}(v)u$ the wave component only belongs to L^2. This problem already appears in the deterministic system (2.3), and one tool to overcome it is the so called normal form transform.

The normal form method was originally introduced by Shatah [50] to analyze the long time behavior of quadratic nonlinear Klein-Gordon equations. The idea was to transform the equation in such a way that the order of the nonlinearity increases, as higher order nonlinearities are favorable for the long time analysis using perturbation techniques. Nowadays, the normal form method is widely used to study both the long time behavior and the low regularity well-posedness of dispersive PDEs.

For the Zakharov system (2.3), the normal form transform was introduced by Guo and Nakanishi in [31] in order to show scattering for small, radial initial data in the energy space. It was subsequently used to advance the understanding of both the low regularity well-posedness and the long-time behavior, see [10, 27, 28, 30, 51].

The underlying idea is to use the resonance structure of the system to reveal subtle features of the nonlinear interactions. To be more precise, an integration by parts argument on the Fourier side transforms the equation in such a way that only low-high and high-high interactions of the quadratic nonlinearity remain. Consequently, derivatives only fall on the Schrödinger component of the quadratic nonlinearity, showing that a loss of derivatives in the nonlinearity $\mathrm{Re}(v)u$ can be avoided. We refer to Section 4 for the details, where we perform the normal form transform for the random system (2.8).

We point out that the previously discussed rescaling transforms, allowing us to analyze the stochastic Zakharov system (1.9) pathwise via the random system (2.8), are the reason that we can employ analytical tools such as the normal form method, ultimately giving us access to the rather low regularity setting of the energy space.

For later reference we note that (2.8) after the normal form transform reads

$$\begin{cases} (\partial_t - \mathrm{i}\Delta)(u + \Omega_\mathrm{b}(v,u)) = -\mathrm{i}(vu)_R + \mathrm{i}(b\cdot\nabla u + cu - \mathcal{T}_t(W_2)u) + \mathrm{i}\Omega_\mathrm{b}(|\nabla||u|^2, u) - \mathrm{i}\Omega_\mathrm{b}(v, vu) \\ \qquad\qquad + \mathrm{i}\Omega_\mathrm{b}(v, b\cdot\nabla u + cu), \\ (\partial_t - \mathrm{i}|\nabla|)v = \mathrm{i}|\nabla||u|^2, \end{cases}$$

where Ω_b is a bilinear Fourier multiplier operator related to quadratic high-low interactions and $(vu)_R$ contains the possibly resonant interactions of vu, see (4.3) and (4.7) below for the precise definitions. When working with the normal form transform, we also drop the real parts in (2.8) for convenience, see Remark 4.1.

Another regularity issue is posed by the first order perturbation $b\cdot\nabla u$ on the right-hand side of (2.8) as ∇u only belongs to L^2 but u must be controlled in H^1. This problem is new since these lower order perturbations, generated by the noise, do not appear in the deterministic setting. We deal with the $b\cdot\nabla u$ term by employing local smoothing estimates for the Schrödinger equation. To make optimal use of the local smoothing effect, we adapt an approach developed in [11] for the Schrödinger maps problem, setting up a functional framework combining Strichartz and local smoothing estimates, see Subsection 5.1 below.

(iii) Further reduction for noise regularization. The structural difference between the stochastic Zakharov systems driven by conservative or non-conservative noise is revealed by the rescaling transform.

For the non-conservative noise in Theorem 1.7, we introduce a new rescaling transformation by

$$\begin{cases} z := e^{\widehat{\mu}t - W_1(t)} X, \\ v := Y, \end{cases} \tag{2.12}$$

where

$$\widehat{\mu} := \frac{1}{2}(|\phi_1^{(1)}|^2 - (\phi_1^{(1)})^2). \tag{2.13}$$

Note that in the current non-conservative case we have

$$\operatorname{Re}\widehat{\mu} = (\operatorname{Im}\phi_1^{(1)})^2 > 0, \tag{2.14}$$

while $\operatorname{Re}\widehat{\mu} = 0$ in the conservative case as in Theorem 1.2 and Theorem 1.5.

The stochastic Zakharov system (1.1) can then be reduced to

$$\begin{cases} \mathrm{i}\partial_t z + \Delta z = \operatorname{Re}(v) z, \\ \mathrm{i}\partial_t v + |\nabla| v = -h|\nabla||z|^2, \\ (u(0), v(0)) = (X_0, Y_0) \in H^1 \times H^1 \end{cases} \tag{2.15}$$

with h being the geometric Brownian motion

$$h(t) := e^{2\operatorname{Re}(W_1(t) - \widehat{\mu}t)} = e^{-2\operatorname{Im}\phi_1^{(1)}\beta_1^{(1)}(t) - 2(\operatorname{Im}\phi_1^{(1)})^2 t}, \tag{2.16}$$

or equivalently in the normal form formulation

$$\begin{cases} (\partial_t - \mathrm{i}\Delta)(z + \Omega_{\mathrm{b}}(v,z)) = -\mathrm{i}(vz)_R + \mathrm{i}\Omega_{\mathrm{b}}(h|\nabla||z|^2, z) - \mathrm{i}\Omega_{\mathrm{b}}(v, vz), \\ (\partial_t - \mathrm{i}|\nabla|)v = \mathrm{i}h|\nabla||z|^2. \end{cases} \tag{2.17}$$

The key fact here is that, because of the iterated logarithmic law of Brownian motion

$$\limsup_{t\to\infty} \frac{\beta_1^{(1)}(t)}{\sqrt{2t\log\log t}} = 1, \quad \liminf_{t\to\infty} \frac{\beta_1^{(1)}(t)}{\sqrt{2t\log\log t}} = -1, \quad \mathbb{P}\text{-a.s.}, \tag{2.18}$$

the geometrical Brownian motion h decays exponentially fast at infinity. Heuristically, we expect that it weakens the nonlinearity in the wave equation and thus stabilizes the system.

In order to explore this damping effect, it is important to extract the free wave from the Schrödinger equation. More precisely, we set

$$v_1(t) := e^{\mathrm{i}t|\nabla|} Y_0, \quad v_2 := v - v_1. \tag{2.19}$$

Using (4.10) and the fact that $(\partial_t - \mathrm{i}|\nabla|)v_1 = 0$, we obtain

$$(\partial_t - \mathrm{i}\Delta)\Omega_{\mathrm{b}}(v_1, z) = \mathrm{i}(v_1 z)_{XL} - \mathrm{i}\Omega_{\mathrm{b}}(v_1, vz). \tag{2.20}$$

Subtracting (2.20) from (2.17), we further reduce (2.17) to the system

$$\begin{cases} (\partial_t - \mathrm{i}\Delta + \mathrm{i}v_1)(z + \Omega_{\mathrm{b}}(v_2, z)) = -\mathrm{i}(v_2 z)_R + \mathrm{i}\Omega_{\mathrm{b}}(h|\nabla||z|^2, z) - \mathrm{i}\Omega_{\mathrm{b}}(v_2, (v_1 + v_2)z) + \mathrm{i}v_1\Omega_{\mathrm{b}}(v_2, z), \\ (\partial_t - \mathrm{i}|\nabla|)v_2 = \mathrm{i}h|\nabla||z|^2, \\ (z(0), v_2(0)) = (X_0, 0). \end{cases} \tag{2.21}$$

The key point is then to prove Strichartz estimates for the Schrödinger equation with potential v_1, which is done in Lemma 8.1.

3. Refined rescaling transformations

In this section we perform the refined rescaling transformations, which convert the original stochastic system into a random system, allowing us to apply the normal form method in order to overcome the analytical difficulties inherent to the Zakharov system in the next section.

We start with the equivalence of the solvability of the two systems (1.1) and (2.7).

3.1. **Equivalence via rescaling transformations.**

Theorem 3.1 (Equivalence of the solvability via rescaling transformations)**.**

(i) Let (X,Y) be a solution to (1.1) on $[0,\tau]$ in the sense of Definition 1.1, where τ is an $\{\mathscr{F}_t\}$-stopping time and $(X,Y)\in C([0,\tau];H^1\times L^2)$ $\mathbb{P}$-a.s. Set $u:=e^{-W_1}X$ and $v:=Y-\mathcal{T}_t(W_2)$. Then, (u,v) is an analytically weak solution to (2.7) on $[0,\tau]$ as equations in $H^{-1}\times H^{-1}$.

(ii) Let (u,v) be an analytically weak solution to (2.7) on $[0,\tau]$ as equations in $H^{-1}\times H^{-1}$, where τ is an $\{\mathscr{F}_t\}$-stopping time, and (u,v) is $\{\mathscr{F}_t\}$-adapted and continuous in $H^1\times L^2$. Set $(X,Y):=(e^{W_1}u,v+\mathcal{T}_t(W_2))$. Then, (X,Y) is a solution of (1.1) on $[0,\tau]$ in the sense of Definition 1.1.

Proof. It is not difficult to prove the statements for the wave component, as $\mathcal{T}_t(W_2)$ satisfies the equation

$$\mathrm{d}\mathcal{T}_t(W_2)=\mathrm{i}|\nabla|\mathcal{T}_t(W_2)\,\mathrm{d}t-\mathrm{i}\,\mathrm{d}W_2(t). \tag{3.1}$$

Hence, we focus on the Schrödinger component u and prove the assertion in (i) as the one in (ii) can be proved similarly.

Let $\{e_j\}_{j=1}^\infty\subseteq\mathcal{S}$ be an orthonormal basis of L^2. Let $J_\varepsilon:=(I-\varepsilon\Delta)^{-1}$, $f_\varepsilon:=J_\varepsilon(f)$ for any $f\in\mathcal{S}'$, and set $\tau_M:=\inf\{t\in[0,T]:\|X(t)\|_{L^2}^2>M\}\wedge\tau$, $M\in\mathbb{N}$, with $T>0$ from Definition 1.1. Note that, for $\mathbb{P}$-a.e. $\omega\in\Omega$, there exists M large enough such that $\tau(\omega)=\tau_M(\omega)$ since $X(\omega)\in C([0,\tau(\omega)];H^1)$. This yields that

$$\bigcup_{M\in\mathbb{N}}\{t\le\tau_M(\omega)\}=\{t\le\tau(\omega)\} \tag{3.2}$$

for $\mathbb{P}$-a.e. $\omega\in\Omega$. For the sake of brevity, we omit the dependence on ω below.

Fix any $M\in\mathbb{N}$. We apply the operator J_ε to both sides of the Schrödinger equation in (1.1) to get

$$\mathrm{d}X_\varepsilon=\mathrm{i}\Delta X_\varepsilon\,\mathrm{d}t-\mathrm{i}(X\operatorname{Re}Y)_\varepsilon\,\mathrm{d}t-(\mu X)_\varepsilon\,\mathrm{d}t+\mathrm{i}\sum_{k=1}^\infty(X\phi_k^{(1)})_\varepsilon\,\mathrm{d}\beta_k^{(1)}(t),\quad t\in[0,\tau_M],$$

with $X_\varepsilon(0)=(X_0)_\varepsilon$. Pairing the above equation in L^2 with e_j, we then obtain that

$$\langle X_\varepsilon(t),e_j\rangle=\langle(X_0)_\varepsilon,e_j\rangle+\int_0^t\langle\mathrm{i}\Delta X_\varepsilon-\mathrm{i}(X\operatorname{Re}Y)_\varepsilon-(\mu X)_\varepsilon,e_j\rangle\,\mathrm{d}s+\sum_{k=1}^\infty\int_0^t\left\langle\mathrm{i}(X\phi_k^{(1)})_\varepsilon,e_j\right\rangle\mathrm{d}\beta_k^{(1)}(s) \tag{3.3}$$

for every $j\ge1$ and $t\in[0,\tau_M]$. Moreover, by Itô's calculus we have

$$e^{W_1(t)}=1-\int_0^t\mu e^{W_1(s)}\,\mathrm{d}s+\sum_{k=1}^\infty\int_0^t\mathrm{i}\phi_k^{(1)}e^{W_1(s)}\,\mathrm{d}\beta_k^{(1)}(s),\quad t\in[0,\tau_M].$$

This yields that for any $\varphi\in\mathcal{S}$ and any $j\ge1$

$$\left\langle e_j,e^{W_1(t)}\varphi\right\rangle=\langle e_j,\varphi\rangle-\int_0^t\left\langle e_j,\mu e^{W_1}\varphi\right\rangle\mathrm{d}s+\sum_{k=1}^\infty\int_0^t\left\langle e_j,\mathrm{i}\phi_k^{(1)}e^{W_1}\varphi\right\rangle\mathrm{d}\beta_k^{(1)}(s),\quad t\in[0,\tau_M], \tag{3.4}$$

where we also interchanged the summation over k and the inner product, which is justified due to

$$\mathbb{E}\sup_{t\in[0,\tau_M]}\left|\sum_{k=1}^\infty\int_0^t\left\langle e_j,\phi_k^{(1)}e^{W_1}\right\rangle\mathrm{d}\beta_k^{(1)}\right|^2\lesssim\mathbb{E}\sum_{k=1}^\infty\int_0^{\tau_M}\left|\left\langle e_j,\phi_k^{(1)}e^{W_1}\right\rangle\right|^2\mathrm{d}s\lesssim\tau_M\|e_j\|_{L^2}^2\sum_{k=1}^\infty\|\phi_k^{(1)}\|_{L^2}^2<\infty.$$

Hence, applying the product rule for scalar valued processes and using (3.3) and (3.4), we compute for $t\in[0,\tau_M]$,

$$\begin{aligned}\langle X_\varepsilon(t),e_j\rangle\left\langle e_j,e^{W_1(t)}\varphi\right\rangle&=\langle(X_0)_\varepsilon,e_j\rangle\langle e_j,\varphi\rangle+\int_0^t\langle\mathrm{i}\Delta X_\varepsilon-\mathrm{i}(X\operatorname{Re}Y)_\varepsilon-(\mu X)_\varepsilon,e_j\rangle\left\langle e_j,e^{W_1}\varphi\right\rangle\mathrm{d}s\\&\quad-\int_0^t\langle X_\varepsilon,e_j\rangle\left\langle e_j,\mu e^{W_1}\varphi\right\rangle\mathrm{d}s+\sum_{k=1}^\infty\int_0^t\left\langle\mathrm{i}(X\phi_k^{(1)})_\varepsilon,e_j\right\rangle\left\langle e_j,\mathrm{i}\phi_k^{(1)}e^{W_1}\varphi\right\rangle\mathrm{d}s\\&\quad+\sum_{k=1}^\infty\int_0^t\left\langle\mathrm{i}(X\phi_k^{(1)})_\varepsilon,e_j\right\rangle\left\langle e_j,e^{W_1}\varphi\right\rangle\mathrm{d}\beta_k^{(1)}(s)\end{aligned}$$

$$+\sum_{k=1}^{\infty}\int_0^t \langle X_\varepsilon, e_j\rangle \left\langle e_j, \mathrm{i}\phi_k^{(1)} e^{W_1}\varphi\right\rangle \,\mathrm{d}\beta_k^{(1)}(s). \tag{3.5}$$

Next, we sum both sides over j in order to derive the evolution formula of the process $\langle X_\varepsilon(t), e^{W_1(t)}\varphi\rangle$. Note that the regularity $(X,Y)\in C([0,\tau_M];H^1)\times C([0,\tau_M];L^2)$ and the summability $\{\|\phi_k^{(1)}\|_{L^\infty}\}\in \ell^2$, implied by (1.10) and the Sobolev embedding $H^3\hookrightarrow L^\infty$ in three dimensions, suffice to justify the application of Fubini's theorem to interchange the sum with integration. We take one stochastic integration as an example to illustrate the arguments. Actually, by the Burkholder-Davis-Gundy inequality and the Cauchy inequality,

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,\tau_M\wedge t]}\left|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\int_0^s \left\langle \mathrm{i}(X\phi_k^{(1)})_\varepsilon, e_j\right\rangle \langle e_j, e^{W_1}\varphi\rangle \,\mathrm{d}\beta_k^{(1)}(r)\right|^2\\
&\lesssim \mathbb{E}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\int_0^{\tau_M\wedge t}\left|\left\langle \mathrm{i}(X\phi_k^{(1)})_\varepsilon, e_j\right\rangle \langle e_j, e^{W_1}\varphi\rangle\right|^2 \mathrm{d}s\\
&\lesssim \sum_{k=1}^{\infty}\mathbb{E}\int_0^{\tau_M\wedge t}\|(X\phi_k^{(1)})_\varepsilon\|_{L^2}^2\|e^{W_1}\varphi\|_{L^2}^2\,\mathrm{d}s\\
&\lesssim tM\|\varphi\|_{L^2}^2\sum_{k=1}^{\infty}\|\phi_k^{(1)}\|_{L^\infty}^2<\infty,
\end{aligned}$$

where we also used $|e^{W_1}|=1$ and

$$\sup_{s\in[0,\tau_M\wedge t]}\|(X(s)\phi_k^{(1)})_\varepsilon\|_{L^2}^2 \le \sup_{s\in[0,\tau_M\wedge t]}\|X(s)\|_{L^2}^2\|\phi_k^{(1)}\|_{L^\infty}^2\le M\|\phi_k^{(1)}\|_{L^\infty}^2$$

in the last step. We thus obtain

$$\begin{aligned}
\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\int_0^t \left\langle \mathrm{i}(X\phi_k^{(1)})_\varepsilon, e_j\right\rangle \langle e_j, e^{W_1}\varphi\rangle \,\mathrm{d}\beta_k^{(1)}(s) &= \sum_{k=1}^{\infty}\int_0^t \sum_{j=1}^{\infty}\left\langle \mathrm{i}(X\phi_k^{(1)})_\varepsilon, e_j\right\rangle \langle e_j, e^{W_1}\varphi\rangle \,\mathrm{d}\beta_k^{(1)}(s)\\
&= \sum_{k=1}^{\infty}\int_0^t \left\langle \mathrm{i}(X\phi_k^{(1)})_\varepsilon, e^{W_1}\varphi\right\rangle \mathrm{d}\beta_k^{(1)}(s)
\end{aligned}$$

for all $t\in[0,\tau_M]$. The other terms in (3.5) can be treated in a similar manner.

Hence, interchanging the summation over j with the integration in time in (3.5), we infer that

$$\begin{aligned}
\left\langle X_\varepsilon(t), e^{W_1(t)}\varphi\right\rangle &= \sum_{j=1}^{\infty}\langle X_\varepsilon(t), e_j\rangle \left\langle e_j, e^{W_1(t)}\varphi\right\rangle\\
&= \langle (X_0)_\varepsilon, \varphi\rangle + \int_0^t \left\langle \mathrm{i}\Delta X_\varepsilon - \mathrm{i}(X\operatorname{Re}Y)_\varepsilon - (\mu X)_\varepsilon, e^{W_1}\varphi\right\rangle \mathrm{d}s\\
&\quad - \int_0^t \left\langle X_\varepsilon, \mu e^{W_1}\varphi\right\rangle \mathrm{d}s + \sum_{k=1}^{\infty}\int_0^t \left\langle (X\phi_k^{(1)})_\varepsilon, \phi_k^{(1)} e^{W_1}\varphi\right\rangle \mathrm{d}s\\
&\quad + \sum_{k=1}^{\infty}\int_0^t \left\langle \mathrm{i}(X\phi_k^{(1)})_\varepsilon, e^{W_1}\varphi\right\rangle \mathrm{d}\beta_k^{(1)}(s) - \sum_{k=1}^{\infty}\int_0^t \left\langle \mathrm{i}\phi_k^{(1)}X_\varepsilon, e^{W_1}\varphi\right\rangle \mathrm{d}\beta_k^{(1)}(s)
\end{aligned}\tag{3.6}$$

for all $t\in[0,\tau_M]$. Next we pass to the limit $\varepsilon\to 0$ in (3.6) above. Note that the operator J_ε is contractive on the spaces H^k and $f_\varepsilon\to f$ in H^k as $\varepsilon\to 0$ for every $k=-1,0,1$. Taking into account the regularity of (X,Y) and the ℓ^2-summability of $\{\|\phi_k^{(1)}\|_{L^\infty}\}$, we can interchange the limit with the sum and the integral. For instance, with the Burkholder-Davis-Gundy inequality we get for the stochastic integration

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,\tau_M\wedge t]}\left|\sum_{k=1}^{\infty}\int_0^{\tau_M\wedge t}\left\langle \mathrm{i}(X\phi_k^{(1)})_\varepsilon - \mathrm{i}X\phi_k^{(1)}, e^{W_1}\varphi\right\rangle \mathrm{d}\beta_k^{(1)}(r)\right|^2\\
&\lesssim \mathbb{E}\int_0^{\tau_M\wedge t}\sum_{k=1}^{\infty}\left|\left\langle \mathrm{i}(X\phi_k^{(1)})_\varepsilon - \mathrm{i}X\phi_k^{(1)}, e^{W_1}\varphi\right\rangle\right|^2 \mathrm{d}s
\end{aligned}$$

$$\lesssim \|\varphi\|_{L^2}^2 \mathbb{E}\int_0^{\tau_M\wedge t}\sum_{k=1}^\infty \|(X\phi_k^{(1)})_\varepsilon - X\phi_k^{(1)}\|_{L^2}^2 \,\mathrm{d}s.$$

Since $\|(X\phi_k^{(1)})_\varepsilon - X\phi_k^{(1)}\|_{L^2}^2 \to 0$, $\mathrm{d}\mathbb{P}\otimes \mathrm{d}t$-a.e. and $\sup_{s\in[0,\tau_M]}\|(X(s)\phi_k^{(1)})_\varepsilon\|_{L^2}^2 \le M\|\phi_k^{(1)}\|_{L^\infty}^2 \in L^1_\Omega L^1_t \ell^1_k$, the dominated convergence theorem yields that the above right-hand side converges to zero as $\varepsilon\to 0$, and thus there is a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that

$$\sum_{k=1}^\infty \int_0^t \left\langle \mathrm{i}(X\phi_k^{(1)})_{\varepsilon_n}, e^{W_1}\varphi\right\rangle \mathrm{d}\beta_k^{(1)}(s) \to \sum_{k=1}^\infty \int_0^t \left\langle \mathrm{i}X\phi_k^{(1)}, e^{W_1}\varphi\right\rangle \mathrm{d}\beta_k^{(1)}(s), \quad \text{as } n\to\infty,\ t\in[0,\tau_M],\ \mathbb{P}-a.s.$$

Analogous arguments allow us to pass to the limit for the other terms in (3.6) and thus we arrive at

$${}_{H^{-1}}\left\langle X(t), e^{W_1(t)}\varphi\right\rangle_{H^1} = {}_{H^{-1}}\langle X_0,\varphi\rangle_{H^1} + \int_0^t {}_{H^{-1}}\langle \mathrm{i}\Delta X - \mathrm{i}X\operatorname{Re}Y, e^{W_1}\varphi\rangle_{H^1}\,\mathrm{d}s, \quad t\in[0,\tau_M].$$

Taking into account $\overline{e^{W_1}} = e^{-W_1}$ and interchanging the integration with the duality pairing, we conclude

$$\begin{aligned}{}_{H^{-1}}\langle u(t),\varphi\rangle_{H^1} &= {}_{H^{-1}}\left\langle X_0 + \int_0^t \mathrm{i}e^{-W_1}\Delta(e^{W_1}u) - \mathrm{i}u\operatorname{Re}Y\,\mathrm{d}s, \varphi\right\rangle_{H^1}\\ &= {}_{H^{-1}}\left\langle X_0 + \int_0^t \mathrm{i}e^{-W_1}\Delta(e^{W_1}u) - \mathrm{i}u\operatorname{Re}v - \mathrm{i}u\operatorname{Re}\mathcal{T}_s(W_2)\,\mathrm{d}s, \varphi\right\rangle_{H^1}\end{aligned}$$

for any $\varphi\in\mathcal{S}$. But since $\mathcal{S}$ is dense in H^1, it follows that that u solves the Schrödinger equation in (2.7) in the space H^{-1} on $[0,\tau_M]$, $\mathbb{P}$-a.s.

Finally, we deduce from (3.2) that u solves the Schrödinger equation (2.7) on the whole interval $[0,\tau]$, $\mathbb{P}$-a.s., completing the proof. □

3.2. **Refined rescaling transformations.** In the proof of the wellposedness theorem we will extend a solution of (2.7) up to its maximal existence time. This requires to solve the system (2.7) also on intervals $[\sigma,\sigma+\tau]$ away from zero. In Proposition 3.2 below we use refined rescaling transforms in order to show that this problem can be reduced to solving an appropriate system on $[0,\tau]$.

Proposition 3.2 (Refined rescaling transformations)**.** *Let $\sigma,\tau\colon\Omega\to[0,T]$ such that $\sigma+\tau\le T$.*

(i) Let $(u_\sigma,v_\sigma)\in C([0,\tau];H^1\times L^2)$ be an analytically weak solution to the system

$$\begin{cases}\partial_t u_\sigma(t) = \mathrm{i}e^{-W_{1,\sigma}(t)}\Delta(e^{W_{1,\sigma}(t)}u_\sigma(t)) - \mathrm{i}u_\sigma(t)\operatorname{Re}v_\sigma(t) - \mathrm{i}u_\sigma(t)\operatorname{Re}\mathcal{T}_{\sigma+t,\sigma}(W_2),\\ \partial_t v_\sigma(t) = \mathrm{i}|\nabla|v_\sigma(t) + \mathrm{i}|\nabla||u_\sigma(t)|^2,\end{cases} \tag{3.7}$$

as equations in $H^{-1}\times H^{-1}$, where $W_{1,\sigma}$ and $\mathcal{T}_{\sigma+t,\sigma}(W_2)$ are increments of the noise defined by

$$W_{1,\sigma}(t) := W_1(\sigma+t) - W_1(\sigma), \tag{3.8}$$

$$\mathcal{T}_{\sigma+t,\sigma}(W_2) := -\mathrm{i}\int_\sigma^{\sigma+t} e^{\mathrm{i}(\sigma+t-s)|\nabla|}\,\mathrm{d}W_2(s) \tag{3.9}$$

for all $t\in[0,\tau]$. For any $t\in[\sigma,\sigma+\tau]$, we set

$$u(t) := e^{-W_1(\sigma)}u_\sigma(t-\sigma), \tag{3.10}$$

$$v(t) := v_\sigma(t-\sigma) - e^{\mathrm{i}(t-\sigma)|\nabla|}\mathcal{T}_\sigma(W_2). \tag{3.11}$$

Then, (u,v) is an analytically weak solution of the system (2.7) on $[\sigma,\sigma+\tau]$ with

$$(u(\sigma),v(\sigma)) = (e^{-W_1(\sigma)}u_\sigma(0), v_\sigma(0) - \mathcal{T}_\sigma(W_2)). \tag{3.12}$$

(ii) Conversely, if $(u,v)\in C([\sigma,\sigma+\tau];H^1\times L^2)$ is an analytically weak solution of the system (2.7) on $[\sigma,\sigma+\tau]$ as equations in $H^{-1}\times H^{-1}$, then

$$(u_\sigma(t),v_\sigma(t)) := (e^{W_1(\sigma)}u(\sigma+t), v(\sigma+t) + e^{\mathrm{i}t|\nabla|}\mathcal{T}_\sigma(W_2)), \quad t\in[0,\tau], \tag{3.13}$$

is an analytically weak solution of the system (3.7) on $[0,\tau]$.

Proof. (i) Let us first consider the wave component. By (3.11) and the equation for v_σ in (3.7), we get for any $t \in [\sigma, \sigma+\tau]$

$$\begin{aligned} v(t) &= v_\sigma(t-\sigma) - e^{\mathrm{i}(t-\sigma)|\nabla|}\mathcal{T}_\sigma(W_2) \\ &= v_\sigma(0) + \int_0^{t-\sigma} \mathrm{i}|\nabla| v_\sigma(s)\,\mathrm{d}s + \int_0^{t-\sigma} \mathrm{i}|\nabla||u_\sigma(s)|^2\,\mathrm{d}s - e^{\mathrm{i}(t-\sigma)|\nabla|}\mathcal{T}_\sigma(W_2). \end{aligned} \tag{3.14}$$

Using (3.11) and a change of variables, we derive

$$\begin{aligned} \int_0^{t-\sigma} \mathrm{i}|\nabla| v_\sigma(s)\,\mathrm{d}s &= \int_0^{t-\sigma} \mathrm{i}|\nabla|(v(s+\sigma) + e^{\mathrm{i}s|\nabla|}\mathcal{T}_\sigma(W_2))\,\mathrm{d}s \\ &= \int_\sigma^t \mathrm{i}|\nabla|(v(s) + e^{\mathrm{i}(s-\sigma)|\nabla|}\mathcal{T}_\sigma(W_2))\,\mathrm{d}s \\ &= \int_\sigma^t \mathrm{i}|\nabla| v(s)\,\mathrm{d}s + (e^{\mathrm{i}(t-\sigma)|\nabla|} - 1)\mathcal{T}_\sigma(W_2). \end{aligned} \tag{3.15}$$

Moreover, we obtain from (3.10) and $|e^{W_1}| = 1$ that

$$\int_0^{t-\sigma} \mathrm{i}|\nabla||u_\sigma(s)|^2\,\mathrm{d}s = \int_0^{t-\sigma} \mathrm{i}|\nabla||u(\sigma+s)|^2\,\mathrm{d}s = \int_\sigma^t \mathrm{i}|\nabla||u(s)|^2\,\mathrm{d}s. \tag{3.16}$$

Hence, plugging (3.15) and (3.16) into (3.14), we conclude

$$\begin{aligned} v(t) &= v_\sigma(0) - \mathcal{T}_\sigma(W_2) + \int_\sigma^t i|\nabla| v(s)\,\mathrm{d}s + \int_\sigma^t i|\nabla||u(s)|^2\,\mathrm{d}s \\ &= v(\sigma) + \int_\sigma^t i|\nabla| v(s) + i|\nabla||u(s)|^2\,\mathrm{d}s, \end{aligned}$$

which yields that v satisfies the wave equation in (2.7) on $[\sigma, \sigma+\tau]$.

Regarding the Schrödinger component, we infer from (3.10) and the equation for u_σ in (3.7) that

$$\begin{aligned} u(\sigma+t) &= e^{-W_1(\sigma)} u_\sigma(t) \\ &= e^{-W_1(\sigma)} u_\sigma(0) + e^{-W_1(\sigma)} \int_0^t \mathrm{i}e^{-W_{1,\sigma}(s)}\Delta(e^{W_{1,\sigma}(s)}u_\sigma(s)) - \mathrm{i}u_\sigma(s)\operatorname{Re}(v_\sigma(s) + \mathcal{T}_{\sigma+s,\sigma}(W_2))\,\mathrm{d}s. \end{aligned}$$

for any $t \in [0, \tau]$. The definition of $W_{1,\sigma}$ in (3.8) then yields

$$\begin{aligned} u(\sigma+t) = e^{-W_1(\sigma)} u_\sigma(0) + \int_0^t &\mathrm{i}e^{-W_1(\sigma+s)}\Delta(e^{W_1(\sigma+s)}e^{-W_1(\sigma)}u_\sigma(s)) \\ &- \mathrm{i}e^{-W_1(\sigma)}u_\sigma(s)\operatorname{Re}(v_\sigma(s) + \mathcal{T}_{\sigma+s,\sigma}(W_2))\,\mathrm{d}s. \end{aligned}$$

Employing (3.10) again, the identity

$$\begin{aligned} v_\sigma(s) + \mathcal{T}_{\sigma+s,\sigma}(W_2) &= v(\sigma+s) + e^{is|\nabla|}\mathcal{T}_\sigma(W_2) + \mathcal{T}_{\sigma+s,\sigma}(W_2) \\ &= v(\sigma+s) + \mathcal{T}_{\sigma+s}(W_2), \end{aligned}$$

and a change of variables, we infer

$$\begin{aligned} u(\sigma+t) &= e^{-W_1(\sigma)} u_\sigma(0) + \int_0^t \mathrm{i}e^{-W_1(\sigma+s)}\Delta(e^{W_1(\sigma+s)}u(\sigma+s)) - \mathrm{i}u(\sigma+s)\operatorname{Re}(v(\sigma+s) + \mathcal{T}_{\sigma+s}(W_2))\,\mathrm{d}s \\ &= e^{-W_1(\sigma)} u_\sigma(0) + \int_\sigma^{\sigma+t} \mathrm{i}e^{-W_1(s)}\Delta(e^{W_1(s)}u(s)) - \mathrm{i}u(s)\operatorname{Re} v(s) - \mathrm{i}u(s)\operatorname{Re}\mathcal{T}_s(W_2)\,\mathrm{d}s \end{aligned}$$

as an equation in H^{-1} for all $t \in [0, \tau]$. We conclude that u is an analytically weak solution of the Schrödinger equation in (2.7) on $[\sigma, \sigma+\tau]$.

(ii) Using (3.13) and the equation for the wave component v in (2.7) on $[\sigma, \sigma+\tau]$, we obtain

$$v_\sigma(t) = v(\sigma) + \int_\sigma^{\sigma+t} \mathrm{i}|\nabla| v(s)\,\mathrm{d}s + \int_\sigma^{\sigma+t} \mathrm{i}|\nabla||u(s)|^2\,\mathrm{d}s + e^{\mathrm{i}t|\nabla|}\mathcal{T}_\sigma(W_2)$$

for all $t \in [0,\tau]$. Employing (3.13) once more, we derive with a change of variables

$$\begin{aligned}\int_\sigma^{\sigma+t} \mathrm{i}|\nabla| v(s)\,\mathrm{d}s &= \int_\sigma^{\sigma+t} \mathrm{i}|\nabla|(v_\sigma(s-\sigma) - e^{\mathrm{i}(s-\sigma)|\nabla|}\mathcal{T}_\sigma(W_2))\,\mathrm{d}s \\ &= \int_0^t \mathrm{i}|\nabla| v_\sigma(s)\,\mathrm{d}s - \int_0^t \mathrm{i}|\nabla| e^{\mathrm{i}s|\nabla|}\mathcal{T}_\sigma(W_2))\,\mathrm{d}s \\ &= \int_0^t \mathrm{i}|\nabla| v_\sigma(s)\,\mathrm{d}s - e^{\mathrm{i}t|\nabla|}\mathcal{T}_\sigma(W_2) + \mathcal{T}_\sigma(W_2).\end{aligned}$$

Moreover, since $|e^{W_1(\sigma)}| = 1$, (3.13) and a change of variables also yield

$$\int_\sigma^{\sigma+t} \mathrm{i}|\nabla||u(s)|^2\,\mathrm{d}s = \int_0^t \mathrm{i}|\nabla||u(\sigma+s)|^2\,\mathrm{d}s = \int_0^t \mathrm{i}|\nabla||u_\sigma(s)|^2\,\mathrm{d}s.$$

Combining the last three identities, we thus arrive at

$$\begin{aligned}v_\sigma(t) &= v(\sigma) + \mathcal{T}_\sigma(W_2) + \int_0^t \mathrm{i}|\nabla| v_\sigma(s)\,\mathrm{d}s + \int_0^t \mathrm{i}|\nabla||u_\sigma(s)|^2\,\mathrm{d}s \\ &= v_\sigma(0) + \int_0^t \mathrm{i}|\nabla| v_\sigma(s)\,\mathrm{d}s + \mathrm{i}|\nabla||u_\sigma(s)|^2\,\mathrm{d}s\end{aligned}$$

for all $t \in [0,\tau]$. Consequently, v_σ satisfies the wave equation in (3.7) on $[0,\tau]$.

Concerning the Schrödinger component, we first note that the equation for u in (2.7) on $[\sigma,\sigma+\tau]$ implies

$$\begin{aligned}u_\sigma(t) &= e^{W_1(\sigma)}u(\sigma+t) \\ &= e^{W_1(\sigma)}u(\sigma) + e^{W_1(\sigma)}\int_\sigma^{\sigma+t} \mathrm{i}e^{-W_1(s)}\Delta(e^{W_1(s)}u(s)) - \mathrm{i}u(s)\operatorname{Re} v(s) - \mathrm{i}u(s)\operatorname{Re}\mathcal{T}_s(W_2)\,\mathrm{d}s \\ &= u_\sigma(0) + e^{W_1(\sigma)}\int_0^t \mathrm{i}e^{-W_1(\sigma+s)}\Delta(e^{W_1(\sigma+s)}u(\sigma+s)) - \mathrm{i}u(\sigma+s)\operatorname{Re} v(\sigma+s) \\ &\qquad\qquad\qquad\qquad - \mathrm{i}u(\sigma+s)\operatorname{Re}\mathcal{T}_{\sigma+s}(W_2)\,\mathrm{d}s\end{aligned}$$

for all $t \in [0,\tau]$. Employing (3.8) and (3.13), we thus obtain

$$\begin{aligned}u_\sigma(t) &= u_\sigma(0) + \int_0^t \mathrm{i}e^{-(W_1(\sigma+s)-W_1(\sigma))}\Delta(e^{W_1(\sigma+s)-W_1(\sigma)}e^{W_1(\sigma)}u(\sigma+s))\,\mathrm{d}s \\ &\qquad - \int_0^t \mathrm{i}e^{W_1(\sigma)}u(\sigma+s)\operatorname{Re}(v(\sigma+s) + \mathcal{T}_{\sigma+s}(W_2))\,\mathrm{d}s \\ &= u_\sigma(0) + \int_0^t \mathrm{i}e^{-W_{1,\sigma}(s)}\Delta(e^{W_{1,\sigma}}u_\sigma(s)) - \mathrm{i}u_\sigma(s)\operatorname{Re}(v_\sigma(s) - e^{\mathrm{i}s|\nabla|}\mathcal{T}_\sigma(W_2) + \mathcal{T}_{\sigma+s}(W_2))\,\mathrm{d}s\end{aligned}$$

for any $t \in [0,\tau]$, where the equation is taken in the space H^{-1}. Taking into account the identity

$$-e^{\mathrm{i}s|\nabla|}\mathcal{T}_\sigma(W_2) + \mathcal{T}_{\sigma+s}(W_2) = \mathcal{T}_{\sigma+s,\sigma}(W_2),$$

we arrive at

$$u_\sigma(t) = u_\sigma(0) + \int_0^t \mathrm{i}e^{-W_{1,\sigma}(s)}\Delta(e^{W_{1,\sigma}(s)}u_\sigma(s)) - \mathrm{i}u_\sigma(s)\operatorname{Re} v_\sigma(s) - \mathrm{i}u_\sigma(s)\operatorname{Re}\mathcal{T}_{\sigma+s,\sigma}(W_2)\,\mathrm{d}s \tag{3.17}$$

for all $t \in [0,\tau]$. We conclude that u_σ solves the Schrödinger equation in (3.7) on $[0,\tau]$, finishing the proof. ☐

3.3. **Gluing solutions.** When we extend a solution of (2.7) up to its maximal existence time, we have to concatenate the solutions provided by Proposition 3.2. The next result explains how we can glue together these solutions.

Proposition 3.3 (Gluing solutions)**.** *Let* $(u_1,v_1) \in C([0,\sigma];H^1)\times C([0,\sigma];L^2)$ *be an analytically weak solution to* (2.7) *on* $[0,\sigma]$*, and let* $(u_\sigma,v_\sigma) \in C([0,\tau];H^1)\times C([0,\tau];L^2)$ *be an analytically weak solution to* (3.7) *on* $[0,\tau]$ *with the initial condition*

$$(u_\sigma(0),v_\sigma(0)) := (e^{W_1(\sigma)}u_1(\sigma), v_1(\sigma) + \mathcal{T}_\sigma(W_2)).$$

For any $t \in [0, \sigma+\tau]$, we set

$$u(t) := \begin{cases} u_1(t), & \text{if } t \in [0,\sigma), \\ e^{-W_1(\sigma)} u_\sigma(t-\sigma), & \text{if } t \in [\sigma, \sigma+\tau], \end{cases} \qquad v(t) := \begin{cases} v_1(t), & \text{if } t \in [0,\sigma), \\ v_\sigma(t-\sigma) - e^{\mathrm{i}(t-\sigma)|\nabla|}\mathcal{T}_\sigma(W_2)), & \text{if } t \in [\sigma, \sigma+\tau]. \end{cases} \tag{3.18}$$

Then, $(u, v) \in C([0, \sigma+\tau]; H^1 \times L^2)$ is an analytically weak solution to the system (2.7) on the time interval $[0, \sigma+\tau]$.

Proof. By (3.18), (u, v) solves (2.7) on $[0, \sigma]$. We thus focus on the case $t \in [\sigma, \sigma+\tau]$ in the following.

Proposition 3.2 (i) yields that for every $t \in [\sigma, \sigma+\tau]$,

$$u(t) = u(\sigma) + \int_\sigma^t \mathrm{i}e^{-W_1(s)}\Delta(e^{W_1(s)}u(s)) - \mathrm{i}u(s)\operatorname{Re} v(s) - \mathrm{i}u(s)\operatorname{Re}\mathcal{T}_s(W_2)\,\mathrm{d}s.$$

Since $u(\sigma) = e^{-W_1(\sigma)}u_\sigma(0) = u_1(\sigma)$ and u_1 solves the Schrödinger equation in (2.7) on $[0, \sigma]$, we also have

$$u(\sigma) = u_1(\sigma) = X_0 + \int_0^\sigma \mathrm{i}e^{-W_1(s)}\Delta(e^{W_1(s)}u(s)) - \mathrm{i}u(s)\operatorname{Re} v(s) - \mathrm{i}u(s)\operatorname{Re}\mathcal{T}_s(W_2)\,\mathrm{d}s.$$

Therefore, combining the previous two equations, we arrive at

$$u(t) = X_0 + \int_0^t \mathrm{i}e^{-W_1(s)}\Delta(e^{W_1(s)}u(s)) - \mathrm{i}u(s)\operatorname{Re} v(s) - \mathrm{i}u(s)\operatorname{Re}\mathcal{T}_s(W_2)\,\mathrm{d}s$$

for all $t \in [\sigma, \sigma+\tau]$, which yields that u solves the Schrödinger equation in (2.7) on $[\sigma, \sigma+\tau]$.

Turning to the wave component, we apply Proposition 3.2 (i) again in order to derive that for every $t \in [\sigma, \sigma+\tau]$,

$$\begin{aligned} v(t) &= v(\sigma) + \int_\sigma^t \mathrm{i}|\nabla|v(s) + \mathrm{i}|\nabla||u(s)|^2\,\mathrm{d}s \\ &= v_\sigma(0) - \mathcal{T}_\sigma(W_2) + \int_\sigma^t \mathrm{i}|\nabla|v(s) + \mathrm{i}|\nabla||u(s)|^2\,\mathrm{d}s. \end{aligned}$$

Since v_1 solves the wave equation in (2.7) on $[0, \sigma]$, we moreover get

$$v_\sigma(0) - \mathcal{T}_\sigma(W_2) = v_1(\sigma) = Y_0 + \int_0^\sigma \mathrm{i}|\nabla|v(s) + \mathrm{i}|\nabla||u(s)|^2\,\mathrm{d}s,$$

which implies

$$v(t) = Y_0 + \int_0^t \mathrm{i}|\nabla||u(s)|^2 + \mathrm{i}|\nabla|v(s)\,\mathrm{d}s$$

for all $t \in [\sigma, \sigma+\tau]$. Consequently, v also satisfies the wave equation in (2.7) on $[\sigma, \sigma+\tau]$, completing the proof. □

4. Normal form reduction

In this section we perform the normal form transform for the stochastic Zakharov system in the formulation (2.8). It is based on an integration by parts on the Fourier side exploiting the resonance structure of the Zakharov system. It transforms the system into a form which can be solved by a fixed point argument based on Strichartz and local smoothing estimates.

We closely follow [31], where the normal form transform for the Zakharov system was introduced. However, we note that in [31] and the subsequent works [27, 28, 30] both the Schrödinger and the wave part of the Zakharov system were transformed. For our purposes here, it is sufficient to only transform the Schödinger part as in [51] so that we content ourselves with this simpler form.

We first fix some notation. Take an even function $\eta_0 \in C_c^\infty(\mathbb{R})$ such that $0 \le \eta_0 \le 1$, $\eta_0(x) = 1$ for $|x| \le \frac{5}{4}$, and $\eta_0(x) = 0$ for $|x| \ge \frac{8}{5}$. For every dyadic number $N \in 2^{\mathbb{Z}}$ we set

$$\chi_N(\xi) = \eta_0(|\xi|/N) - \eta_0(2|\xi|/N), \qquad \chi_{\le N}(\xi) = \eta_0(|\xi|/N)$$

for all $\xi \in \mathbb{R}^3$. To each of these symbols we associate the Littlewood-Paley projectors

$$P_N f = \mathcal{F}^{-1}(\chi_N \hat{f}), \qquad P_{\le N} f = \mathcal{F}^{-1}(\chi_{\le N}\hat{f}),$$

where $\hat{f} = \mathcal{F}f$ denotes the Fourier transform of f.

We next define different frequency interactions by

$$(fg)_{LH} := \sum_{N\in 2^{\mathbb{Z}}} P_{\le K^{-1}N} f\, P_N g, \qquad (fg)_{HL} := (gf)_{LH},$$
$$(fg)_{HH} := \sum_{\substack{N_1/N_2, N_2/N_1 < K,\\ N_1,N_2\in 2^{\mathbb{Z}}}} P_{N_1} f\, P_{N_2} g, \tag{4.1}$$

where $K \in 2^{\mathbb{Z}}$ with $K \ge 2^5$ is a dyadic integer which will be fixed in the contraction argument. To distinguish resonant interactions, we also introduce

$$(fg)_{1L} := \sum_{\substack{2^{-1}\le N\le 2,\\ N\in 2^{\mathbb{Z}}}} P_N f\, P_{\le K^{-1}N} g, \qquad (fg)_{XL} := \sum_{\substack{|\log_2 N|>1,\\ N\in 2^{\mathbb{Z}}}} P_N f\, P_{\le K^{-1}N} g. \tag{4.2}$$

Note that

$$fg = (fg)_{HL} + (fg)_{HH} + (fg)_{LH} = (fg)_{XL} + (fg)_{1L} + (fg)_{HH} + (fg)_{LH}.$$

For later use we set

$$(fg)_R := (fg)_{1L+HH+LH}. \tag{4.3}$$

Finally, we write $\mathcal{P}_*$ for the symbol of the bilinear operator with index $*$, i.e.,

$$\mathcal{F}(fg)_*(\xi) := \int_{\mathbb{R}^3} \mathcal{P}_*(\xi-\eta,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,\mathrm{d}\eta$$

with $* \in \{1L, XL, HH, LH, R\}$.

Remark 4.1. Since $\mathrm{Re}(v) = \frac{1}{2}(v+\overline{v})$, the nonlinearity $\mathrm{Re}(v)u$ in (2.8) can be written as $\frac{1}{2}vu + \frac{1}{2}\overline{v}u$. The term $\overline{v}u$ can be treated in the same way as vu so that we replace $\mathrm{Re}(v)u$ by vu in (2.8) for simplicity. Moreover, our arguments work for $\mathrm{Re}(\mathcal{T}.(W_2))$ in the same way as for $\mathcal{T}.(W_2)$, so that we also drop the real part here for notational convenience.

We write (2.8) with initial data $(u_0, v_0) \in H^1 \times L^2$ in the Duhamel formulation

$$\begin{cases} u(t) = e^{\mathrm{i}t\Delta}u_0 - \mathrm{i}\displaystyle\int_0^t e^{\mathrm{i}(t-s)\Delta}(vu - b\cdot\nabla u - cu + \mathcal{T}.(W_2)u)(s)\,\mathrm{d}s, \\ v(t) = e^{\mathrm{i}t|\nabla|}v_0 + \mathrm{i}\displaystyle\int_0^t e^{\mathrm{i}(t-s)|\nabla|}(|\nabla||u|^2)(s)\,\mathrm{d}s. \end{cases} \tag{4.4}$$

The normal form transform (for the Schrödinger part) builds on the fact that the resonance function

$$\omega_{\mathrm{r}}(\xi-\eta,\eta) := |\xi|^2 + |\xi-\eta| - |\eta|^2$$

does not vanish on the support of $\mathcal{P}_{XL}$, since $1 \nsim |\xi-\eta| \sim |\xi| \gg |\eta|$ for $(\xi-\eta,\eta) \in \operatorname{supp}\mathcal{P}_{XL}$. Taking the Fourier transform on both sides of (4.4), we obtain

$$\hat{u}(t) = e^{-\mathrm{i}t|\xi|^2}\hat{u}_0 - \mathrm{i}\int_0^t e^{-\mathrm{i}(t-s)|\xi|^2}\mathcal{F}(vu)(s)\,\mathrm{d}s + \mathrm{i}\int_0^t e^{-\mathrm{i}(t-s)|\xi|^2}\mathcal{F}(b\cdot\nabla u + cu - \mathcal{T}.(W_2)u)(s)\,\mathrm{d}s,$$
$$\hat{v}(t) = e^{\mathrm{i}t|\xi|}\hat{v}_0 + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)|\xi|}|\xi||\mathcal{F}(|u|^2)(s)\,\mathrm{d}s,$$

and thus

$$\begin{cases} \partial_t(e^{\mathrm{i}t|\xi|^2}\hat{u}(t,\xi)) = -\mathrm{i}e^{\mathrm{i}t|\xi|^2}\mathcal{F}(vu)(t,\xi) + \mathrm{i}e^{\mathrm{i}t|\xi|^2}\mathcal{F}(b\cdot\nabla u + cu - \mathcal{T}.(W_2)u)(t,\xi), \\ \partial_t(e^{-\mathrm{i}t|\xi|}\hat{v}(t,\xi)) = \mathrm{i}e^{-\mathrm{i}t|\xi|}|\xi|\mathcal{F}(|u|^2)(t,\xi). \end{cases} \tag{4.5}$$

Writing

$$\begin{aligned} \hat{u}(t) = e^{-\mathrm{i}t|\xi|^2}\hat{u}_0 &- \mathrm{i}\int_0^t e^{-\mathrm{i}(t-s)|\xi|^2}\mathcal{F}(vu)_{XL}(s)\,\mathrm{d}s - \mathrm{i}\int_0^t e^{-\mathrm{i}(t-s)|\xi|^2}\mathcal{F}(vu)_R(s)\,\mathrm{d}s \\ &+ \mathrm{i}\int_0^t e^{-\mathrm{i}(t-s)|\xi|^2}\mathcal{F}(b\cdot\nabla u + cu - \mathcal{T}.(W_2)u)(s)\,\mathrm{d}s =: I + II + III + IV, \end{aligned} \tag{4.6}$$

we obtain

$$
\begin{aligned}
II &= -\mathrm{i}\int_0^t \int_{\mathbb{R}^3} e^{-\mathrm{i}(t-s)|\xi|^2} \mathcal{P}_{XL}(\xi-\eta,\eta)\hat{v}(s,\xi-\eta)\hat{u}(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \\
&= -\mathrm{i}e^{-\mathrm{i}t|\xi|^2}\int_0^t \int_{\mathbb{R}^3} e^{\mathrm{i}s(|\xi|^2+|\xi-\eta|-|\eta|^2)}\mathcal{P}_{XL}(\xi-\eta,\eta)[e^{-\mathrm{i}s|\xi-\eta|}\hat{v}(s,\xi-\eta)][e^{\mathrm{i}s|\eta|^2}\hat{u}(s,\eta)]\,\mathrm{d}\eta\,\mathrm{d}s \\
&= -e^{-\mathrm{i}t|\xi|^2}\int_0^t \int_{\mathbb{R}^3} \frac{\mathcal{P}_{XL}(\xi-\eta,\eta)}{\omega_{\mathrm{r}}(\xi-\eta,\eta)}\partial_s e^{\mathrm{i}s\omega_{\mathrm{r}}(\xi-\eta,\eta)}[e^{-\mathrm{i}s|\xi-\eta|}\hat{v}(s,\xi-\eta)][e^{\mathrm{i}s|\eta|^2}\hat{u}(s,\eta)]\,\mathrm{d}\eta\,\mathrm{d}s.
\end{aligned}
$$

Integrating by parts, using (4.4) and writing Ω_{b} for the bilinear operator

$$
\Omega_{\mathrm{b}}(f,g) := \mathcal{F}^{-1}\int_{\mathbb{R}^3} \frac{\mathcal{P}_{XL}(\xi-\eta,\eta)}{\omega_{\mathrm{r}}(\xi-\eta,\eta)}\hat{f}(\xi-\eta)\hat{g}(\eta)\,\mathrm{d}\eta, \tag{4.7}
$$

we get

$$
\begin{aligned}
II =& -\mathcal{F}\Omega_{\mathrm{b}}(v,u)(t,\xi) + e^{-\mathrm{i}t|\xi|^2}\mathcal{F}\Omega_{\mathrm{b}}(v_0,u_0)(\xi) \\
&+ e^{-\mathrm{i}t|\xi|^2}\int_0^t \int_{\mathbb{R}^3} \frac{\mathcal{P}_{XL}(\xi-\eta,\eta)}{\omega_{\mathrm{r}}(\xi-\eta,\eta)} e^{\mathrm{i}s\omega_{\mathrm{r}}(\xi-\eta,\eta)}[e^{-\mathrm{i}s|\xi-\eta|}\mathrm{i}|\xi-\eta|\mathcal{F}(|u|^2)(s,\xi-\eta)][e^{\mathrm{i}s|\eta|^2}\hat{u}(s,\eta)]\,\mathrm{d}\eta\,\mathrm{d}s \\
&+ e^{-\mathrm{i}t|\xi|^2}\int_0^t \int_{\mathbb{R}^3} \frac{\mathcal{P}_{XL}(\xi-\eta,\eta)}{\omega_{\mathrm{r}}(\xi-\eta,\eta)} e^{\mathrm{i}s\omega_{\mathrm{r}}(\xi-\eta,\eta)}[e^{-\mathrm{i}s|\xi-\eta|}\hat{v}(s,\xi-\eta)] \\
&\qquad\qquad \cdot [e^{\mathrm{i}s|\eta|^2}(-\mathrm{i})\mathcal{F}(vu - b\cdot\nabla u - cu + \mathcal{T}_{\cdot}(W_2)u)(s,\eta)]\,\mathrm{d}\eta\,\mathrm{d}s \\
=& -\mathcal{F}\Omega_{\mathrm{b}}(v,u)(t,\xi) + e^{-\mathrm{i}t|\xi|^2}\mathcal{F}\Omega_{\mathrm{b}}(v_0,u_0)(\xi) + \mathrm{i}\mathcal{F}\Big(\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(|\nabla||u|^2,u)(s)\,\mathrm{d}s\Big)(\xi) \\
&+ \mathrm{i}\mathcal{F}\Big(\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(v,-vu+b\cdot\nabla u + cu - \mathcal{T}_{\cdot}(W_2)u)(s)\,\mathrm{d}s\Big)(\xi).
\end{aligned} \tag{4.8}
$$

Inserting this formula into (4.6), we have transformed system (4.4) into

$$
\begin{aligned}
u(t) =& e^{\mathrm{i}t\Delta}u_0 + e^{\mathrm{i}t\Delta}\Omega_{\mathrm{b}}(v_0,u_0) - \Omega_{\mathrm{b}}(v,u)(t) - \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}(vu)_R(s)\,\mathrm{d}s \\
&+ \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}(b\cdot\nabla u + cu - \mathcal{T}_{\cdot}(W_2)u)(s)\,\mathrm{d}s + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(|\nabla||u|^2,u)(s)\,\mathrm{d}s \\
&+ \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(v,-vu+b\cdot\nabla u + cu - \mathcal{T}_{\cdot}(W_2)u)(s)\,\mathrm{d}s, \\
v(t) =& e^{\mathrm{i}t|\nabla|}v_0 + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)|\nabla|}(|\nabla||u|^2)(s)\,\mathrm{d}s.
\end{aligned} \tag{4.9}
$$

Correspondingly we also have

$$
\begin{cases}
(\partial_t - \mathrm{i}\Delta)(u + \Omega_{\mathrm{b}}(v,u)) = -\mathrm{i}(vu)_R + \mathrm{i}(b\cdot\nabla u + cu - \mathcal{T}_{\cdot}(W_2)u) + \mathrm{i}\Omega_{\mathrm{b}}(|\nabla||u|^2,u) \\
\qquad\qquad + \mathrm{i}\Omega_{\mathrm{b}}(v,-vu+b\cdot\nabla u + cu - \mathcal{T}_{\cdot}(W_2)u), \\
(\partial_t - \mathrm{i}|\nabla|)v = \mathrm{i}|\nabla||u|^2.
\end{cases}
$$

Remark 4.2. (i) The above computations in (4.8) also show that

$$
(\partial_t - \mathrm{i}\Delta)\Omega_{\mathrm{b}}(v,u) = \mathrm{i}(vu)_{XL} + \Omega_{\mathrm{b}}((\partial_t - \mathrm{i}|\nabla|)v,u) + \Omega_{\mathrm{b}}(v,(\partial_t - \mathrm{i}\Delta)u). \tag{4.10}
$$

(ii) Note that also (3.7) is of the form (2.8) with $\mathcal{T}_t(W_2)$ replaced by $\mathcal{T}_{\sigma+t,\sigma}(W_2)$. In particular, the normal form for (3.7) is given by (4.9) with $\mathcal{T}_t(W_2)$ replaced by $\mathcal{T}_{\sigma+t,\sigma}(W_2)$.

(iii) While we did not specify the regularity of the involved functions in the derivation above and one might think of sufficiently smooth solutions at first, we rigorously prove in Proposition B.1 in the appendix that the normal form (4.9) is equivalent to the Duhamel formulation (2.8) in the energy space.

5. Multilinear estimates

In this section we prove the key multilinear estimates which we will use to solve the Zakharov system in its normal form formulation (4.9). We first introduce our functional framework and then estimate the boundary terms, the quadratic interactions, and the cubic interactions on the right-hand side of (4.9).

Throughout this section, I denotes a bounded subinterval of $[0,\infty)$ and $t_0 := \inf I$.

5.1. **Functional setting.** Our functional framework combines Strichartz and local smoothing norms. We start by introducing the lateral spaces we will use to capture the local smoothing effect of the Schrödinger flow.

Let $\mathbf{e} \in \mathbb{S}^2$ and $\mathcal{P}_{\mathbf{e}} = \{\xi \in \mathbb{R}^3 \,|\, \xi \cdot \mathbf{e} = 0\}$ with the induced Euclidean measure. For $p, q \in [1,\infty]$, we define

$$\|f\|_{L^{p,q}_{\mathbf{e}}(I\times\mathbb{R}^3)} := \Big(\int_{\mathbb{R}} \Big(\int_{I\times\mathcal{P}_{\mathbf{e}}} |f(t, r\mathbf{e}+y)|^q \,\mathrm{d}t\,\mathrm{d}y\Big)^{p/q} \mathrm{d}r\Big)^{1/p} \tag{5.1}$$

with the usual adaptions if $p=\infty$ or $q=\infty$.

We fix a nonnegative and symmetric function $\phi \in C_0^\infty(\mathbb{R})$ such that $\phi(r) = 0$ if $|r| \le \frac{1}{8}$ or $|r| > 4$ and $\phi(r) = 1$ if $\frac{1}{4} \le |r| \le 2$, and set $\phi_N(r) = \phi(r/N)$. Consequently, we have

$$\prod_{j=1}^{3} (1-\phi_N(\xi_j)) = 0 \tag{5.2}$$

for all $\xi \in \mathbb{R}^3$ with $N/2 < |\xi| < 2N$. We finally define $P_{N,\mathbf{e}} := \mathcal{F}_x^{-1}\phi_N(\xi\cdot\mathbf{e})\mathcal{F}_x$. Note that (5.2) implies

$$P_N f = \sum_{j=1}^{3} P_{N,\mathbf{e}_j}\Big[\prod_{l=1}^{j-1}(1-P_{N,\mathbf{e}_l})\Big] P_N f, \tag{5.3}$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ denotes the standard basis of $\mathbb{R}^3$.

We introduce the following norms. For every $N \in 2^{\mathbb{Z}}$ we set

$$\|f\|_{\mathbb{X}(I)} := \|f\|_{L^\infty(I;H^1_x)} + \Big(\sum_{N\in 2^{\mathbb{Z}}} \|P_N f\|^2_{\mathbb{X}_N(I)}\Big)^{\frac{1}{2}} \tag{5.4}$$

with

$$\|f\|_{\mathbb{X}_N(I)} := \langle N\rangle \|f\|_{L^2(I;L^6_x)} + \sum_{j=1}^{3} N^{\frac{3}{2}} \|P_{N,\mathbf{e}_j} f\|_{L^{\infty,2}_{\mathbf{e}_j}(I\times\mathbb{R}^3)},$$

and

$$\|f\|_{\mathbb{Y}(I)} := \|f\|_{L^\infty(I;L^2)}.$$

In order to estimate the nonlinear terms, we introduce

$$\|g\|_{\mathbb{G}(I)} := \Big(\sum_{N\in 2^{Z}} \|P_N g\|^2_{\mathbb{G}_N(I)}\Big)^{\frac{1}{2}} \tag{5.5}$$

with

$$\|g\|_{\mathbb{G}_N(I)} := \inf_{g=g_1+g_2+g_3} \Big(\langle N\rangle \|g_1\|_{L^1(I;L^2_x)} + \langle N\rangle \|g_2\|_{L^{\frac{8}{5}}(I;L^{\frac{4}{3}}_x)} + \sum_{j=1}^{3} N^{\frac{1}{2}} \|g_3\|_{L^{1,2}_{\mathbf{e}_j}(I\times\mathbb{R}^3)}\Big), \quad N \in 2^{\mathbb{Z}}.$$

We further set

$$\mathbb{X}(I) := \{f \in C(I;H^1_x) \colon \|f\|_{\mathbb{X}(I)} < \infty\}, \qquad \mathbb{Y}(I) := L^\infty(I;L^2_x), \qquad \mathbb{G}(I) := \{g \in L^1(I;L^2_x) \colon \|g\|_{\mathbb{G}(I)} < \infty\}. \tag{5.6}$$

We next provide the estimates for the linear flow of the Schrödinger equation which we will use in the following. Part (i) contains the classical Strichartz estimate, while the local smoothing estimate in part (ii) follows from (4.18) in [35].

Lemma 5.1 (Strichartz and local smoothing estimates)**.** *Let $N \in 2^{\mathbb{Z}}$, $\mathbf{e} \in \mathbb{S}^2$, and (q,p) be Schrödinger admissible, i.e., $p, q \in [2,\infty]$ with $\frac{2}{q} + \frac{3}{p} = \frac{3}{2}$. Then, for any $f \in L^2$ we have*

(i) *Strichartz estimate:*

$$\|e^{\mathrm{i}(\cdot-t_0)\Delta}P_N f\|_{L^q(I;L^p_x)} \lesssim \|P_N f\|_{L^2}.$$

(ii) *Local smoothing estimate:*

$$\|e^{\mathrm{i}(\cdot-t_0)\Delta}P_{N,\mathbf{e}} f\|_{L^{\infty,2}_{\mathbf{e}}(I\times\mathbb{R}^d)} \lesssim N^{-\frac{1}{2}}\|f\|_{L^2}.$$

Here and in the following we write $A \lesssim B$ if there is a constant $C > 0$ such that $A \leq CB$. Note that Lemma 5.1 immediately implies

$$\|e^{\mathrm{i}(\cdot-t_0)\Delta} f\|_{\mathbb{X}(I)} \lesssim \|f\|_{H^1_x} \tag{5.7}$$

for all $f \in H^1$. We also need to estimate the inhomogeneous terms in our functions spaces.

Lemma 5.2 (Inhomogeneous estimate)**.** *For any $g \in \mathbb{G}(I)$ we have*

$$\Big\| \int_{t_0}^{\cdot} e^{\mathrm{i}(\cdot-s)\Delta} g(s)\,\mathrm{d}s \Big\|_{\mathbb{X}(I)} \lesssim \|g\|_{\mathbb{G}(I)}. \tag{5.8}$$

Proof. Note that it is enough to prove for any $N \in 2^{\mathbb{Z}}$

$$\Big\| \int_{t_0}^{\cdot} e^{\mathrm{i}(\cdot-s)\Delta} P_N g(s)\,\mathrm{d}s \Big\|_{\mathbb{X}_N(I)} \lesssim \|P_N g\|_{L^{q'}(I;L^{p'}_x)},$$

$$\Big\| \int_{t_0}^{\cdot} e^{\mathrm{i}(\cdot-s)\Delta} P_N g(s)\,\mathrm{d}s \Big\|_{\mathbb{X}_N(I)} \lesssim \sum_{j=1}^{3} N^{\frac{1}{2}} \|P_N g\|_{L^{1,2}_{\mathbf{e}_j}(I\times\mathbb{R}^3)}$$

for $(q', p') = (1, 2)$ and for $(q', p') = (\frac{8}{5}, \frac{4}{3})$. The first estimate (both for $(q', p') = (1, 2)$ and for $(q', p') = (\frac{8}{5}, \frac{4}{3})$) follows from Lemma 5.1, the dual Strichartz estimate, and the Christ-Kiselev lemma in the form of Lemma B.3 in [57]. The second estimate follows from Proposition 3.8 in [11]. □

5.2. **Boundary estimates.** We next derive the multilinear estimates for the terms appearing on the right-hand side of (4.9). We start with the boundary terms from the integration by parts argument. To that purpose, we first introduce the bilinear operators

$$T_m(f, g)(x) = \int_{\mathbb{R}^6} m(\xi, \eta)\hat{f}(\xi)\hat{g}(\eta)e^{\mathrm{i}x(\xi+\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta, \quad x \in \mathbb{R}^3, \tag{5.9}$$

for $m \in L^\infty(\mathbb{R}^6)$ and $f, g \in \mathcal{S}(\mathbb{R}^3)$.

A crucial tool in estimating the boundary terms is the following Coifman-Meyer-type bilinear multiplier estimate, which was proven in [31, Lemma 3.5].

Lemma 5.3 (Bilinear multiplier estimate)**.** *Let $m \in C^\infty(\mathbb{R}^6)$ be bounded and satisfy*

$$|\partial^\alpha_\xi \partial^\beta_\eta m(\xi, \eta)| \lesssim_{\alpha,\beta} |\xi|^{-|\alpha|}|\eta|^{-|\beta|}$$

for all $\xi, \eta \in \mathbb{R}^3$ and $\alpha, \beta \in \mathbb{N}^3_0$. Take $p, q, r \in [1, \infty]$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then, for any $f \in L^p(\mathbb{R}^3)$, $g \in L^q(\mathbb{R}^3)$, and $N_1, N_2 \in 2^{\mathbb{Z}}$, we have

$$\|T_m(P_{N_1} f, P_{N_2} g)\|_{L^r} \lesssim \|P_{N_1} f\|_{L^p}\|P_{N_2} g\|_{L^q},$$

where T_m is the operator defined in (5.9)*.*

We will use the above lemma to estimate the bilinear operator Ω_{b} given by (4.7). Roughly speaking, Lemma 5.3 yields that Ω_{b} acts like

$$\Omega_{\mathrm{b}}(f, g) \sim |\nabla|^{-1}\langle\nabla\rangle^{-1}(fg)_{XL}. \tag{5.10}$$

We obtain the following estimates for the boundary terms.

Lemma 5.4 (Boundary estimates)**.** *We have*

$$\|e^{\mathrm{i}(\cdot-t_0)\Delta}\Omega_{\mathrm{b}}(v_0, u_0)\|_{\mathbb{X}(I)} \lesssim K^{-1}\|v_0\|_{L^2}\|u_0\|_{H^1}, \tag{5.11}$$

$$\|\Omega_{\mathrm{b}}(v, u)\|_{\mathbb{X}(I)} \lesssim (K^{-1} + |I|^{\frac{1}{8}})\|v\|_{\mathbb{Y}(I)}\|u\|_{\mathbb{X}(I)}. \tag{5.12}$$

Proof. By (5.7), it is enough to show

$$\|\Omega_{\mathrm{b}}(v_0,u_0)\|_{H^1} \lesssim K^{-1}\|v_0\|_{L^2}\|u_0\|_{H^1} \tag{5.13}$$

in order to obtain (5.11). To this end, employing Lemma 5.3, we infer

$$\begin{aligned}
\|\Omega_{\mathrm{b}}(v_0,u_0)\|_{H^1} &\lesssim \Big(\sum_{N\in 2^{\mathbb{Z}}} \|\langle\nabla\rangle\Omega_{\mathrm{b}}(P_N v_0, P_{\le K^{-1}N}u_0)\|_{L^2}^2\Big)^{\frac12}\\
&\lesssim \Big(\sum_{N\in 2^{\mathbb{Z}}}\Big(\sum_{N_1\le K^{-1}N} \||\nabla|\langle\nabla\rangle\Omega_{\mathrm{b}}(N^{-1}P_N v_0, P_{N_1}u_0)\|_{L^2}\Big)^2\Big)^{\frac12}\\
&\lesssim \Big(\sum_{N\in 2^{\mathbb{Z}}}\Big(\sum_{N_1\le K^{-1}N} N^{-1}\|P_N v_0\|_{L^2}\|P_{N_1}u_0\|_{L^\infty}\Big)^2\Big)^{\frac12}\\
&\lesssim \|v_0\|_{L^2}\Big(\sum_{N\in 2^{\mathbb{Z}}}\Big(\sum_{N_1\le K^{-1}N} N^{-1}N_1\|N_1^{\frac12}P_{N_1}u_0\|_{L^2}\Big)^2\Big)^{\frac12}\\
&\lesssim K^{-1}\|v_0\|_{L^2}\|u_0\|_{H^1},
\end{aligned} \tag{5.14}$$

where we used Young's inequality for series in the last step.

In order to prove (5.12), we first note that arguing as in (5.14) we get

$$\|\Omega_{\mathrm{b}}(v,u)\|_{L^\infty(I;H^1_x)} \lesssim K^{-1}\|v\|_{L^\infty(I;L^2_x)}\|u\|_{L^\infty(I;H^1_x)} \lesssim K^{-1}\|v\|_{\mathbb{Y}(I)}\|u\|_{\mathbb{X}(I)}. \tag{5.15}$$

To control the second component of the $\mathbb{X}(I)$-norm, we use Lemma 5.3 again to derive

$$\begin{aligned}
\|\langle\nabla\rangle\Omega_{\mathrm{b}}(v,u)\|_{L^2(I;\dot B^0_{6,2})} &\lesssim \Big\|\Big(\sum_{N\in 2^{\mathbb{Z}}}\Big(\sum_{N_1\le K^{-1}N} \||\nabla|\langle\nabla\rangle\Omega_{\mathrm{b}}(N^{-1}P_N v, P_{N_1}u)\|_{L^6_x}\Big)^2\Big)^{\frac12}\Big\|_{L^2(I)}\\
&\lesssim \Big\|\Big(\sum_{N\in 2^{\mathbb{Z}}}\Big(\sum_{N_1\le K^{-1}N} \|N^{-1}P_N v\|_{L^6_x}\|P_{N_1}u\|_{L^\infty_x}\Big)^2\Big)^{\frac12}\Big\|_{L^2(I)}\\
&\lesssim \Big\|\Big(\sum_{N\in 2^{\mathbb{Z}}}\|P_N v\|_{L^2_x}^2\Big(\sum_{N_1\le K^{-1}N} N_1^{\frac34}\|P_{N_1}u\|_{L^4_x}\Big)^2\Big)^{\frac12}\Big\|_{L^2(I)}\\
&\lesssim \|v\|_{\mathbb{Y}(I)}\|\langle\nabla\rangle u\|_{L^2(I;L^4_x)}\\
&\lesssim |I|^{\frac18}\|v\|_{\mathbb{Y}(I)}\|u\|_{\mathbb{X}(I)},
\end{aligned} \tag{5.16}$$

where we exploited that $\sum_{N_1\in 2^{\mathbb{Z}}} N_1^{\frac34}\|P_{N_1}u\|_{L^4_x} \lesssim \|\langle\nabla\rangle u\|_{L^4_x}$ and $\dot B^0_{4,2}\hookrightarrow L^4$. It remains to control the local smoothing component of the $\mathbb{X}(I)$-norm. For this purpose, we first note that $\mathcal{F}_{x_j}((P_{N,\mathbf{e}_j}g)(x'))$ is supported in an interval of length $4N$ for any $g\in L^2(\mathbb{R}^3)$, where $x'=(x_2,x_3)$, $x'=(x_1,x_3)$, respectively $x'=(x_1,x_2)$ if $j=1,2,3$. Fix $j\in\{1,2,3\}$. Using Minkowski's inequality and Bernstein's inequality in one dimension, we obtain

$$\begin{aligned}
&\Big(\sum_{N\in 2^{\mathbb{Z}}}(N^{\frac32}\|P_N P_{N,\mathbf{e}_j}\Omega_{\mathrm{b}}(v,u)\|_{L^{\infty,2}_{\mathbf{e}_j}(I\times\mathbb{R}^3)})^2\Big)^{\frac12}\\
&\lesssim \Big(\sum_{N\in 2^{\mathbb{Z}}}(N^{\frac32}\|P_{N,\mathbf{e}_j}\Omega_{\mathrm{b}}(P_N v, P_{\le K^{-1}N}u)\|_{L^2_{t,x'}L^\infty_{x_j}})^2\Big)^{\frac12}\\
&\lesssim \Big(\sum_{N\in 2^{\mathbb{Z}}}(N^2\|\Omega_{\mathrm{b}}(P_N v, P_{\le K^{-1}N}u)\|_{L^2(I\times\mathbb{R}^3)}\Big)^2\Big)^{\frac12}\\
&\lesssim \Big\|\Big(\sum_{N\in 2^{\mathbb{Z}}}\Big(\sum_{N_1\le K^{-1}N} N^2\|\Omega_{\mathrm{b}}(P_N v, P_{N_1}u)\|_{L^2}\Big)^2\Big)^{\frac12}\Big\|_{L^2(I)}.
\end{aligned}$$

Applying Lemma 5.3 once more and arguing as in (5.16), we infer that the right-hand side is bounded by

$$\Big\|\Big(\sum_{N\in 2^{\mathbb{Z}}}\Big(\sum_{N_1\le K^{-1}N} \|P_N v\|_{L^2_x}\|P_{N_1}u\|_{L^\infty}\Big)^2\Big)^{\frac12}\Big\|_{L^2(I)}$$

$$
\begin{aligned}
&\lesssim \Big\| \Big(\sum_{N\in 2^{\mathbb{Z}}} \|P_N v\|_{L^2}^2 \Big(\sum_{N_1 \leq K^{-1}N} N_1^{\frac{3}{4}} \|P_{N_1} u\|_{L^4} \Big)^2 \Big)^{\frac{1}{2}} \Big\|_{L^2(I)} \\
&\lesssim |I|^{\frac{1}{8}} \|v\|_{\mathbb{Y}(I)} \|u\|_{\mathbb{X}(I)}. \qquad (5.17)
\end{aligned}
$$

Combining (5.15), (5.16), and (5.17), we conclude (5.12). □

5.3. **Bilinear estimates.** We first derive the bilinear estimates for the Schrödinger operator.

Lemma 5.5. *We have*

$$\Big\| \int_{t_0}^{\cdot} e^{\mathrm{i}(\cdot - s)\Delta} (vu)_R(s)\, \mathrm{d}s \Big\|_{\mathbb{X}(I)} \lesssim |I|^{\frac{1}{4}} K \log_2 K \|v\|_{\mathbb{Y}(I)} \|u\|_{\mathbb{X}(I)}, \tag{5.18}$$

$$\Big\| \int_{t_0}^{\cdot} e^{\mathrm{i}(\cdot - s)\Delta} (b\cdot \nabla u)(s)\, \mathrm{d}s \Big\|_{\mathbb{X}(I)} \lesssim \Big(|I|^{\frac{1}{4}} \|b\|_{L^\infty(I;H^1_x)} + \sum_{j=1}^{3} \|b\|_{L^{1,\infty}_{\mathbf{e}_j}(I\times\mathbb{R}^3)} \Big) \|u\|_{\mathbb{X}(I)}, \tag{5.19}$$

$$\Big\| \int_{t_0}^{\cdot} e^{\mathrm{i}(\cdot - s)\Delta} (cu)(s)\, \mathrm{d}s \Big\|_{\mathbb{X}(I)} \lesssim |I|^{\frac{1}{4}} \|c\|_{L^\infty(I;H^1_x)} \|u\|_{\mathbb{X}(I)}, \tag{5.20}$$

$$\Big\| \int_{t_0}^{\cdot} e^{\mathrm{i}(\cdot - s)\Delta} (\mathcal{T}_\cdot(W_2)u)(s)\, \mathrm{d}s \Big\|_{\mathbb{X}(I)} \lesssim |I|^{\frac{1}{4}} \|\mathcal{T}_\cdot(W_2)\|_{L^\infty(I;H^1_x)} \|u\|_{\mathbb{X}(I)}. \tag{5.21}$$

Proof. We start by proving (5.18). By (5.8) and the definition of $\|\cdot\|_{\mathbb{G}(I)}$ we obtain

$$\Big\| \int_{t_0}^{\cdot} e^{\mathrm{i}(\cdot - s)\Delta} (vu)_R(s)\, \mathrm{d}s \Big\|_{\mathbb{X}(I)} \lesssim \| \|\langle \nabla \rangle P_N (vu)_R\|_{L^{\frac{8}{5}}(I;L_x^{\frac{4}{3}})} \|_{l^2_N} \lesssim \|\langle\nabla\rangle (vu)_R\|_{L^{\frac{8}{5}}(I;\dot{B}^0_{\frac{4}{3},2})},$$

where we applied Minkowski's inequality in the last step.

In order to estimate the right-hand side above, we start with the LH-component of $(vu)_R$. Employing Hölder's inequality, we derive

$$
\begin{aligned}
\|\langle\nabla\rangle (vu)_{LH}\|_{L^{\frac{8}{5}}(I;\dot{B}^0_{\frac{4}{3},2})} &\lesssim \Big\| \Big(\sum_{N\in 2^{\mathbb{Z}}} \langle N\rangle^2 \|P_{\leq K^{-1}N} v P_N u\|^2_{L_x^{\frac{4}{3}}} \Big)^{\frac{1}{2}} \Big\|_{L^{\frac{8}{5}}(I)} \\
&\lesssim \Big\| \Big(\sum_{N\in 2^{\mathbb{Z}}} \langle N\rangle^2 \|P_{\leq K^{-1}N} v\|^2_{L^2_x} \|P_N u\|^2_{L^4_x} \Big)^{\frac{1}{2}} \Big\|_{L^{\frac{8}{5}}(I)} \\
&\lesssim |I|^{\frac{1}{4}} \|v\|_{\mathbb{Y}(I)} \|u\|_{\mathbb{X}(I)}. \qquad (5.22)
\end{aligned}
$$

Similarly, we obtain for the HH- and $1L$-components

$$
\begin{aligned}
\|\langle\nabla\rangle (vu)_{HH}\|_{L^{\frac{8}{5}}(I;\dot{B}^0_{\frac{4}{3},2})} &\lesssim \|\langle\nabla\rangle (vu)_{HH}\|_{L^{\frac{8}{5}}(I;L_x^{\frac{4}{3}})} \\
&\lesssim \Big\| \sum_{N\in 2^{\mathbb{Z}}} \sum_{\substack{N_1\in 2^{\mathbb{Z}} \\ K^{-1}N < N_1 < KN}} \langle\nabla\rangle (P_{N_1} v P_N u) \Big\|_{L^{\frac{8}{5}}(I;L_x^{\frac{4}{3}})} \\
&\lesssim \Big\| \sum_{\substack{M\in 2^{\mathbb{Z}} \\ K^{-1}<M<K}} \sum_{N\in 2^{\mathbb{Z}}} \langle MN + N\rangle \|P_{MN} v\|_{L^2_x} \|P_N u\|_{L^4_x} \Big\|_{L^{\frac{8}{5}}(I)} \\
&\lesssim |I|^{\frac{1}{4}} K \log_2 K \|v\|_{\mathbb{Y}(I)} \|u\|_{\mathbb{X}(I)}, \qquad (5.23)
\end{aligned}
$$

and

$$
\begin{aligned}
\|\langle\nabla\rangle (vu)_{1L}\|_{L^{\frac{8}{5}}(I;\dot{B}^0_{\frac{4}{3},2})} &\lesssim \Big\| \sum_{\substack{2^{-1}\leq N\leq 2 \\ N\in 2^{\mathbb{Z}}}} \langle\nabla\rangle (P_N v P_{\leq K^{-1}N} u) \Big\|_{L^{\frac{8}{5}}(I;L_x^{\frac{4}{3}})} \\
&\lesssim |I|^{\frac{1}{4}} \|v\|_{\mathbb{Y}(I)} \|u\|_{L^{\frac{8}{3}}(I;L^4_x)} \\
&\lesssim |I|^{\frac{1}{4}} \|v\|_{\mathbb{Y}(I)} \|u\|_{\mathbb{X}(I)}. \qquad (5.24)
\end{aligned}
$$

Estimate (5.18) then follows immediately from (5.22), (5.23), and (5.24).

We next prove (5.19). To that purpose, we decompose

$$b \cdot \nabla u = (b \cdot \nabla u)_{HL} + (b \cdot \nabla u)_{HH} + (b \cdot \nabla u)_{LH}$$

with $K = 2^5$. Estimate (5.8) and the definition of $\| \cdot \|_{\mathbb{G}(I)}$, again combined with Minkowski's inequality, yield

$$\begin{aligned} \Big\| \int_0^t e^{\mathrm{i}(t-s)\Delta}(b \cdot \nabla u)(s) \, \mathrm{d}s \Big\|_{\mathbb{X}(I)} &\lesssim \|\langle \nabla \rangle (b \cdot \nabla u)_{HL+HH}\|_{L^{\frac{8}{5}}(I;\dot{B}^0_{\frac{4}{3},2})} \\ &\quad + \sum_{j=1}^3 \Big(\sum_{N \in 2^{\mathbb{Z}}} N \|P_N (b \cdot \nabla u)_{LH}\|^2_{L^{1,2}_{\mathbf{e}_j}(I \times \mathbb{R}^d)} \Big)^{\frac{1}{2}}. \end{aligned} \tag{5.25}$$

We start with the first summand on the right-hand side. We infer

$$\begin{aligned} \|\langle \nabla \rangle (b \cdot \nabla u)_{HL}\|_{L^{\frac{8}{5}}(I;\dot{B}^0_{\frac{4}{3},2})} &\lesssim \Big\| \Big(\sum_{N \in 2^{\mathbb{Z}}} \langle N \rangle^2 \|P_N b\|^2_{L^2_x} \|P_{\leq K^{-1}N} (\nabla u)\|^2_{L^4_x} \Big)^{\frac{1}{2}} \Big\|_{L^{\frac{8}{5}}(I)} \\ &\lesssim \|b\|_{L^\infty(I;H^1_x)} \|\nabla u\|_{L^{\frac{8}{5}}(I;L^4_x)} \\ &\lesssim |I|^{\frac{1}{4}} \|b\|_{L^\infty(I;H^1_x)} \|\langle \nabla \rangle u\|_{L^{\frac{8}{3}}(I;\dot{B}^0_{4,2})} \\ &\lesssim |I|^{\frac{1}{4}} \|b\|_{L^\infty(I;H^1_x)} \|u\|_{\mathbb{X}(I)}. \end{aligned} \tag{5.26}$$

With the same adaptions as in (5.23), we also obtain

$$\|\langle \nabla \rangle (b \cdot \nabla u)_{HH}\|_{L^{\frac{8}{5}}(I;\dot{B}^0_{\frac{4}{3},2})} \lesssim |I|^{\frac{1}{4}} \|b\|_{L^\infty(I;H^1_x)} \|u\|_{\mathbb{X}(I)}. \tag{5.27}$$

It remains to estimate the sum on the right-hand side of (5.25). Fix $j \in \{1,2,3\}$. We then derive

$$\begin{aligned} \Big(\sum_{N \in 2^{\mathbb{Z}}} N \|P_N (b \cdot \nabla u)_{LH}\|^2_{L^{1,2}_{\mathbf{e}_j}(I \times \mathbb{R}^3)} \Big)^{\frac{1}{2}} &\lesssim \Big(\sum_{N \in 2^{\mathbb{Z}}} N \|P_{\leq K^{-1}N} b P_N (\nabla u)\|^2_{L^{1,2}_{\mathbf{e}_j}(I \times \mathbb{R}^3)} \Big)^{\frac{1}{2}} \\ &\lesssim \Big(\sum_{N \in 2^{\mathbb{Z}}} N \|P_{\leq K^{-1}N} b\|^2_{L^{1,\infty}_{\mathbf{e}_j}(I \times \mathbb{R}^3)} \|P_N (\nabla u)\|^2_{L^{\infty,2}_{\mathbf{e}_j}(I \times \mathbb{R}^3)} \Big)^{\frac{1}{2}} \\ &\lesssim \|b\|_{L^{1,\infty}_{\mathbf{e}_j}(I \times \mathbb{R}^3)} \Big(\sum_{N \in 2^{\mathbb{Z}}} N^3 \|P_N u\|^2_{L^{\infty,2}_{\mathbf{e}_j}(I \times \mathbb{R}^3)} \Big)^{\frac{1}{2}} \\ &\lesssim \|b\|_{L^{1,\infty}_{\mathbf{e}_j}(I \times \mathbb{R}^3)} \|u\|_{\mathbb{X}(I)}, \end{aligned} \tag{5.28}$$

where we used that P_N has a smooth and compactly supported kernel. Inserting (5.26), (5.27), and (5.28) into (5.25), we conclude (5.19).

Finally, to get (5.20) we estimate as above

$$\begin{aligned} \Big\| \int_{t_0}^{\cdot} e^{\mathrm{i}(\cdot - s)\Delta}(cu)(s) \, \mathrm{d}s \Big\|_{\mathbb{X}(I)} &\lesssim \|\langle \nabla \rangle (cu)\|_{L^{\frac{8}{5}}(I;\dot{B}^0_{\frac{4}{3},2})} \lesssim |I|^{\frac{1}{4}} \|\langle \nabla \rangle c\|_{L^\infty(I;L^2_x)} \|\langle \nabla \rangle u\|_{L^{\frac{8}{3}}(I;L^4_x)} \\ &\lesssim |I|^{\frac{1}{4}} \|c\|_{L^\infty(I;H^1_x)} \|u\|_{\mathbb{X}(I)}. \end{aligned}$$

Estimate (5.21) is shown in the same way as (5.20) by replacing c with $\mathcal{T}.(W_2)u$. Therefore, the proof is complete. □

Regarding the wave operator, we have the following bilinear estimate.

Lemma 5.6. *We have*

$$\Big\| \int_{t_0}^{\cdot} e^{\mathrm{i}(\cdot - s)|\nabla|} |\nabla| (u_1 u_2)(s) \, \mathrm{d}s \Big\|_{\mathbb{Y}(I)} \lesssim |I|^{\frac{1}{4}} \|u_1\|_{\mathbb{X}(I)} \|u_2\|_{\mathbb{X}(I)}. \tag{5.29}$$

Proof. The energy-estimate, dyadic decomposition and interpolation yield

$$\Big\| \int_{t_0}^{\cdot} e^{\mathrm{i}(\cdot - s)|\nabla|} |\nabla| (u_1 u_2)(s) \, \mathrm{d}s \Big\|_{\mathbb{Y}(I)} \lesssim \||\nabla| (u_1 u_2)\|_{L^1(I;L^2_x)}$$

$$\lesssim |I|^{\frac14}\|\langle\nabla\rangle u_1\|_{L^2(I;\dot B^0_{6,2})}\|\langle\nabla\rangle u_2\|_{L^4(I;\dot B^0_{3,2})}$$
$$\lesssim |I|^{\frac14}\|u_1\|_{\mathbb{X}(I)}\|u_2\|_{\mathbb{X}(I)}.$$

□

5.4. **Trilinear estimates.**

Lemma 5.7 (Trilinear estimates)**.** *We have*

$$\Big\|\int_{t_0}^{\cdot} e^{\mathrm{i}(\cdot-s)\Delta}\Omega_{\mathrm b}(|\nabla|(u_1u_2),u_3)(s)\,\mathrm ds\Big\|_{\mathbb{X}(I)} \lesssim |I|^{\frac14}\|u_1\|_{\mathbb{X}(I)}\|u_2\|_{\mathbb{X}(I)}\|u_3\|_{\mathbb{X}(I)}, \tag{5.30}$$

$$\Big\|\int_{t_0}^{\cdot} e^{\mathrm{i}(\cdot-s)\Delta}\Omega_{\mathrm b}(v_1,v_2u)(s)\,\mathrm ds\Big\|_{\mathbb{X}(I)} \lesssim |I|^{\frac18}\|v_1\|_{\mathbb{Y}(I)}\|v_2\|_{\mathbb{Y}(I)}\|u\|_{\mathbb{X}(I)}, \tag{5.31}$$

$$\Big\|\int_{t_0}^{\cdot} e^{\mathrm{i}(\cdot-s)\Delta}\Omega_{\mathrm b}(v,b\cdot\nabla u+cu-\mathcal{T}_\cdot(W_2)u)(s)\,\mathrm ds\Big\|_{\mathbb{X}(I)} \lesssim |I|^{\frac18}\Big(\|b\|_{L^\infty(I;H^1_x)}+\|c\|_{\mathbb{Y}(I)} + \|\mathcal{T}_\cdot(W_2)\|_{\mathbb{Y}(I)}\Big)\|v\|_{\mathbb{Y}(I)}\|u\|_{\mathbb{X}(I)}. \tag{5.32}$$

Proof. We start by proving (5.30). Using estimate (5.8) and Lemma 5.3, we derive

$$\begin{aligned}
\Big\|\int_{t_0}^{\cdot} e^{\mathrm{i}(\cdot-s)\Delta}\Omega_{\mathrm b}(|\nabla|(u_1u_2),u_3)(s)\,\mathrm ds\Big\|_{\mathbb{X}(I)} &\lesssim \|\Omega_{\mathrm b}(|\nabla|(u_1u_2),u_3)\|_{L^1(I;H^1_x)}\\
&\lesssim \Big\|\Big(\sum_{N\in 2^{\mathbb Z}}\Big(\sum_{N_1\le K^{-1}N}\|P_N(u_1u_2)\|_{L^2_x}\|P_{N_1}u_3\|_{L^\infty_x}\Big)^2\Big)^{\frac12}\Big\|_{L^1(I)}\\
&\lesssim |I|^{\frac14}\big\|\|u_1u_2\|_{L^2_x}\|\langle\nabla\rangle u_3\|_{L^4_x}\big\|_{L^{\frac43}(I)}\\
&\lesssim |I|^{\frac14}\|u_1\|_{\mathbb{Y}(I)}\|\langle\nabla\rangle u_2\|_{L^{\frac83}(I;L^4_x)}\|\langle\nabla\rangle u_3\|_{L^{\frac83}(I;L^4_x)}\\
&\lesssim |I|^{\frac14}\|u_1\|_{\mathbb{X}(I)}\|u_2\|_{\mathbb{X}(I)}\|u_3\|_{\mathbb{X}(I)},
\end{aligned}$$

where we once more used that, via Bernstein's inequality,

$$\sum_{N_1\le K^{-1}N}\|P_{N_1}u\|_{L^\infty_x} \lesssim \sum_{N_1\in 2^{\mathbb Z}} N_1^{\frac34}\|P_{N_1}u\|_{L^4_x}\lesssim \|\langle\nabla\rangle u\|_{L^4_x}.$$

We proceed in the same way to show (5.31). Estimate (5.8) and Lemma 5.3 imply

$$\begin{aligned}
\Big\|\int_{t_0}^{\cdot} e^{\mathrm{i}(\cdot-s)\Delta}\Omega_{\mathrm b}(v_1,v_2u)(s)\,\mathrm ds\Big\|_{\mathbb{X}(I)} &\lesssim \|\langle\nabla\rangle\Omega_{\mathrm b}(v_1,v_2u)\|_{L^{\frac85}(I;\dot B^0_{\frac43,2})}\\
&\lesssim \Big\|\Big(\sum_{N\in 2^{\mathbb Z}}\Big(\sum_{N_1\le K^{-1}N}N^{-1}\|P_Nv_1\|_{L^2_x}\|P_{N_1}(v_2u)\|_{L^4_x}\Big)^2\Big)^{\frac12}\Big\|_{L^{\frac85}(I)}\\
&\lesssim \|v_1\|_{\mathbb{Y}(I)}\Big\|\Big(\sum_{N\in 2^{\mathbb Z}}\Big(\sum_{N_1\le K^{-1}N}N^{-1}N_1\|P_{N_1}(v_2u)\|_{L^{\frac{12}{7}}_x}\Big)^2\Big)^{\frac12}\Big\|_{L^{\frac85}(I)}.
\end{aligned}$$

By Young's inequality for series and the embedding $L^{\frac{12}{7}}\hookrightarrow \dot B^0_{\frac{12}{7},2}$ we thus conclude

$$\begin{aligned}
\Big\|\int_{t_0}^{\cdot} e^{\mathrm{i}(\cdot-s)\Delta}\Omega_{\mathrm b}(v_1,v_2u)(s)\,\mathrm ds\Big\|_{\mathbb{X}(I)} &\lesssim \|v_1\|_{\mathbb{Y}(I)}\|v_2u\|_{L^{\frac85}(I;L_x^{\frac{12}{7}})}\\
&\lesssim \|v_1\|_{\mathbb{Y}(I)}\|v_2\|_{L^\infty(I;L^2_x)}\|u\|_{L^{\frac85}(I;L^{12}_x)}\\
&\lesssim |I|^{\frac18}\|v_1\|_{\mathbb{Y}(I)}\|v_2\|_{\mathbb{Y}(I)}\|u\|_{\mathbb{X}(I)}.
\end{aligned}\tag{5.33}$$

Replacing v_2 in the second component of $\Omega_{\mathrm b}$ by $c-\mathcal{T}_\cdot(W_2)$, we obtain from (5.33)

$$\Big\|\int_{t_0}^{\cdot} e^{\mathrm{i}(\cdot-s)\Delta}\Omega_{\mathrm b}(v,(c-\mathcal{T}_\cdot(W_2))u)(s)\,\mathrm ds\Big\|_{\mathbb{X}(I)} \lesssim |I|^{\frac18}\|v\|_{\mathbb{Y}(I)}(\|c\|_{\mathbb{Y}(I)}+\|\mathcal{T}_\cdot(W_2)\|_{\mathbb{Y}(I)})\|u\|_{\mathbb{X}(I)}. \tag{5.34}$$

Similarly, replacing $v_2 u$ by $b\cdot\nabla u$, using Sobolev's embedding $H^1 \hookrightarrow L^{\frac{12}{5}}$ and arguing as in the proof of (5.33), we obtain

$$\begin{aligned}\Big\| \int_{t_0}^{\cdot} e^{\mathrm{i}(\cdot-s)\Delta}\Omega_{\mathrm{b}}(v, b\cdot\nabla u)(s)\,\mathrm{d}s\Big\|_{\mathbb{X}(I)} \lesssim& \|v\|_{L^\infty(I;L^2_x)}\|b\cdot\nabla u\|_{L^{\frac{8}{5}}(I;L_x^{\frac{12}{7}})} \\ \lesssim& \|v\|_{L^\infty(I;L^2_x)}\|b\|_{L^8(I;L_x^{\frac{12}{5}})}\|\langle\nabla\rangle u\|_{L^2(I;L^6_x)} \\ \lesssim& |I|^{\frac{1}{8}}\|b\|_{L^\infty(I;H^1_x)}\|v\|_{\mathbb{Y}(I)}\|u\|_{\mathbb{X}(I)}. \end{aligned} \tag{5.35}$$

Finally, estimate (5.32) follows from estimates (5.34) and (5.35). □

6. Well-posedness up to the maximal existence time

We prepare the proof of the well-posedness theorem by showing several properties of the noise which we use in the construction of the solutions.

Lemma 6.1. *Let $T \in (0,\infty)$ and $\kappa \in (0,1/2)$. Then, W_1 is C^κ-Hölder continuous in H^3 and W_2 and the process $t \mapsto \int_0^t e^{\mathrm{i}(t-s)|\nabla|}\,\mathrm{d}W_2(s)$ are C^κ-Hölder continuous in H^1. Moreover, for every $j = 1,2,3$ and for $\mathbb{P}$-a.e. $\omega\in\Omega$, there exists a sequence $(n_l(\omega))_{l\in\mathbb{N}}$ in $\mathbb{N}$ with $n_l(\omega)\to\infty$ as $l\to\infty$ such that*

$$\sum_{k=n_l}^{\infty}\int \sup_{y\in\mathbb{R}^2}|\phi_k^{(1)}(r\mathbf{e}_j+y)|\,\mathrm{d}r \sup_{t\in[0,T]}|\beta_k^{(1)}(t,\omega)| \longrightarrow 0, \quad \text{as } l\to\infty. \tag{6.1}$$

Proof. We first note that under condition (1.10), W_1 is a continuous and $\{\mathscr{F}_t\}$-adapted Wiener process in H^3 (see, e.g., [39]). Moreover, for any $p\geq 1$, using the Gaussianity, we obtain for $0\leq s<t\leq T$

$$\mathbb{E}\|W_1(t)-W_1(s)\|^p_{H^3} \lesssim (\mathbb{E}\|W_1(t)-W_1(s)\|^2_{H^3})^{\frac{p}{2}} \lesssim \Big(\sum_{k=1}^\infty \|\phi_k^{(1)}\|^2_{H^3}\Big)^{\frac{p}{2}}(t-s)^{\frac{p}{2}}.$$

The Kolmogorov continuity criterion thus implies the C^κ-Hölder continuity of W_1 in H^3.

The assertion for W_2 and $\int_0^\cdot e^{\mathrm{i}(\cdot-s)|\nabla|}\,\mathrm{d}W_2(s)$ can be proved similarly using condition (1.10) and the fact that $e^{\mathrm{i}t|\nabla|}$ is unitary in H^1.

In order to prove (6.1), we note that

$$\begin{aligned}&\mathbb{E}\sum_{k=n}^{\infty}\int \sup_{y\in\mathbb{R}^2}|\phi_k^{(1)}(r\mathbf{e}_j+y)|\,\mathrm{d}r \sup_{t\in[0,T]}|\beta_k^{(1)}(t)| \\ &= \sum_{k=n}^{\infty}\int \sup_{y\in\mathbb{R}^2}|\phi_k^{(1)}(r\mathbf{e}_j+y)|\,\mathrm{d}r\,\mathbb{E}\sup_{t\in[0,T]}|\beta_k^{(1)}(t)| \lesssim T^{\frac{1}{2}}\sum_{k=n}^{\infty}\int \sup_{y\in\mathbb{R}^2}|\phi_k^{(1)}(r\mathbf{e}_j+y)|\,\mathrm{d}r,\end{aligned}$$

which tends to zero as $n\to\infty$ due to the summability in (1.10). We infer that

$$\sum_{k=n}^{\infty}\int \sup_{y\in\mathbb{R}^2}|\phi_k^{(1)}(r\mathbf{e}_j+y)|\,\mathrm{d}r \sup_{t\in[0,T]}|\beta_k^{(1)}(t)| \longrightarrow 0 \ \text{ in probability as } n\to\infty,$$

which yields the almost sure convergence up to some subsequence $(n_l)_{l\in\mathbb{N}}$. □

We are now ready to prove the local well-posedness result in Theorem 1.2.

Proof of Theorem 1.2. Fix any $T\in(0,\infty)$. Write $\mathbb{X}(0,\tau) := \mathbb{X}([0,\tau])$ and $\mathbb{Y}(0,\tau) := \mathbb{Y}([0,\tau])$ for $\tau\in(0,\infty)$. The proof proceeds in four steps.

Step 1. We define the fixed point operator $\Phi = (\Phi_1,\Phi_2)$ by the right-hand side of (4.9). More precisely, let $\mathcal{U}_t$ and $\mathcal{I}_t$ denote the homogeneous and inhomogeneous flow operators, respectively, for the linear Schrödinger equation. In view of (4.9), we define

$$\begin{aligned}\Phi_1(u,v)(t) :=& \,\mathcal{U}_t(u_0+\Omega_{\mathrm{b}}(v_0,u_0)) - \Omega_{\mathrm{b}}(v,u) - \mathrm{i}\mathcal{I}_t(vu)_R + \mathrm{i}\mathcal{I}_t(b\cdot\nabla u + cu - \mathcal{T}_\cdot(W_2)u) \\ &+ \mathrm{i}\mathcal{I}_t\Omega_{\mathrm{b}}(|\nabla||u|^2,u) - \mathrm{i}\mathcal{I}_t\Omega_{\mathrm{b}}(v,vu) + \mathrm{i}\mathcal{I}_t\Omega_{\mathrm{b}}(v,b\cdot\nabla u + cu - \mathcal{T}_\cdot(W_2)u) \\ =:& \,\mathcal{U}_t(u_0+\Omega_{\mathrm{b}}(v_0,u_0)) + \sum_{j=1}^{6}\Phi_{1,j}(v,u)(t),\end{aligned} \tag{6.2}$$

and

$$\Phi_2(u,v)(t) := \mathcal{V}_t v_0 + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)|\nabla|}(|\nabla||u|^2)(s)\,\mathrm{d}s \tag{6.3}$$

for $t \in [0,T]$, with $\mathcal{V}_t = e^{\mathrm{i}t|\nabla|}$ and initial data

$$(u_0, v_0) := (X_0, Y_0). \tag{6.4}$$

First, we note that Lemma 5.6 yields

$$\|\Phi_2(u,v)\|_{\mathbb{Y}(0,\tau)} \le \|Y_0\|_{L^2} + C\tau^{\frac14}\|u\|^2_{\mathbb{X}(0,\tau)}, \tag{6.5}$$

where C is the implicit constant from (5.29).

For the first component of the fixed point operator, the multilinear estimates in Section 5 imply

$$\begin{aligned}\|\Phi_1(u,v)\|_{\mathbb{X}(0,\tau)} \lesssim{}& (1+K^{-1}\|Y_0\|_{L^2})\|X_0\|_{H^1} + (K^{-1}+\tau^{\frac18})\|v\|_{\mathbb{Y}(0,\tau)}\|u\|_{\mathbb{X}(0,\tau)} \\ &+ \tau^{\frac14}K\log_2 K\|v\|_{\mathbb{Y}(0,\tau)}\|u\|_{\mathbb{X}(0,\tau)} + \Big(\tau^{\frac14}\|b\|_{L^\infty((0,\tau),H^1_x)} + \sum_{j=1}^3 \|b\|_{L^{1,\infty}_{\mathbf{e}_j}((0,\tau)\times\mathbb{R}^3)} \\ &+ \tau^{\frac14}\|c\|_{L^\infty((0,\tau),H^1_x)} + \|\mathcal{T}_\cdot(W_2)\|_{L^\infty((0,\tau),H^1_x)}\Big)\|u\|_{\mathbb{X}(0,\tau)} \\ &+ \tau^{\frac14}\|u\|^3_{\mathbb{X}(0,\tau)} + \tau^{\frac18}\|v\|^2_{\mathbb{Y}(0,\tau)}\|u\|_{\mathbb{X}(0,\tau)} \\ &+ \tau^{\frac18}(\|b\|_{L^\infty((0,\tau);H^1_x)} + \|c\|_{\mathbb{Y}(0,\tau)} + \|\mathcal{T}_\cdot(W_2)\|_{\mathbb{Y}(0,\tau)})\|v\|_{\mathbb{Y}(0,\tau)}\|u\|_{\mathbb{X}(0,\tau)}.\end{aligned} \tag{6.6}$$

We note that by (2.9) and (2.10), we have

$$\begin{aligned}&\|b(t)\|_{H^1} + \sum_{j=1}^3 \|b\|_{L^{1,\infty}_{\mathbf{e}_j}((0,t)\times\mathbb{R}^3)} + \|c(t)\|_{H^1} + \|\mathcal{T}_t(W_2)\|_{H^1} \\ &\lesssim \|\nabla W_1(t)\|_{H^1} + \sum_{j=1}^3\sum_{k=1}^\infty \int \sup_{y\in\mathbb{R}^2}|\phi_k^{(1)}(r\mathbf{e}_j+y)|\,\mathrm{d}r \sup_{s\in[0,t]}|\beta_k^{(1)}(s)| + \|W_1(t)\|^2_{H^3} + \|W_1(t)\|_{H^3} \\ &\quad + \|\mathcal{T}_t(W_2)\|_{H^1} =: W^*(t).\end{aligned} \tag{6.7}$$

Recall that K is a dyadic integer with $K \ge 2^5$. We thus obtain for $\tau \le 1$

$$\begin{aligned}\|\Phi_1(u,v)\|_{\mathbb{X}(0,\tau)} \lesssim{}& (1+K^{-1}\|Y_0\|_{L^2})\|X_0\|_{H^1} + (K^{-1}+\tau^{\frac18}K\log_2 K + \|W^*\|_{C([0,\tau])})\|v\|_{\mathbb{Y}(0,\tau)}\|u\|_{\mathbb{X}(0,\tau)} \\ &+ \tau^{\frac18}\|v\|^2_{\mathbb{Y}(0,\tau)}\|u\|_{\mathbb{X}(0,\tau)} + \tau^{\frac18}\|u\|^3_{\mathbb{X}(0,\tau)}\end{aligned} \tag{6.8}$$

where the implicit constant is independent of $\tau, K, \|X_0\|_{H^1}$, and $\|Y_0\|_{L^2}$.

We next define

$$M := \max\{10C_0\|X_0\|_{H^1}, 10\|Y_0\|_{L^2}, 1\},$$

where C_0 is the maximum of the implicit constant in (6.8) and the constant in (6.5), as well as

$$B_{\mathbb{X}(0,\tau)\times\mathbb{Y}(0,\tau)}(M) := \{(u,v) \in \mathbb{X}(0,\tau)\times\mathbb{Y}(0,\tau) \colon \|u\|_{\mathbb{X}(0,\tau)} + \|v\|_{\mathbb{Y}(0,\tau)} \le M\}.$$

We then fix a dyadic integer $K = K(\|X_0\|_{H^1}, \|Y_0\|_{L^2}) \ge 2^5$ large enough such that $K^{-1}\|Y_0\|_{L^2} < \frac{1}{10}$ and $K^{-1}C_0M < \frac{1}{10}$, and subsequently define the $\{\mathscr{F}_t\}$-stopping time

$$\tau := \inf\left\{t\in[0,T] \colon t^{\frac18}C_0(K\log_2(K)M + M^2) \ge \frac{1}{10},\ W^*(t)C_0M \ge \frac{1}{10}\right\} \wedge T.$$

Employing Lemma 6.1, we note that $W^*(t) \to 0$ as $t \to 0$ $\mathbb{P}$-a.s., so that $\mathbb{P}$-a.s. $\tau > 0$. The estimates in (6.8) and (6.5) thus yield

$$\Phi(B_{\mathbb{X}(0,\tau)\times\mathbb{Y}(0,\tau)}(M)) \subseteq B_{\mathbb{X}(0,\tau)\times\mathbb{Y}(0,\tau)}(M).$$

Step 2. The contraction of the operator Φ can be proved similarly. Actually, take any $(u_1,v_1),(u_2,v_2) \in B_{\mathbb{X}(0,\tau)\times\mathbb{Y}(0,\tau)}(M)$. We then have

$$\|u_j\|_{\mathbb{X}(0,\tau)} + \|v_j\|_{\mathbb{Y}(0,\tau)} \le M = C(\|X_0\|_{H^1}, \|Y_0\|_{L^2}) \quad \text{for } j = 1,2.$$

Lemma 5.6 yields

$$\begin{aligned}\|\Phi_2(u_1,v_1)-\Phi_2(u_2,v_2)\|_{\mathbb{Y}(0,\tau)} &\le \left\| \int_0^t e^{\mathrm{i}(t-s)|\nabla|}(|\nabla||u_1|^2-|\nabla||u_2|^2)(s)\,\mathrm{d}s\right\|_{\mathbb{Y}(0,\tau)}\\ &\le C_0\tau^{\frac14}(\|u_1\|_{\mathbb{X}(0,\tau)}+\|u_2\|_{\mathbb{X}(0,\tau)})\|u_1-u_2\|_{\mathbb{X}(0,\tau)}\\ &\le \tau^{\frac18}\|u_1-u_2\|_{\mathbb{X}(0,\tau)}, \end{aligned} \tag{6.9}$$

where we used that $C_0\tau^{\frac18}M\le 1/10$ in the last step, due to the definition of τ.

In the following we will frequently use the multilinear estimates from Section 5 and that $\|u_j\|_{\mathbb{X}(0,\tau)}+\|v_j\|_{\mathbb{Y}(0,\tau)}\le C(\|X_0\|_{H^1},\|Y_0\|_{L^2})$. Hence, the implicit constants below may depend on $\|X_0\|_{H^1}$ and $\|Y_0\|_{L^2}$.

Applying Lemma 5.4, we get

$$\begin{aligned}\|\Phi_{1,1}(u_1,v_1)-\Phi_{1,1}(u_2,v_2)\|_{\mathbb{X}(0,\tau)} &= \|\Omega_{\mathrm{b}}(v_1-v_2,u_1)+\Omega_{\mathrm{b}}(v_2,u_1-u_2)\|_{\mathbb{X}(0,\tau)}\\ &\lesssim (K^{-1}+\tau^{\frac18})(\|v_1-v_2\|_{\mathbb{Y}(0,\tau)}\|u_1\|_{\mathbb{X}(0,\tau)}+\|v_2\|_{\mathbb{Y}(0,\tau)}\|u_1-u_2\|_{\mathbb{X}(0,\tau)})\\ &\lesssim (K^{-1}+\tau^{\frac18})\|(u_1,v_1)-(u_2,v_2)\|_{\mathbb{X}(0,\tau)\times\mathbb{Y}(0,\tau)}.\end{aligned}$$

In the same way, Lemma 5.5 yields

$$\begin{aligned}\|\Phi_{1,2}(u_1,v_1)-\Phi_{1,2}(u_2,v_2)\|_{\mathbb{X}(0,\tau)} &\le \|\mathcal{I}_t((v_1-v_2)u_1)_R\|_{\mathbb{X}(0,\tau)}+\|\mathcal{I}_t(v_2(u_1-u_2))_R\|_{\mathbb{X}(0,\tau)}\\ &\lesssim \tau^{\frac14}K\log_2 K\|(u_1,v_1)-(u_2,v_2)\|_{\mathbb{X}(0,\tau)\times\mathbb{Y}(0,\tau)},\end{aligned}$$

and

$$\|\Phi_{1,3}(u_1,v_1)-\Phi_{1,3}(u_2,v_2)\|_{\mathbb{X}(0,\tau)}\lesssim \|W^*\|_{C([0,\tau])}\|u_1-u_2\|_{\mathbb{X}(0,\tau)}.$$

We also derive via Lemma 5.7 that

$$\|\Phi_{1,4}(u_1,v_1)-\Phi_{1,4}(u_2,v_2)\|_{\mathbb{X}(0,\tau)}\lesssim \tau^{\frac14}(\|u_1\|^2_{\mathbb{X}(0,\tau)}+\|u_2\|^2_{\mathbb{X}(0,\tau)})\|u_1-u_2\|_{\mathbb{X}(0,\tau)}\lesssim \tau^{\frac14}\|u_1-u_2\|_{\mathbb{X}(0,\tau)}$$

and

$$\begin{aligned}\|\Phi_{1,5}(u_1,v_1)-\Phi_{1,5}(u_2,v_2)\|_{\mathbb{X}(0,\tau)} &\lesssim \|\mathcal{I}.\Omega_{\mathrm{b}}(v_1-v_2,v_1u_1)\|_{\mathbb{X}(0,\tau)}+\|\mathcal{I}.\Omega_{\mathrm{b}}(v_2,(v_1-v_2)u_1)\|_{\mathbb{X}(0,\tau)}\\ &\quad+\|\mathcal{I}.\Omega_{\mathrm{b}}(v_2,v_2(u_1-u_2))\|_{\mathbb{X}(0,\tau)}\\ &\lesssim \tau^{\frac18}\big(\|v_1-v_2\|_{\mathbb{Y}(0,\tau)}\|v_1\|_{\mathbb{Y}(0,\tau)}\|u_1\|_{\mathbb{X}(0,\tau)}\\ &\qquad+\|v_2\|_{\mathbb{Y}(0,\tau)}\|v_1-v_2\|_{\mathbb{Y}(0,\tau)}\|u_1\|_{\mathbb{X}(0,\tau)}\\ &\qquad+\|v_2\|_{\mathbb{Y}(0,\tau)}\|v_2\|_{\mathbb{Y}(0,\tau)}\|u_1-u_2\|_{\mathbb{X}(0,\tau)}\big)\\ &\lesssim \tau^{\frac18}\|(u_1,v_1)-(u_2,v_2)\|_{\mathbb{X}(0,\tau)\times\mathbb{Y}(0,\tau)}.\end{aligned}$$

Regarding the last term $\Phi_{1,6}$, we obtain

$$\begin{aligned}\|\Phi_{1,6}(u_1,v_1)-\Phi_{1,6}(u_2,v_2)\|_{\mathbb{X}(0,\tau)} &\lesssim \|\mathcal{I}.\Omega_{\mathrm{b}}(v_1-v_2,b\cdot\nabla u_1+cu_1-\mathcal{T}.(W_2)u_1)\|_{\mathbb{X}(0,\tau)}\\ &\quad+\|\mathcal{I}.\Omega_{\mathrm{b}}(v_2,b\cdot\nabla(u_1-u_2)+c(u_1-u_2)-\mathcal{T}.(W_2)(u_1-u_2))\|_{\mathbb{X}(0,\tau)}\\ &\lesssim \tau^{\frac18}\|v_1-v_2\|_{\mathbb{Y}(0,\tau)}\|W^*\|_{C([0,\tau])}\|u_1\|_{\mathbb{X}(0,\tau)}\\ &\quad+\tau^{\frac18}\|v_2\|_{\mathbb{Y}(0,\tau)}\|W^*\|_{C([0,\tau])}\|u_1-u_2\|_{\mathbb{X}(0,\tau)}\\ &\lesssim \tau^{\frac18}\|(u_1,v_1)-(u_2,v_2)\|_{\mathbb{X}(0,\tau)\times\mathbb{Y}(0,\tau)},\end{aligned}$$

where we also used the fact that $\|W^*\|_{C([0,\tau])}\lesssim 1$ in the last step.

We thus showed that

$$\begin{aligned}&\|\Phi(u_1,v_1)-\Phi(u_2,v_2)\|_{\mathbb{X}(0,\tau)\times\mathbb{Y}(0,\tau)}\\ &\le C'(\|X_0\|_{H^1},\|Y_0\|_{L^2})(K^{-1}+\tau^{\frac18}+\tau^{\frac14}K\log_2 K+\|W^*\|_{C([0,\tau])})\|(u_1,v_1)-(u_2,v_2)\|_{\mathbb{X}(0,\tau)\times\mathbb{Y}(0,\tau)}.\end{aligned}$$

Taking $K=K(\|X_0\|_{H^1},\|Y_0\|_{L^2})$ possibly larger such that $K^{-1}C'(\|X_0\|_{H^1},\|Y_0\|_{L^2})<\frac{1}{10}$, updating the definition of τ, and defining the $\{\mathscr{F}_t\}$-stopping time

$$\tau_1:=\inf\{t\in[0,T]\colon C'(\|X_0\|_{H^1},\|Y_0\|_{H^1})(t^{\frac18}K\log_2 K+W^*(t))\ge\frac{1}{10}\}\wedge\tau, \tag{6.10}$$

we infer that Φ is a contractive selfmapping on $B_{\mathbb{X}(0,\tau_1)\times\mathbb{Y}(0,\tau_1)}(M)$. Note that $\tau_1>0$ $\mathbb{P}$-a.s. as $W^*(t)\to 0$ for $t\to 0$ $\mathbb{P}$-a.s.

In conclusion, we obtain that there exists a small constant $\varepsilon_*(\|X_0\|_{H^1},\|Y_0\|_{L^2})>0$ which is decreasing with respect to its arguments such that Φ is a contractive selfmapping on a closed ball in $\mathbb{X}(0,\tau_1)\times\mathbb{Y}(0,\tau_1)$, where

$$\tau_1=\inf\{t\in[0,T]:t\geq\varepsilon_*(\|X_0\|_{H^1},\|Y_0\|_{L^2}),W^*(t)\geq\varepsilon_*(\|X_0\|_{H^1},\|Y_0\|_{L^2})\}\wedge T \tag{6.11}$$

is an $\{\mathscr{F}_t\}$-stopping time. Hence, there exists $(\widetilde{u}_1,\widetilde{v}_1)\in C([0,\tau_1];H^1\times L^2)\cap(\mathbb{X}(0,\tau_1)\times\mathbb{Y}(0,\tau_1))$ such that $\Phi_1(\widetilde{u}_1,\widetilde{v}_1)=(\widetilde{u}_1,\widetilde{v}_1)$. Letting $(u_1(t),v_1(t)):=(\widetilde{u}_1(t\wedge\tau_1),\widetilde{v}_1(t\wedge\tau_1))$ for all $t\in[0,T]$, we infer that (u_1,v_1) is an $\{\mathscr{F}_t\}$-adapted continuous process in $H^1\times L^2$ (see e.g. [2] for the relevant arguments) and solves (4.9) on $[0,\tau_1]$, thus also (2.7) due to the equivalence results in Propositions A.1 and B.1.

Step 3. In order to extend the solution to the maximal existence time, we use the refined rescaling transforms and the gluing procedure combined with induction arguments.

Suppose that for some $n\geq 1$, (u_n,v_n) is an $\{\mathscr{F}_t\}$-adapted continuous process in $H^1\times L^2$ which solves (1.1) on $[0,\sigma_n]$ and satisfies $(u_n,v_n)\equiv(u_n(\sigma_n),v_n(\sigma_n))$ on $[\sigma_n,T]$, where $\sigma_n(\leq T)$ is an $\{\mathscr{F}_t\}$-stopping time.

Analogous to (6.2) and (6.3), we define the operator $\Phi_{n+1}=(\Phi_{n+1,1},\Phi_{n+1,2})$ on $\mathbb{X}(0,T)\times\mathbb{Y}(0,T)$ by

$$\begin{aligned}\Phi_{n+1,1}(u,v)(t):=&\,\mathcal{U}_t(u_{0,n}+\Omega_{\mathrm{b}}(v_{0,n},u_{0,n}))-\Omega_{\mathrm{b}}(v,u)-\mathrm{i}\mathcal{I}_t(vu)_R+\mathrm{i}\mathcal{I}_t(b_{\sigma_n}\cdot\nabla u+c_{\sigma_n}u-\mathcal{T}_{\sigma_n+\cdot,\sigma_n}(W_2)u)\\&+\mathrm{i}\mathcal{I}_t\Omega_{\mathrm{b}}(|\nabla||u|^2,u)-\mathrm{i}\mathcal{I}_t\Omega_{\mathrm{b}}(v,vu)+\mathrm{i}\mathcal{I}_t\Omega_{\mathrm{b}}(v,b_{\sigma_n}\cdot\nabla u+c_{\sigma_n}u-\mathcal{T}_{\sigma_n+\cdot,\sigma_n}(W_2)u)\end{aligned} \tag{6.12}$$

and

$$\Phi_{n+1,2}(u,v)(t):=\mathcal{V}_tv_0+\mathrm{i}\int_0^t e^{\mathrm{i}(t-s)|\nabla|}(|\nabla||u|^2)(s)\,\mathrm{d}s \tag{6.13}$$

for all $t\in[0,T]$, where $\mathcal{T}_{\sigma_n+\cdot,\sigma_n}(W_2)$ is given by (3.9), the new perturbation coefficients b_{σ_n} and c_{σ_n} are defined as in (2.9) and (2.10), respectively, with W_1 replaced by W_{1,σ_n} defined in (3.8), i.e.,

$$b_{\sigma_n}(t)=2\nabla W_{1,\sigma_n}(t),\quad c_{\sigma_n}(t)=|\nabla W_{1,\sigma_n}(t)|^2+\Delta W_{1,\sigma_n}(t),\quad t\in[0,T],$$

and the initial data is given by

$$(u_{0,n},v_{0,n}):=(e^{W_1(\sigma_n)}u_n(\sigma_n),v_n(\sigma_n)+\mathcal{T}_{\sigma_n}(W_2)).$$

Furthermore, we set

$$\begin{aligned}W^*_{\sigma_n}(t):=&\,\|W_{1,\sigma_n}(t)\|_{H^3}+\sum_{j=1}^3\sum_{k=1}^\infty\int\sup_{y\in\mathbb{R}^2}|\phi_k^{(1)}(r\mathbf{e}_j+y)|\,\mathrm{d}r\sup_{s\in[0,t]}|\beta_k^{(1)}(\sigma_n+s)-\beta_k^{(1)}(\sigma_n)|\\&+\|W_{1,\sigma_n}(t)\|^2_{H^3}+\|\mathcal{T}_{\sigma_n+t,\sigma_n}(W_2)\|_{H^1}.\end{aligned} \tag{6.14}$$

We define the $\{\mathcal{F}_{\sigma_n+t}\}$-stopping time

$$\tau_{n+1}:=\inf\{t\in[0,T]\colon t\geq\varepsilon_*(\|u_{0,n}\|_{H^1},\|v_{0,n}\|_{L^2}),W^*_{\sigma_n}(t)\geq\varepsilon_*(\|u_{0,n}\|_{H^1},\|v_{0,n}\|_{L^2})\}\wedge(T-\sigma_n).$$

and

$$\sigma_{n+1}:=\sigma_n+\tau_{n+1}.$$

Then σ_{n+1} is an $\{\mathcal{F}_t\}$-stopping time, see e.g. [2] for the relevant arguments, and $\sigma_{n+1}\leq T$.

Proceeding as in Step 1 and Step 2, we infer that Φ_{n+1} is a contractive selfmapping on a closed subset of $\mathbb{X}(0,\tau_{n+1})\times\mathbb{Y}(0,\tau_{n+1})$, and thus there exists $(\widetilde{u}_{\sigma_{n+1}},\widetilde{v}_{\sigma_{n+1}})\in\mathbb{X}(0,\tau_{n+1})\times\mathbb{Y}(0,\tau_{n+1})$ such that $\Phi_{n+1}(\widetilde{u}_{\sigma_{n+1}},\widetilde{v}_{\sigma_{n+1}})=(\widetilde{u}_{\sigma_{n+1}},\widetilde{v}_{\sigma_{n+1}})$. In particular, setting

$$(u_{\sigma_{n+1}}(t),v_{\sigma_{n+1}}(t)):=(\widetilde{u}_{\sigma_{n+1}}(t\wedge\tau_{n+1}),\widetilde{v}_{\sigma_{n+1}}(t\wedge\tau_{n+1}))$$

for all $t\in[0,T]$, we infer that $(u_{\sigma_{n+1}},v_{\sigma_{n+1}})$ is an $\{\mathscr{F}_{\sigma_n+t}\}$-adapted continuous process in $H^1\times L^2$ and solves (3.7) on $[0,\tau_{n+1}]$, due to Propositions A.1 and B.1.

We then use Proposition 3.3 to glue together (u_n,v_n) and $(u_{\sigma_{n+1}},v_{\sigma_{n+1}})$, i.e.,

$$\begin{aligned}u_{n+1}(t)&:=u_n(t)\chi_{[0,\sigma_n)}(t)+e^{-W_1(\sigma_n)}u_{\sigma_{n+1}}((t-\sigma_n)\wedge\tau_{n+1})\chi_{[\sigma_n,T]}(t),\\v_{n+1}(t)&:=v_n(t)\chi_{[0,\sigma_n)}(t)+\Big(v_{\sigma_{n+1}}((t-\sigma_n)\wedge\tau_{n+1})-e^{\mathrm{i}((t-\sigma_n)\wedge\tau_{n+1})|\nabla|}\mathcal{T}_{\sigma_n}(W_2)\Big)\chi_{[\sigma_n,T]}(t).\end{aligned}$$

We obtain that (u_{n+1}, v_{n+1}) is an $\{\mathscr{F}_t\}$-adapted continuous process in $H^1 \times L^2$ (see [2] for the relevant arguments), $(u_{n+1}, v_{n+1}) = (u_n, v_n)$ on $[0, \sigma_n)$, and $(u_{n+1}, v_{n+1}) \equiv (u_{n+1}(\sigma_{n+1}), v_{n+1}(\sigma_{n+1}))$ on $[\sigma_{n+1}, T]$. Moreover, in view of Proposition 3.3, (u_{n+1}, v_{n+1}) solves (2.7) on the larger time interval $[0, \sigma_{n+1}]$.

Proceeding inductively, we thus get an increasing sequence of $\{\mathscr{F}_t\}$-stopping times $\{\sigma_n\}$ and an $\{\mathscr{F}_t\}$-stopping time $\tau_T^*(\leq T)$, as well as corresponding $\{\mathscr{F}_t\}$-adapted continuous processes $(u_n, v_n) \in C([0,T]; H^1 \times L^2)$, $n \geq 1$, such that $\sigma_n \to \tau_T^*$ as $n \to \infty$, (u_n, v_n) solves (1.1) on $[0, \sigma_n]$, and (u_{n+1}, v_{n+1}) coincides with (u_n, v_n) on $[0, \sigma_n]$ for all $n \geq 1$. In particular, setting $u^T := \lim_{n\to\infty} u_n \chi_{[0,\tau_T^*)}$ and $v^T := \lim_{n\to\infty} v_n \chi_{[0,\tau_T^*)}$, we obtain an analytically weak solution (u^T, v^T) to (2.7) on $[0, \tau_T^*)$. As the uniqueness of solutions is a local property, it can be derived using similar arguments as in Step 2.

It is clear that τ_T^* is increasing with T, and consequently there exists an $\{\mathscr{F}_t\}$-stopping time τ^* such that $\tau_T^* \to \tau^*$ as $T \to \infty$, $\mathbb{P}$-a.s. We set $u := \lim_{T\to\infty} u^T \chi_{[0,\tau^*)}$ and $v := \lim_{T\to\infty} v^T \chi_{[0,\tau^*)}$. By the uniqueness of solutions on $[0, \tau_T^*)$, we thus obtain a unique solution (u, v) to (2.7) on $[0, \tau^*)$.

Using Theorem 3.1, we finally get a unique solution (X, Y) on $[0, \tau^*)$ in the sense of Definition 1.1 by setting $(X, Y) = (e^{W_1} u, v + \mathcal{T}_\cdot(W_2))$.

Step 4. It remains to prove the blowup in (1.12) if $\tau^* < \infty$. Note that by the boundedness of $\|e^{W_1}\|_{H^2}$ and of $\|\mathcal{T}_\cdot(W_2)\|_{L^2}$ on bounded time intervals, the blow-up alternative (1.12) is equivalent to $\limsup_{t\to\tau^*} (\|u(t)\|_{H^1} + \|v(t)\|_{L^2}) = \infty$.

Below we consider $\omega \in \Omega$ such that $\tau^*(\omega) < \infty$ and the convergence (6.1) holds. For simplicity, the dependence on ω is omitted in the following.

We argue by contradiction. Suppose that $\tau^* < \infty$ and

$$\limsup_{t\to\tau^*} (\|u(t)\|_{H^1} + \|v(t)\|_{L^2}) < \infty.$$

Then there exists a constant C^* such that

$$\begin{aligned}\|u_{0,n}\|_{H^1} + \|v_{0,n}\|_{L^2} &= \|e^{W_1(\sigma_n)} u_n(\sigma_n)\|_{H^1} + \|v_n(\sigma_n) + \mathcal{T}_{\sigma_n}(W_2))\|_{L^2} \\ &\leq (\|W_1(\sigma_n)\|_{H^2} + 1)\|u_n(\sigma_n)\|_{H^1} + \|v_n(\sigma_n)\|_{L^2} + \|\mathcal{T}_{\sigma_n}(W_2)\|_{L^2} \leq C^* < \infty\end{aligned}$$

for all $n \in \mathbb{N}$. Let $T(= T(\omega)) \in (0, \infty)$ be such that $\tau^* < T$ and $\{\sigma_n, \tau_n\}$ be constructed as above. Then, by construction, $\sigma_n < T$ for any $n \geq 1$, and since $\varepsilon_*(\cdot, \cdot)$ is decreasing with respect to the arguments, we have either

$$\tau_{n+1} = \varepsilon_*(\|u_{0,n}\|_{H^1}, \|v_{0,n}\|_{L^2}) \geq \varepsilon_*(C^*, C^*), \tag{6.15}$$

or

$$W^*_{\sigma_n}(\tau_{n+1}) = \varepsilon_*(\|u_{0,n}\|_{H^1}, \|v_{0,n}\|_{L^2}) \geq \varepsilon_*(C^*, C^*). \tag{6.16}$$

We first discuss the case (6.16). In view of the convergence (6.1), there exists $n_l \in \mathbb{N}$ such that

$$\sum_{j=1}^{3} \sum_{k=n_l+1}^{\infty} \int \sup_{y\in\mathbb{R}^2} |\phi_k^{(1)}(r\mathbf{e}_j + y)| \, dr \sup_{t\in[0,T]} |\beta_k^{(1)}(t)| \leq \frac{1}{20} \varepsilon_*(C^*, C^*).$$

Moreover, by the C^κ-Hölder continuity of Brownian motions, we infer for the first n_l modes of the noise

$$\begin{aligned}&\sum_{j=1}^{3} \sum_{k=1}^{n_l} \int \sup_{y\in\mathbb{R}^2} |\phi_k^{(1)}(r\mathbf{e}_j + y)| \, dr \sup_{s\in[0,\tau_{n+1}]} |\beta_k^{(1)}(\sigma_n + s) - \beta_k^{(1)}(\sigma_n)| \\ &\leq \sum_{j=1}^{3} \sum_{k=1}^{n_l} \int \sup_{y\in\mathbb{R}^2} |\phi_k^{(1)}(r\mathbf{e}_j + y)| \, dr \, \widetilde{C}(k, \kappa, T) \tau_{n+1}^\kappa =: \widetilde{C}(n_l, \kappa, T) \tau_{n+1}^\kappa,\end{aligned} \tag{6.17}$$

where $\widetilde{C}(k, \kappa, T)$ is the (random) C^κ-Hölder norm of $\beta_k^{(1)}$ for $1 \leq k \leq n_l$. Hence, we obtain

$$\begin{aligned}&\sum_{j=1}^{3} \sum_{k=1}^{\infty} \int \sup_{y\in\mathbb{R}^2} |\phi_k^{(1)}(r\mathbf{e}_j + y)| \, dr \sup_{s\in[0,\tau_{n+1}]} |\beta_k^{(1)}(\sigma_n + s) - \beta_k^{(1)}(\sigma_n)| \\ &\leq \frac{1}{10} \varepsilon_*(C^*, C^*) + \widetilde{C}(n_l, \kappa, T) \tau_{n+1}^\kappa.\end{aligned}$$

The Hölder continuity of the noise in Lemma 6.1 also yields that there exists $\widetilde{C}'(\kappa,T)$ such that

$$\begin{aligned}
&\|W_{1,\sigma_n}(\tau_{n+1})\|_{H^3} + \|W_{1,\sigma_n}(\tau_{n+1})\|_{H^3}^2 + \|\mathcal{T}_{\sigma_n+\tau_{n+1},\sigma_n}(W_2)\|_{H^1} \\
&= \|W_1(\sigma_n+\tau_{n+1}) - W_1(\sigma_n)\|_{H^3} + \|W_1(\sigma_n+\tau_{n+1}) - W_1(\sigma_n)\|_{H^3}^2 \\
&\quad + \Big\| \int_{\sigma_n}^{\sigma_n+\tau_{n+1}} e^{\mathrm{i}(\sigma_n+\tau_{n+1}-s)|\nabla|} \, \mathrm{d}W_2(s) \Big\|_{H^1} \\
&\leq \widetilde{C}'(\kappa,T)\tau_{n+1}^{\kappa}.
\end{aligned}$$

Combining the above two estimates and the definition of $W^*_{\sigma_n}$ in (6.14), we infer

$$W^*_{\sigma_n}(\tau_{n+1}) \leq \frac{1}{10}\varepsilon_*(C^*,C^*) + (\widetilde{C}^*(n_l,\kappa,T) + \widetilde{C}'(\kappa,T))\tau_{n+1}^{\kappa}.$$

If (6.16) holds, we thus get the uniform lower bound

$$\tau_{n+1} \geq \left(\frac{9}{10}\varepsilon_*(C^*,C^*)(\widetilde{C}^*(n_l,\kappa,T) + \widetilde{C}'(\kappa,T))^{-1} \right)^{\frac{1}{\kappa}}. \tag{6.18}$$

Since (6.15) also gives such a lower bound for the extension time τ_{n+1}, we conclude that σ_{n+1} reaches T after finitely many steps, i.e., $\sigma_N = T$ for some N large enough. Consequently, we have the contradiction $\tau^* < T = \sigma_N < \tau^*$, which proves (1.12).

Therefore, the proof of Theorem 1.2 is complete. □

7. Well-posedness below the ground state

In this section we prove Theorem 1.5 using the variational analysis of the ground state Q.

Recall that E_Z and E_S, introduced in (1.4) and (1.7), are the energies of the Zakharov system (2.3) and the focusing cubic NLS (1.5), respectively. Also recall the action functional J from (1.14) and the corresponding scaling derivative functional K from (1.15).

For any $\lambda > 0$, set $Q_\lambda(x) := \lambda Q(\lambda x)$, where Q is the ground state from (1.8). Then Q_λ minimizes the action

$$J_\lambda := E_S + \lambda^2 M,$$

that is, Q has the variational characterization

$$\lambda J(Q) = J_\lambda(Q_\lambda) = \inf\{J_\lambda(\varphi) : \varphi \neq 0,\ K(\varphi) = 0\}, \tag{7.1}$$

see [30].

In order to prove Theorem 1.5, we adapt arguments from [30], which are based on the variational characterization (7.1) of the ground state and the mass conservation of the Schrödinger component.

Proof of Theorem 1.5. In view of the blow-up alternative in Theorem 1.2, it is sufficient to prove for $\mathbb{P}$-a.e. $\omega \in \Omega$ the boundedness of the $H^1 \times L^2$-norm of the solution $(u(t,\omega), v(t,\omega))$ on $[0, \sigma_*(\omega) \wedge \tau^*(\omega))$, i.e.,

$$\sup_{t \in [0,\sigma_*(\omega)\wedge\tau^*(\omega))} (\|u(t,\omega)\|_{H^1} + \|v(t,\omega)\|_{L^2}) < \infty. \tag{7.2}$$

For simplicity, we omit the dependence on ω in the following.

Without loss of generality we assume $u_0 \neq 0$, as otherwise the uniqueness statement in Theorem 1.2 leads to $u \equiv 0$ which in turn implies (7.2). We next note that the conservation of mass (1.2) yields for any $\lambda > 0$ and $t \in [0, \sigma_* \wedge \tau^*)$

$$E_Z(u(t),v(t)) + \lambda^2 M(u(t)) - \lambda J(Q) = M(u_0)\left(\left(\lambda - \frac{J(Q)}{2M(u_0)}\right)^2 + \frac{4E_Z(u(t),v(t))M(u_0) - J^2(Q)}{4M^2(u_0)} \right).$$

Using the definition of the stopping time σ_* in (1.16) and the conservation of mass again, we derive for any $t \in [0, \sigma_* \wedge \tau^*)$

$$4E_Z(u(s),v(s))M(u_0) = 4E_Z(u(s),v(s))M(u(s)) < 4E_S(Q)M(Q) = J^2(Q),$$

where the last identity is a property of the ground state which follows from the variational characterization in (7.1), see [30, p. 417].

Hence, we derive that for $\lambda_* := \frac{J(Q)}{2M(u_0)}$, independent of ω and t,

$$E_Z(u(t), v(t)) + \lambda_*^2 M(u(t)) < \lambda_* J(Q) = J_{\lambda_*}(Q_{\lambda_*}) \tag{7.3}$$

for all $t \in [0, \sigma_* \wedge \tau^*)$, which yields via (1.6)

$$J_{\lambda_*}(u(t)) \leq E_Z(u(t), v(t)) + \lambda_*^2 M(u(t)) < J_{\lambda_*}(Q_{\lambda_*}) \tag{7.4}$$

for all $t \in [0, \sigma_* \wedge \tau^*)$. Suppose that $K(u(t_0)) = 0$ for some $t_0 \in [0, \sigma_* \wedge \tau^*)$ and $u(t_0) \neq 0$. Then the variational characterization (7.1) of Q_{λ_*} implies that $J_{\lambda_*}(Q_{\lambda_*}) \leq J_{\lambda_*}(u(t_0))$, which, however, leads to a contradiction with (7.4).

Consequently, we obtain that for any $t \in [0, \sigma_* \wedge \tau^*)$,

$$K(u(t)) = 0 \Leftrightarrow u(t) = 0. \tag{7.5}$$

If $K(u_0) = 0$, (7.5) yields a contradiction to $u_0 \neq 0$.

In the case $K(u_0) > 0$, we infer $K(u(t)) > 0$ for all $t \in [0, \sigma_* \wedge \tau^*)$ from (7.5). Combining (1.6) and (1.15) with (7.3), we thus arrive at

$$\frac{1}{6}\|\nabla u(s)\|_{L^2}^2 + \frac{\lambda_*^2}{2}\|u(s)\|_{L^2} + \frac{1}{4}\|v(s) + |u(s)|^2\|_{L^2}^2 = E_Z(u(s), v(s)) + \lambda_*^2 M(u(s)) - \frac{1}{3}K(u(s)) < \lambda_* J(Q)$$

for all $t \in [0, \sigma_* \wedge \tau^*)$. In particular, we get the uniform bound

$$\|u(t)\|_{H^1} + \|v(t) + |u(t)|^2\|_{L^2}^2 \lesssim 1 \tag{7.6}$$

on $[0, \sigma_* \wedge \tau^*)$. Employing the Sobolev embedding $H^1 \hookrightarrow L^4$, we further derive

$$\|v(t)\|_{L^2}^2 \lesssim \|v(t) + |u(t)|^2\|_{L^2}^2 + \|u(t)\|_{L^4}^4 \lesssim \|v(t) + |u(t)|^2\|_{L^2}^2 + \|u(t)\|_{H^1}^4 \lesssim 1, \tag{7.7}$$

for all $t \in [0, \sigma_* \wedge \tau^*)$. The estimates in (7.6) and (7.7) thus imply the $H^1 \times L^2$-boundedness (7.2) of the solution. ☐

8. Regularization by noise

In this section we prove Theorem 1.7 concerning the regularization effect of the non-conservative noise on the well-posedness of the Zakharov system.

As mentioned in Section 1, the key point is to prove Strichartz estimates for the Schrödinger equation with the potential $v_1(t) := e^{\mathrm{i}t|\nabla|}Y_0$ on a bounded time interval. Let $\mathcal{U}_{v_1}$ and $\mathcal{I}_{v_1}$ denote the evolution operators of the homogeneous and inhomogeneous flows generated by the operator $(\partial_t - \mathrm{i}\Delta + \mathrm{i}v_1)$, respectively.

For any interval $I \subseteq [0, T]$, we define the spaces

$$S^1(I) := \{f \in C(I; H_x^1) \colon \|f\|_{S^1(I)} < \infty\}, \quad N^1(I) := \{g \in L^1(I; L_x^2) \colon \|g\|_{N^1(I)} < \infty\}$$

endowed with the norms

$$\|f\|_{S^1(I)} := \|f\|_{L^\infty(I; H_x^1)} + \Big(\sum_{N \in 2^{\mathbb{Z}}} \langle N \rangle^2 \|P_N f\|_{L^2(I; L_x^6)}^2\Big)^{\frac{1}{2}},$$

$$\|g\|_{N^1(I)} := \Big(\sum_{N \in 2^{\mathbb{Z}}} \|P_N g\|_{G_N(I)}^2\Big)^{\frac{1}{2}},$$

where

$$\|g\|_{G_N(I)} := \inf_{g = g_1 + g_2} \Big(\langle N \rangle \|g_1\|_{L^1(I; L_x^2)} + \langle N \rangle \|g_2\|_{L^{\frac{8}{5}}(I; L_x^{\frac{4}{3}})}\Big), \quad N \in 2^{\mathbb{Z}}.$$

We point out that $S^1(0, T) \subseteq C(I; H^1)$. Moreover, we use the space $Y^1(I) = C(I; H^1)$ for the wave component and still denote $C(I; L^2)$ by $\mathbb{Y}(I)$.

Note that compared with the $\mathbb{X}(I)$- and the $\mathbb{G}(I)$-norm from Section 5, we just dropped the local smoothing component. The local smoothing component was necessary to treat the lower order perturbation terms arising from the conservative noise in (2.8), which do not appear in (2.15) as the nonconservative noise is a one-dimensional Brownian motion.

In the next lemma we provide Strichartz estimates for the operators $\mathcal{U}_{v_1}$ and $\mathcal{I}_{v_1}$.

Lemma 8.1. *Let $T \in (0,\infty)$, $Y_0 \in H^1$, and $v_1(t) = e^{\mathrm{i}t|\nabla|}Y_0$ for all $t \in [0,T]$. Then, there is a constant $C = C(T, \|Y_0\|_{H^1}) > 0$, increasing in its arguments, such that*

$$\|\mathcal{U}_{v_1} f\|_{S^1(0,T)} \le C\|f\|_{H^1}, \tag{8.1}$$

$$\|\mathcal{I}_{v_1} g\|_{S^1(0,T)} \le C\|g\|_{N^1(0,T)} \tag{8.2}$$

for all $f \in H^1$ and $g \in N^1(0,T)$.

Proof. We define a partition $0 = t_0 < t_1 < \ldots < t_L < t_{L+1} = T$ of $[0,T]$ by setting $t_0 := 0$ and

$$t_{n+1} := \inf\left\{ t \in (t_n, T] : C_0(t-t_n)^{\frac14}\|Y_0\|_{H^1} \ge \frac12 \right\} \wedge T, \tag{8.3}$$

where $C_0(\ge 1)$ is the constant appearing in (8.6) and (8.7) below. Note that,

$$L \le (2C_0\|Y_0\|_{H^1})^4 T. \tag{8.4}$$

Set $u := \mathcal{U}_{v_1} f$. Then, u satisfies the equation

$$(\partial_t - \mathrm{i}\Delta)u = -\mathrm{i}v_1 u$$

on $[0,T]$ with $u(0) = f \in H^1$. Writing the Duhamel formulation for the above equation, we can apply Lemma 5.1 (i) for the homogeneous part and argue as in the proof of (5.20) for the inhomogeneous part, also using

$$\|v_1\|_{Y^1(0,T)} = \|Y_0\|_{H^1}, \tag{8.5}$$

in order to derive

$$\begin{aligned} \|u\|_{S^1(t_n,t_{n+1})} &\le C_0\|u(t_n)\|_{H^1} + C_0(t_{n+1}-t_n)^{\frac14}\|v_1\|_{Y^1(t_n,t_{n+1})}\|u\|_{S^1(t_n,t_{n+1})} \\ &\le C_0\|u(t_n)\|_{H^1} + C_0(t_{n+1}-t_n)^{\frac14}\|Y_0\|_{H^1}\|u\|_{S^1(t_n,t_{n+1})} \\ &\le C_0\|u(t_n)\|_{H^1} + \frac12\|u\|_{S^1(t_n,t_{n+1})}, \end{aligned} \tag{8.6}$$

for all $0 \le n \le L$, where we also used (8.3) in the last step. Consequently, we get

$$\|u\|_{S^1(t_n,t_{n+1})} \le 2C_0\|u(t_n)\|_{H^1}, \quad 0 \le n \le L,$$

and thus

$$\|u\|_{S^1(t_n,t_{n+1})} \le (2C_0)^{n+1}\|f\|_{H^1}, \quad 0 \le n \le L.$$

Taking into account (8.4), we obtain

$$\|u\|_{S^1(0,T)} \le \sum_{n=0}^{L}\|u\|_{S^1(t_n,t_{n+1})} \le \sum_{n=0}^{L}(2C_0)^{n+1}\|f\|_{H^1} \le \frac{1}{2C_0-1}(2C_0)^{(2C_0\|Y_0\|_{H^1})^4T+2}\|f\|_{H^1},$$

which proves (8.1).

Regarding (8.2), we set $w := \mathcal{I}_{v_1} g$ and note that w solves the equation

$$(\partial_t - \mathrm{i}\Delta)w = -\mathrm{i}v_1 w + g$$

with $w(0) = 0$. Proceeding as above, i.e. employing Lemma 5.1 (i), arguing as in the proof of (5.20), and using (8.3) and (8.5), we obtain

$$\begin{aligned} \|w\|_{S^1(t_n,t_{n+1})} &\le C_0\|w(t_n)\|_{H^1} + C_0(t_{n+1}-t_n)^{\frac14}\|v_1\|_{Y^1(t_n,t_{n+1})}\|w\|_{S^1(t_n,t_{n+1})} + C_0\|g\|_{N^1(t_n,t_{n+1})} \\ &\le C_0\|w(t_n)\|_{H^1} + C_0(t_{n+1}-t_n)^{\frac14}\|Y_0\|_{H^1}\|w\|_{S^1(t_n,t_{n+1})} + C_0\|g\|_{N^1(t_n,t_{n+1})} \\ &\le C_0\|w(t_n)\|_{H^1} + \frac12\|w\|_{S^1(t_n,t_{n+1})} + C_0\|g\|_{N^1(t_n,t_{n+1})}, \end{aligned} \tag{8.7}$$

which implies that for every $0 \le n \le L$

$$\|w\|_{S^1(t_n,t_{n+1})} \le 2C_0\|w(t_n)\|_{H^1} + 2C_0\|g\|_{N^1(t_n,t_{n+1})}. \tag{8.8}$$

We claim that

$$\|w\|_{S^1(t_n,t_{n+1})} \le (4C_0)^{n+1}\|g\|_{N^1(0,t_{n+1})} \quad \text{for all } 0 \le n \le L. \tag{8.9}$$

As a matter of fact, the base estimate (i.e., (8.9) with $n=0$) follows immediately from (8.8) and the fact that $w(0)=0$. Suppose that (8.9) is valid for some $0\le n<L$. We then derive from (8.8) and the imbedding $S^1(t_n,t_{n+1})\hookrightarrow C([t_n,t_{n+1}];H^1)$ that

$$\begin{aligned}\|w\|_{S^1(t_{n+1},t_{n+2})}&\le 2C_0\|w(t_{n+1})\|_{H^1}+2C_0\|g\|_{N^1(t_{n+1},t_{n+2})}\\&\le 2C_0(4C_0)^{n+1}\|g\|_{N^1(0,t_{n+1})}+2C_0\|g\|_{N^1(t_{n+1},t_{n+2})}\\&\le (4C_0)^{n+2}\|g\|_{N^1(0,t_{n+2})},\end{aligned}$$

which yields (8.9) at step $n+1$. By induction, we conclude (8.9) for every $0\le n\le L$ as claimed.

Therefore, we obtain from (8.4) and (8.9) that

$$\|w\|_{S^1(0,T)}\le\sum_{n=0}^{L}\|w\|_{S^1(t_n,t_{n+1})}\le\sum_{n=0}^{L}(4C_0)^{n+1}\|g\|_{N^1(0,t_{n+1})}\le C(T,\|Y_0\|_{H^1})\|g\|_{N^1(0,T)},$$

i.e. (8.2), finishing the proof. □

Using the above Strichartz estimates and arguing as in the proofs of Lemma 5.4, Lemma 5.5, and Lemma 5.7, we infer corresponding multilinear estimates for the flow operators $\mathcal{U}_{v_1}$ and $\mathcal{I}_{v_1}$ of the Schrödinger equation with potential v_1.

Corollary 8.2. *Let $T\in(0,\infty)$, $Y_0\in H^1$, and $v_1(t)=e^{\mathrm{i}t|\nabla|}Y_0$ for all $t\in[0,T]$. We then have*

$$\|\Omega_{\mathrm{b}}(v,u)\|_{S^1(0,T)}\lesssim(K^{-1}+T^{\frac18})\|v\|_{\mathbb{Y}(0,T)}\|u\|_{S^1(0,T)},\tag{8.10}$$

$$\Big\|\mathcal{I}_{v_1}(vu)_R\Big\|_{S^1(0,T)}\lesssim T^{\frac14}K\log_2 K\|v\|_{\mathbb{Y}(0,T)}\|u\|_{S^1(0,T)},\tag{8.11}$$

$$\Big\|\mathcal{I}_{v_1}\Omega_{\mathrm{b}}(w_1,w_2u)\Big\|_{S^1(0,T)}\lesssim T^{\frac18}\|w_1\|_{\mathbb{Y}(0,T)}\|w_2\|_{\mathbb{Y}(0,T)}\|u\|_{S^1(0,T)}.\tag{8.12}$$

Moreover, for any $h\in L^4(0,T)$ we have

$$\Big\|\mathcal{I}_{v_1}\Omega_{\mathrm{b}}(h|\nabla|(u_1u_2),u_3)\Big\|_{S^1((0,T)}\lesssim\|h\|_{L^4(0,T)}\|u_1\|_{S^1(0,T)}\|u_2\|_{S^1(0,T)}\|u_3\|_{S^1(0,T)},\tag{8.13}$$

$$\Big\|\int_0^{\cdot}e^{\mathrm{i}(\cdot-s)|\nabla|}(h(s)|\nabla|(u_1u_2)(s))\,\mathrm{d}s\Big\|_{\mathbb{Y}(0,T)}\lesssim\|h\|_{L^4(0,T)}\|u_1\|_{S^1(0,T)}\|u_2\|_{S^1(0,T)}.\tag{8.14}$$

In the above estimates, the implicit constant depends on T and $\|Y_0\|_{H^1}$ and is increasing in both variables.

We are now ready to prove Theorem 1.7.

Proof of Theorem 1.7. Fix $T\in(0,\infty)$. We first write system (2.21) in its mild form

$$\begin{cases}z=\mathcal{U}_{v_1}X_0-\Omega_{\mathrm{b}}(v_2,z)-\mathrm{i}\mathcal{I}_{v_1}(v_2z)_R+\mathrm{i}\mathcal{I}_{v_1}\Omega_{\mathrm{b}}(h|\nabla||z|^2,z)\\ \qquad -\mathrm{i}\mathcal{I}_{v_1}\Omega_{\mathrm{b}}(v_2,(v_1+v_2)z)+\mathrm{i}\mathcal{I}_{v_1}(v_1\Omega_{\mathrm{b}}(v_2,z)),\\ v_2=\mathrm{i}\displaystyle\int_0^{\cdot}e^{\mathrm{i}(\cdot-s)|\nabla|}(h|\nabla||z|^2)(s)\,\mathrm{d}s,\end{cases}\tag{8.15}$$

with h defined in (2.16) and v_1 and v_2 as in (2.19).

In order to solve (8.15), we define the fixed point operator Ψ on $S^1(0,T)$ by

$$\begin{aligned}\Psi(z)&=\mathcal{U}_{v_1}X_0-\Omega_{\mathrm{b}}(v_2(z),z)-\mathrm{i}\mathcal{I}_{v_1}(v_2(z)z)_R+\mathrm{i}\mathcal{I}_{v_1}\Omega_{\mathrm{b}}(h|\nabla||z|^2,z)\\&\quad-\mathrm{i}\mathcal{I}_{v_1}\Omega_{\mathrm{b}}(v_2(z),(v_1+v_2(z))z)+\mathrm{i}\mathcal{I}_{v_1}(v_1\Omega_{\mathrm{b}}(v_2(z),z))\\&=:\mathcal{U}_{v_1}X_0+\sum_{j=1}^{5}\Psi_j(z),\end{aligned}\tag{8.16}$$

where $v_2(z)$ is defined by

$$v_2(z)=\mathrm{i}\int_0^{\cdot}e^{\mathrm{i}(\cdot-s)|\nabla|}(h|\nabla||z|^2)(s)\,\mathrm{d}s.\tag{8.17}$$

We set $M:=2C_0(T,\|Y_0\|_{H^1})\|X_0\|_{H^1}$, where $C_0(T,\|Y_0\|_{H^1})(\ge 1)$ is the constant from Lemma 8.1. We will prove that Ψ is a contraction on the ball

$$B_{S^1(0,\tau)}(M):=\{z\in S^1(0,\tau):\|z\|_{S^1(0,\tau)}\le M\}$$

for some stopping time τ, which satisfies $\tau = T$ with high probability if ϕ_1 is large enough. The proof consists of the following three steps.

Step 1: We first show that $\Psi(B_{S^1(0,\tau)}(M)) \subseteq B_{S^1(0,\tau)}(M)$. For this purpose, we note that

$$\|v_2(z)\|_{\mathbb{Y}(0,\tau)} \leq C\|h\|_{L^4(0,\tau)}\|z\|^2_{S^1(0,\tau)}, \tag{8.18}$$

by (8.14) and that

$$\|\mathcal{U}_{v_1}X_0\|_{S^1(0,\tau)} \leq C_0(\tau, \|Y_0\|_{H^1})\|X_0\|_{H^1} \leq C_0(T, \|Y_0\|_{H^1})\|X_0\|_{H^1} \tag{8.19}$$

by Lemma 8.1 and $\tau \leq T$. We next estimate each $\Psi_j(z)$ for every $1 \leq j \leq 5$. The implicit constants may depend on T and $\|Y_0\|_{H^1}$.

By the estimate for the boundary term in (8.10) and (8.18), we infer

$$\begin{aligned} \|\Psi_1(z)\|_{S^1(0,\tau)} &\lesssim (K^{-1} + \tau^{\frac{1}{8}})\|v_2(z)\|_{\mathbb{Y}(0,\tau)}\|z\|_{S^1(0,\tau)} \\ &\lesssim (K^{-1} + \tau^{\frac{1}{8}})\|h\|_{L^4(0,\tau)}\|z\|^3_{S^1(0,\tau)}. \end{aligned} \tag{8.20}$$

Using (8.11), we also get

$$\|\Psi_2(z)\|_{S^1(0,\tau)} \lesssim \tau^{\frac{1}{4}} K\log_2 K\|v_2(z)\|_{\mathbb{Y}(0,\tau)}\|z\|_{S^1(0,\tau)} \lesssim \tau^{\frac{1}{4}} K\log_2 K\|h\|_{L^4(0,\tau)}\|z\|^3_{S^1(0,\tau)}, \tag{8.21}$$

while (8.13) gives

$$\|\Psi_3(z)\|_{S^1(0,\tau)} \lesssim \|h\|_{L^4(0,\tau)}\|z\|^3_{S^1(0,\tau)}. \tag{8.22}$$

Moreover, combining (8.12) with (8.18) and using $\|v_1\|_{\mathbb{Y}(0,\tau)} = \|Y_0\|_{L^2}$, we derive that

$$\begin{aligned} \|\Psi_4(z)\|_{S^1(0,\tau)} &\lesssim \tau^{\frac{1}{8}}\|v_2(z)\|_{\mathbb{Y}(0,\tau)}\|v_1 + v_2(z)\|_{\mathbb{Y}(0,\tau)}\|z\|_{S^1(0,\tau)} \\ &\lesssim \tau^{\frac{1}{8}}(\|Y_0\|_{L^2} + \|h\|_{L^4(0,\tau)}\|z\|^2_{S^1(0,\tau)})\|h\|_{L^4(0,\tau)}\|z\|^3_{S^1(0,\tau)}. \end{aligned} \tag{8.23}$$

Regarding the last term $\Psi_5(z)$, we apply Lemma 8.1 to estimate

$$\begin{aligned} \|\Psi_5(z)\|_{S^1(0,\tau)} &\lesssim \|v_1\Omega_{\mathrm{b}}(v_2(z), z)\|_{L^{\frac{4}{3}}(0,\tau;W_x^{1,\frac{3}{2}})} \\ &\lesssim \tau^{\frac{1}{4}}\|v_1\|_{C([0,\tau];H_x^1)}\|\Omega_{\mathrm{b}}(v_2(z), z)\|_{L^2(0,\tau;W_x^{1,6})}. \end{aligned}$$

Estimate (5.16) and (8.18) together with $\|v_1\|_{C([0,\tau];H_x^1)} = \|Y_0\|_{H^1}$ thus yield

$$\begin{aligned} \|\Psi_5(z)\|_{S^1(0,\tau)} &\lesssim \tau^{\frac{1}{4}}(K^{-1} + \tau^{\frac{1}{8}})\|v_1\|_{C([0,\tau];H^1)}\|v_2(z)\|_{\mathbb{Y}(0,\tau)}\|z\|_{S^1(0,\tau)} \\ &\lesssim \tau^{\frac{1}{4}}(K^{-1} + \tau^{\frac{1}{8}})\|Y_0\|_{H^1}\|h\|_{L^4(0,\tau)}\|z\|^3_{S^1(0,\tau)}. \end{aligned} \tag{8.24}$$

We fix $K = 2^5$ in the following. Combining estimates (8.19) to (8.24) and using $\|z\|_{S^1(0,\tau)} \leq M$, we thus conclude

$$\begin{aligned} \|\Psi(z)\|_{S^1(0,\tau)} &\leq C_0(T, \|Y_0\|_{H^1})\|X_0\|_{H^1} + C_1(\tau, \|Y_0\|_{H^1})(1 + \|h\|_{L^4(0,\tau)}\|z\|^2_{S^1(0,\tau)})\|h\|_{L^4(0,\tau)}\|z\|^3_{S^1(0,\tau)} \\ &\leq C_0(T, \|Y_0\|_{H^1})\|X_0\|_{H^1} + C_1(T, \|Y_0\|_{H^1})M^2\|h\|_{L^4(0,\tau)}\|z\|_{S^1(0,\tau)} \\ &\quad + C_1(T, \|Y_0\|_{H^1})M^4\|h\|^2_{L^4(0,\tau)}\|z\|_{S^1(0,\tau)} \end{aligned}$$

with $C_1(T, \|Y_0\|_{H^1}) \geq 1$. Taking the stopping time

$$\tau := \inf\left\{t \in [0,T] : C_1(T, \|Y_0\|_{H^1})M^2\|h\|_{L^4(0,t)} > \frac{1}{4}\right\} \wedge T, \tag{8.25}$$

we then obtain

$$\|\Psi(z)\|_{S^1(0,\tau)} \leq C_0(T, \|Y_0\|_{H^1})\|X_0\|_{H^1} + \frac{1}{2}\|z\|_{S^1(0,\tau)} \leq \frac{M}{2} + \frac{M}{2} = M. \tag{8.26}$$

Consequently, Ψ maps $B_{S^1(0,\tau)}(M)$ into itself.

Step 2: Next we show that $\Psi\colon B_{S^1(0,\tau)}(M) \to B_{S^1(0,\tau)}(M)$ is contractive. The proof is again based on the multilinear estimates. The implicit constants in this step may depend on T, $\|X_0\|_{H^1}$ and $\|Y_0\|_{H^1}$.

We take any $\widetilde{z} \in B_{S^1(0,\tau)}(M)$ and set $v_2(\widetilde{z}) := \mathrm{i}\int_0^{\cdot} e^{\mathrm{i}(\cdot-s)|\nabla|}(h|\nabla||\widetilde{z}|^2)(s)\,\mathrm{d}s$. As in (8.18), we then have

$$\|v_2(\widetilde{z})\|_{\mathbb{Y}(0,\tau)} \lesssim \|h\|_{L^4(0,\tau)}\|\widetilde{z}\|^2_{S^1(0,\tau)}. \tag{8.27}$$

Moreover, (8.14) and the definition of $B_{S^1(0,\tau)}(M)$ imply

$$\begin{aligned}\|v_2(z) - v_2(\widetilde{z})\|_{\mathbb{Y}(0,\tau)} &\lesssim \|h\|_{L^4(0,\tau)}(\|z\|_{S^1(0,\tau)} + \|\widetilde{z}\|_{S^1(0,\tau)})\|z-\widetilde{z}\|_{S^1(0,\tau)}\\ &\lesssim \|h\|_{L^4(0,\tau)}\|z-\widetilde{z}\|_{S^1(0,\tau)}.\end{aligned} \tag{8.28}$$

In order to estimate the difference between $\Psi(z)$ and $\Psi(\widetilde{z})$ in $S^1(0,\tau)$, we first note that by (8.10), (8.27) and (8.28), we infer

$$\begin{aligned}\|\Psi_1(z) - \Psi_1(\widetilde{z})\|_{S^1(0,\tau)} &= \|\Omega_{\mathrm{b}}(v_2(z)-v_2(\widetilde{z}), z) + \Omega_{\mathrm{b}}(v_2(\widetilde{z}), z-\widetilde{z})\|_{S^1(0,\tau)}\\ &\lesssim (K^{-1}+\tau^{\frac18})(\|v_2(z)-v_2(\widetilde{z})\|_{\mathbb{Y}(0,\tau)}\|z\|_{S^1(0,\tau)} + \|v_2(\widetilde{z})\|_{\mathbb{Y}(0,\tau)}\|z-\widetilde{z}\|_{S^1(0,\tau)})\\ &\lesssim \|h\|_{L^4(0,\tau)}\|z-\widetilde{z}\|_{S^1(0,\tau)}.\end{aligned} \tag{8.29}$$

Similarly, using (8.11) we get

$$\begin{aligned}\|\Psi_2(z) - \Psi_2(\widetilde{z})\|_{S^1(0,\tau)} &= \|\mathcal{I}_{v_1}((v_2(z)-v_2(\widetilde{z}))z)_R + \mathcal{I}_{v_1}(v_2(\widetilde{z})(z-\widetilde{z}))_R\|_{S^1(0,\tau)}\\ &\lesssim \tau^{\frac14}K\log_2 K(\|v_2(z)-v_2(\widetilde{z})\|_{\mathbb{Y}(0,\tau)}\|z\|_{S^1(0,\tau)} + \|v_2(\widetilde{z})\|_{\mathbb{Y}(0,\tau)}\|z-\widetilde{z}\|_{S^1(0,\tau)})\\ &\lesssim \|h\|_{L^4(0,\tau)}\|z-\widetilde{z}\|_{S^1(0,\tau)},\end{aligned} \tag{8.30}$$

where we still keep $K = 2^5$ fixed. Applying (8.13), we also derive

$$\begin{aligned}\|\Psi_3(z) - \Psi_3(\widetilde{z})\|_{S^1(0,\tau)} &\lesssim \|h\|_{L^4(0,\tau)}(\|z\|^2_{S^1(0,\tau)} + \|\widetilde{z}\|^2_{S^1(0,\tau)})\|z-\widetilde{z}\|_{S^1(0,\tau)}\\ &\lesssim \|h\|_{L^4(0,\tau)}\|z-\widetilde{z}\|_{S^1(0,\tau)}.\end{aligned} \tag{8.31}$$

Regarding the difference $\Psi_4(z) - \Psi_4(\widetilde{z})$, we use (8.12) and that $\|h\|_{L^4(0,\tau)} \lesssim 1$ to infer

$$\begin{aligned}\|\Psi_4(z) - \Psi_4(\widetilde{z})\|_{S^1(0,\tau)} &= \|\mathcal{I}_{v_1}\Omega_{\mathrm{b}}(v_2(z)-v_2(\widetilde{z}), (v_1+v_2(z))z) + \mathcal{I}_{v_1}\Omega_{\mathrm{b}}(v_2(\widetilde{z}), (v_2(z)-v_2(\widetilde{z}))z)\\ &\quad + \mathcal{I}_{v_1}\Omega_{\mathrm{b}}(v_2(\widetilde{z}), (v_1+v_2(\widetilde{z}))(z-\widetilde{z}))\|_{S^1(0,\tau)}\\ &\lesssim \tau^{\frac18}(\|v_2(z)-v_2(\widetilde{z})\|_{\mathbb{Y}(0,\tau)}\|v_1+v_2(z)\|_{\mathbb{Y}(0,\tau)}\|z\|_{S^1(0,\tau)}\\ &\quad + \|v_2(\widetilde{z})\|_{\mathbb{Y}(0,\tau)}\|v_2(z)-v_2(\widetilde{z})\|_{\mathbb{Y}(0,\tau)}\|z\|_{S^1(0,\tau)}\\ &\quad + \|v_2(\widetilde{z})\|_{\mathbb{Y}(0,\tau)}\|v_1+v_2(\widetilde{z})\|_{\mathbb{Y}(0,\tau)}\|z-\widetilde{z}\|_{S^1(0,\tau)})\\ &\lesssim \|h\|_{L^4(0,\tau)}\|z-\widetilde{z}\|_{S^1(0,\tau)}.\end{aligned} \tag{8.32}$$

Finally, proceeding as in the proof of (8.24) and using (8.27) and (8.28) once more, we derive

$$\begin{aligned}\|\Psi_5(z) - \Psi_5(\widetilde{z})\|_{S^1(0,\tau)} &\lesssim \|\mathcal{I}_{v_1}(v_1\Omega_{\mathrm{b}}(v_2(z)-v_2(\widetilde{z}), z)) + \mathcal{I}_{v_1}(v_1\Omega_{\mathrm{b}}(v_2(\widetilde{z}), z-\widetilde{z}))\|_{S^1(0,\tau)}\\ &\lesssim \tau^{\frac14}(K^{-1}+\tau^{\frac18})\|Y_0\|_{H^1}(\|v_2(z)-v_2(\widetilde{z})\|_{\mathbb{Y}(0,\tau)}\|z\|_{S^1(0,\tau)} + \|v_2(\widetilde{z})\|_{\mathbb{Y}(0,\tau)}\|z-\widetilde{z}\|_{S^1(0,\tau)})\\ &\lesssim \|h\|_{L^4(0,\tau)}\|z-\widetilde{z}\|_{S^1(0,\tau)}.\end{aligned} \tag{8.33}$$

Consequently, estimates (8.29) to (8.33) show that

$$\begin{aligned}\|\Psi(z) - \Psi(\widetilde{z})\|_{S^1(0,\tau)} &\le \sum_{j=1}^{5}\|\Psi_j(z) - \Psi_j(\widetilde{z})\|_{S^1(0,\tau)}\\ &\le C_2(T, \|X_0\|_{H^1}, \|Y_0\|_{H^1})\|h\|_{L^4(0,\tau)}\|z-\widetilde{z}\|_{S^1(0,\tau)}\end{aligned} \tag{8.34}$$

with a constant $C_2(T, \|X_0\|_{H^1}, \|Y_0\|_{H^1}) \ge 1$. We thus define the stopping time

$$\widetilde{\tau} := \inf\left\{t \in [0,T] : C_2(T, \|X_0\|_{H^1}, \|Y_0\|_{H^1})\|h\|_{L^4(0,t)} \ge \frac12\right\} \wedge \tau, \tag{8.35}$$

which yields

$$\|\Psi(z) - \Psi(\widetilde{z})\|_{S^1(0,\widetilde{\tau})} \le \frac12\|z-\widetilde{z}\|_{S^1(0,\widetilde{\tau})}. \tag{8.36}$$

Consequently, the above two steps show that $\Psi\colon B_{S^1(0,\widetilde{\tau})}(M) \to B_{S^1(0,\widetilde{\tau})}(M)$ is a contractive self-map. The Banach fixed point theorem thus gives a unique $z \in B_{S^1(0,\widetilde{\tau})}(M)$ such that $z = \Psi(z)$. In view of (8.17), we obtain that $(z, v_2(z))$ solves (8.15) on $[0, \widetilde{\tau}]$. The uniqueness of this solution can be proved using standard arguments and the estimates from Step 2.

Step 3: In this step we prove that

$$\mathbb{P}(\widetilde{\tau} = T) \longrightarrow 1 \quad \text{as } \phi_1 \to \infty. \tag{8.37}$$

In view of (8.25) and (8.35), it suffices to prove that

$$\mathbb{P}\left(\|h\|^4_{L^4(0,\infty)} \geq C_3^{-4}(T, \|X_0\|_{H^1}, \|Y_0\|_{H^1})\right) \longrightarrow 0, \quad \text{as } \phi_1 \to \infty, \tag{8.38}$$

where $C_3(T, \|X_0\|_{H^1}, \|Y_0\|_{H^1})$ is defined by $C_3 = \max\{4C_1M^2, 2C_2\}$ with $C_1 = C_1(T, \|X_0\|_{H^1}, \|Y_0\|_{H^1})$ and $C_2 = C_2(T, \|X_0\|_{H^1}, \|Y_0\|_{H^1})$ from the construction of τ and $\widetilde{\tau}$ in (8.25) and (8.35), respectively.

In order to prove (8.38), we note that (2.16) yields the identity

$$\|h\|^4_{L^4(0,\infty)} = \int_0^\infty e^{-8 \operatorname{Im} \phi_1^{(1)} \beta_1^{(1)}(t) - 8(\operatorname{Im} \phi_1^{(1)})^2 t} \, \mathrm{d}t.$$

By virtue of the scaling property of Brownian motion, i.e., $\mathbb{P} \circ (\operatorname{Im} \phi_1^{(1)} \beta_1^{(1)}(\cdot))^{-1} = \mathbb{P} \circ (\beta_1^{(1)}((\operatorname{Im} \phi_1^{(1)})^2 \cdot))^{-1}$, we then derive

$$\begin{aligned}
&\mathbb{P}(\|h\|^4_{L^4(0,\infty)} \geq C_3(T, \|X_0\|_{H^1}, \|Y_0\|_{H^1})^{-4}) \\
&= \mathbb{P}\left(\int_0^\infty e^{-8\beta_1^{(1)}((\operatorname{Im} \phi_1^{(1)})^2 t) - 8(\operatorname{Im} \phi_1^{(1)})^2 t} \, \mathrm{d}t \geq C_3(T, \|X_0\|_{H^1}, \|Y_0\|_{H^1})^{-4}\right) \\
&= \mathbb{P}\left(\int_0^\infty e^{-8\beta_1^{(1)}(t) - 8t} \, \mathrm{d}t \geq (\operatorname{Im} \phi_1^{(1)})^2 C_3(T, \|X_0\|_{H^1}, \|Y_0\|_{H^1})^{-4}\right),
\end{aligned} \tag{8.39}$$

where we also used a change of variables in the last step.

By the law of iterated logarithm for Brownian motion (2.18), we infer that there exists $t_0 = t_0(\omega)$ large enough such that $-\beta_1^{(1)}(t) - t \leq -\frac{1}{2}t$ for $t \geq t_0$, implying that $\int_0^\infty e^{-8\beta_1^{(1)}(t) - 8t} dt < \infty$, $\mathbb{P}$-a.s. Since $C_3(T, \|X_0\|_{H^1}, \|Y_0\|_{H^1})$ is a fixed deterministic constant, independent of $\phi_1^{(1)}$, we obtain that the right-hand side of (8.39) converges to zero as $\phi_1^{(1)} \to \infty$, showing (8.38) and completing the proof. □

Appendix

In this appendix we show that the different notions of solution for the Zakharov system which we use in this paper are all equivalent. More precisely, we prove that the analytically weak, the mild, and the normal form solutions all coincide.

Appendix A. Equivalence between weak and mild solutions

We begin with the equivalence between the weak and the mild solutions. The proof presented below follows the strategy by Lions and Masmoudi [38] in the context of the Navier-Stokes equations.

Proposition A.1. *Let $(u, v) \in C([0, \tau]; H^1) \times C([0, \tau]; L^2)$. If (u, v) is a mild solution of* (2.8)*, i.e. it satisfies* (4.4)*, then (u, v) is also an analytically weak solution of* (2.8) *in $H^{-1} \times H^{-1}$, and vice versa.*

Proof. We only prove the statement for the Schrödinger component u, as the case of the wave component v can be proved analogously.

(I) We first assume that u is a mild solution of (2.8), i.e. u satisfies the Schrödinger equation in (4.4). We write $u_\varepsilon := u * \phi_\varepsilon$ for the standard spatial mollification and use the same notation for the other appearing functions. Since $(u, v) \in C([0, \tau]; H^1) \times C([0, \tau]; L^2)$, we deduce that for any $1 \leq q < \infty$,

$$u_\varepsilon \longrightarrow u \quad \text{in } L^q(0, \tau; H^1), \quad v_\varepsilon \longrightarrow v \quad \text{in } L^q(0, \tau; L^2), \quad \text{as } \varepsilon \to 0. \tag{A.1}$$

As $b \in L^\infty(0, \tau; H^1)$ and $c, \mathcal{T}_\cdot(W_2) \in L^\infty(0, \tau; L^2)$, we obtain analogous convergence results for $b_\varepsilon, c_\varepsilon$, and $(\mathcal{T}_\cdot(W_2))_\varepsilon$. Taking into account the Sobolev embedding $H^1 \hookrightarrow L^4$, we thus infer

$$v_\varepsilon u_\varepsilon \longrightarrow vu, \quad b_\varepsilon \cdot \nabla u_\varepsilon \longrightarrow b \cdot \nabla v, \quad c_\varepsilon u_\varepsilon \longrightarrow cu, \quad (\mathcal{T}_\cdot(W_2))_\varepsilon u_\varepsilon \longrightarrow \mathcal{T}_\cdot(W_2) u \quad \text{in } L^q(0, \tau; H^{-1}). \tag{A.2}$$

Set

$$g_\varepsilon(t) := e^{\mathrm{i}t\Delta}u_{0,\varepsilon} - \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\big(v_\varepsilon u_\varepsilon - b_\varepsilon \cdot \nabla u_\varepsilon - c_\varepsilon u_\varepsilon + (\mathcal{T}_s(W_2))_\varepsilon u_\varepsilon\big)\,\mathrm{d}s \tag{A.3}$$

for all $t \in [0,\tau]$. Then, g_ε satisfies the equation

$$\mathrm{i}\partial_t g_\varepsilon + \Delta g_\varepsilon = v_\varepsilon u_\varepsilon - b_\varepsilon \cdot \nabla u_\varepsilon - c_\varepsilon u_\varepsilon + (\mathcal{T}_s(W_2))_\varepsilon u_\varepsilon \tag{A.4}$$

with $g_\varepsilon(0) = u_{0,\varepsilon}$ and (A.2) implies that for any $q \in [1,\infty)$,

$$g_\varepsilon \longrightarrow u \ \text{ in } L^q(0,\tau;H^{-1}), \quad \text{as } \varepsilon \to 0. \tag{A.5}$$

Testing (A.4) with any Schwartz function $\varphi \in \mathcal{S}$ and integrating in time, we obtain

$$\langle g_\varepsilon(t),\varphi\rangle = \langle u_{0,\varepsilon},\varphi\rangle + \int_0^t \langle g_\varepsilon, -\mathrm{i}\Delta\varphi\rangle\,\mathrm{d}s + \int_0^t \langle -\mathrm{i}v_\varepsilon u_\varepsilon + \mathrm{i}b_\varepsilon\cdot\nabla u_\varepsilon + \mathrm{i}c_\varepsilon u_\varepsilon - \mathrm{i}(\mathcal{T}_s(W_2))_\varepsilon u_\varepsilon, \varphi\rangle\,\mathrm{d}s.$$

Using (A.2) and (A.5) to pass to the limit $\varepsilon \to 0$, we get for $\mathrm{d}t$-a.e. $t \in [0,\tau]$

$$\begin{aligned} {}_{H^{-1}}\langle u(t),\varphi\rangle_{H^1} = {}_{H^{-1}}\langle u_0,\varphi\rangle_{H^1} &+ \int_0^t {}_{H^{-1}}\langle u, -\mathrm{i}\Delta\varphi\rangle_{H^1}\,\mathrm{d}s \\ &+ \int_0^t {}_{H^{-1}}\langle -\mathrm{i}vu + \mathrm{i}b\cdot\nabla u + \mathrm{i}cu - \mathrm{i}\mathcal{T}_s(W_2)u, \varphi\rangle_{H^1}\,\mathrm{d}s. \end{aligned}$$

Taking into account the continuity of both sides in t, ${}_{H^{-1}}\langle u, -\mathrm{i}\Delta\varphi\rangle_{H^1} = {}_{H^{-1}}\langle \mathrm{i}\Delta u, \varphi\rangle_{H^1}$, and the density of $\mathcal{S}$ in H^1, we conclude that u is an analytically weak solution of the Schrödinger equation in (2.8) in H^{-1}.

(II) Now we assume that $u \in C([0,\tau];H^1)$ is an analytically weak solution of the Schrödinger equation in (2.8). Using this property and mollifying in space again, we get

$$\partial_t u_\varepsilon = \mathrm{i}\Delta u_\varepsilon - \mathrm{i}\big(vu - b\cdot\nabla u - cu + \mathcal{T}_\cdot(W_2)u\big)_\varepsilon \tag{A.6}$$

with $u_\varepsilon(0) = u_{0,\varepsilon}$. Consequently, for any $\phi \in C^\infty([0,\tau)\times\mathbb{R}^3)$ we get

$$\int_0^\tau \langle \partial_t u_\varepsilon - \mathrm{i}\Delta u_\varepsilon + \mathrm{i}(v_\varepsilon u_\varepsilon - b_\varepsilon\cdot\nabla u_\varepsilon - c_\varepsilon u_\varepsilon + (\mathcal{T}_t(W_2))_\varepsilon u_\varepsilon, \phi\rangle\,\mathrm{d}t = r_\varepsilon(\phi), \tag{A.7}$$

where

$$\begin{aligned} r_\varepsilon(\phi) := -\int_0^\tau \big\langle &\mathrm{i}((vu)_\varepsilon - v_\varepsilon u_\varepsilon) - \mathrm{i}((b\cdot\nabla u)_\varepsilon - b_\varepsilon\cdot\nabla u_\varepsilon) - \mathrm{i}((cu)_\varepsilon - c_\varepsilon u_\varepsilon) \\ &+ \mathrm{i}((\mathcal{T}_s(W_2)u)_\varepsilon - (\mathcal{T}_s(W_2))_\varepsilon u_\varepsilon), \phi\big\rangle\,\mathrm{d}t. \end{aligned}$$

We infer from (A.2) that

$$r_\varepsilon(\phi) \longrightarrow 0, \quad \text{as } \varepsilon \to 0. \tag{A.8}$$

Next, we set

$$\widetilde{g}_\varepsilon(t) := u_\varepsilon(t) - e^{\mathrm{i}t\Delta}u_{0,\varepsilon} + \int_0^t e^{\mathrm{i}(t-s)\Delta}(\mathrm{i}v_\varepsilon u_\varepsilon - \mathrm{i}b_\varepsilon\cdot\nabla u_\varepsilon - \mathrm{i}c_\varepsilon u_\varepsilon + \mathrm{i}(\mathcal{T}_s(W_2))_\varepsilon u_\varepsilon)\,\mathrm{d}s \tag{A.9}$$

for all $t \in [0,\tau]$. Then $\widetilde{g}_\varepsilon$ satisfies the equation

$$(\partial_t - \mathrm{i}\Delta)\widetilde{g}_\varepsilon = (\partial_t - \mathrm{i}\Delta)u_\varepsilon + \mathrm{i}v_\varepsilon u_\varepsilon - \mathrm{i}b_\varepsilon\cdot\nabla u_\varepsilon - \mathrm{i}c_\varepsilon u_\varepsilon + \mathrm{i}(\mathcal{T}_\cdot(W_2))_\varepsilon u_\varepsilon \tag{A.10}$$

with $\widetilde{g}_\varepsilon(0) = 0$. Combining (A.7) and (A.10), we thus obtain

$$\begin{aligned} \int_0^\tau \langle \partial_t\widetilde{g}_\varepsilon - \mathrm{i}\Delta\widetilde{g}_\varepsilon, \phi\rangle\,\mathrm{d}t &= \int_0^\tau \langle \partial_t u_\varepsilon - \mathrm{i}\Delta u_\varepsilon + \mathrm{i}(v_\varepsilon u_\varepsilon - b_\varepsilon\cdot\nabla u_\varepsilon - c_\varepsilon u_\varepsilon + (\mathcal{T}_t(W_2))_\varepsilon u_\varepsilon), \phi\rangle\,\mathrm{d}t \\ &= r_\varepsilon(\phi), \end{aligned} \tag{A.11}$$

so that an integration by parts gives

$$\int_0^\tau \langle \widetilde{g}_\varepsilon, -\partial_t\phi + \mathrm{i}\Delta\phi\rangle\,\mathrm{d}t = r_\varepsilon(\phi). \tag{A.12}$$

For any $F \in C_c^\infty([0,\tau] \times \mathbb{R}^d)$, let $\phi \in C^\infty([0,\tau) \times \mathbb{R}^3)$ be the solution to the inhomogeneous Schrödinger equation

$$-\partial_t \phi + \mathrm{i}\Delta\phi = F \quad \text{with } \phi(\tau) = 0.$$

Hence, we deduce from (A.12) that

$$\int_0^\tau \langle \widetilde{g}_\varepsilon, F \rangle \,\mathrm{d}t = r_\varepsilon(\phi).$$

In view of (A.8), we infer that the righ-hand side above converges to zero as $\varepsilon \to 0$, and thus $\widetilde{g}_\varepsilon$ converges weakly to zero as $\varepsilon \to 0$. On the other hand, using (A.1) and (A.2) to pass to the limit $\varepsilon \to 0$ in (A.9), we also get

$$\widetilde{g}_\varepsilon(t) \longrightarrow u(t) - e^{\mathrm{i}t\Delta} u_0 + \int_0^t e^{\mathrm{i}(t-s)\Delta}\big(\mathrm{i}vu - \mathrm{i}b \cdot \nabla u - \mathrm{i}cu + \mathrm{i}\mathcal{T}_s(W_2)u\big)\,\mathrm{d}s \tag{A.13}$$

in $L^q(0,\tau; H^{-1})$ as $\varepsilon \to 0$. Combining (A.13) with the weak convergence of $\widetilde{g}_\varepsilon$ to zero, we conclude that

$$u(t) - e^{\mathrm{i}t\Delta} u_0 + \int_0^t e^{\mathrm{i}(t-s)\Delta}\big(\mathrm{i}vu - \mathrm{i}b \cdot \nabla u - \mathrm{i}cu + \mathrm{i}\mathcal{T}_s(W_2)u\big)\,\mathrm{d}s = 0 \tag{A.14}$$

for any $t \in [0,\tau]$, which shows that u is a mild solution of the Schrödinger equation in (4.4). □

Appendix B. Equivalence between mild and normal form solutions

In this appendix we show that mild solutions of (4.4) are also solutions of the normal form formulation (4.9) and vice versa. The precise formulation is contained in Proposition B.1 below.

Proposition B.1. *Let $(u_0, v_0) \in H^1 \times L^2$. Then, the following holds:*

(i) Let $(u,v) \in \mathbb{X}(0,\tau) \times \mathbb{Y}(0,\tau)$ be a mild solution of (4.4) on $[0,\tau]$ for some $\tau > 0$. Then (u,v) is also a solution of the normal form formulation (4.9) on $[0,\tau]$ for every $K \in 2^{\mathbb{N}}$ with $K \geq 2^5$.

(ii) There exist $K' \in 2^{\mathbb{N}}$ and $\tau' > 0$ with the following property: If $K \in 2^{\mathbb{N}}$ with $K \geq K'$, $\tau \in (0,\tau']$, and $(u,v) \in \mathbb{X}(0,\tau) \times \mathbb{Y}(0,\tau)$ is a solution of (4.9) with parameter K on $[0,\tau]$, then (u,v) is also a mild solution of (4.4) on $[0,\tau]$.

(iii) Let K' and τ' be the parameters from (ii). If $(u,v) \in \mathbb{X}(0,\tau) \times \mathbb{Y}(0,\tau)$ satisfies (4.9) with a dyadic number $K_1 \in 2^{\mathbb{N}}$ with $K_1 \geq K'$ on a time interval $[0,\tau]$ with $\tau \leq \tau'$, then (u,v) also satisfies (4.9) for every $K_2 \in 2^{\mathbb{N}}$ with $K_2 \geq K'$ on $[0,\tau]$.

Proof. (i) Fix $K \in 2^{\mathbb{N}}$ with $K \geq 2^5$. Let (u,v) be a mild solution of (4.4) on $[0,\tau]$, where the equations are understood in $H^{-1} \times H^{-1}$. Let $u_{0,\varepsilon}, v_{0,\varepsilon}, u_\varepsilon, v_\varepsilon, b_\varepsilon, c_\varepsilon, (\mathcal{T}_\cdot(W_2))_\varepsilon$ denote the mollification of the corresponding functions as in the proof of Proposition A.1. We note that $u_\varepsilon, v_\varepsilon, b_\varepsilon, c_\varepsilon, (\mathcal{T}_\cdot(W_2))_\varepsilon \in C([0,\tau], H^k)$ for all $k \in \mathbb{N}$ and $\varepsilon > 0$. Then the convergence results in (A.1) and (A.2) are still valid.

We now define U_ε and V_ε by

$$U_\varepsilon(t) := e^{\mathrm{i}t\Delta} u_{0,\varepsilon} - \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}(v_\varepsilon u_\varepsilon - b_\varepsilon \cdot \nabla u_\varepsilon - c_\varepsilon u_\varepsilon + (\mathcal{T}_\cdot(W_2))_\varepsilon u_\varepsilon)(s)\,\mathrm{d}s, \tag{B.1}$$

$$V_\varepsilon(t) := e^{\mathrm{i}t|\nabla|} v_{0,\varepsilon} + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)|\nabla|} |\nabla| |u_\varepsilon|^2(s)\,\mathrm{d}s \tag{B.2}$$

for all $t \in [0,\tau]$. Then, by (A.1) and (A.2), for any $q \in [1,2]$,

$$U_\varepsilon \longrightarrow e^{\mathrm{i}(\cdot)\Delta} u_0 - \mathrm{i}\int_0^\cdot e^{\mathrm{i}(\cdot-s)\Delta}(vu - b \cdot \nabla u - cu + \mathcal{T}_\cdot(W_2)u)(s)\,\mathrm{d}s = u \quad \text{in } L^q(0,\tau; H^{-1}), \tag{B.3}$$

$$V_\varepsilon \longrightarrow e^{\mathrm{i}(\cdot)|\nabla|} v_0 + \mathrm{i}\int_0^\cdot e^{\mathrm{i}(\cdot-s)|\nabla|} |\nabla| |u|^2(s)\,\mathrm{d}s = v \quad \text{in } L^q(0,\tau; L^2). \tag{B.4}$$

as $\varepsilon \to 0$, where we also used that $u \in \langle\nabla\rangle^{-1} L^2(0,\tau; \dot{B}^0_{6,2})$ since $u \in \mathbb{X}(0,\tau)$.

Note that $U_\varepsilon, V_\varepsilon \in C^1([0,\tau], H^k)$ for all $k \in \mathbb{N}$. Moreover, we have as in (4.5),

$$\begin{cases} \partial_t (e^{\mathrm{i}t|\xi|^2} \hat{U}_\varepsilon(t,\xi)) = -\mathrm{i}e^{\mathrm{i}t|\xi|^2} \mathcal{F}(v_\varepsilon u_\varepsilon)(t,\xi) + \mathrm{i}e^{\mathrm{i}t|\xi|^2} \mathcal{F}(b_\varepsilon \cdot \nabla u_\varepsilon + c_\varepsilon u_\varepsilon - (\mathcal{T}_s(W_2))_\varepsilon u_\varepsilon)(t,\xi), \\ \partial_t (e^{-\mathrm{i}t|\xi|} \hat{V}_\varepsilon(t,\xi)) = \mathrm{i}e^{-\mathrm{i}t|\xi|} |\xi| \mathcal{F}(|u_\varepsilon|^2)(t,\xi). \end{cases} \tag{B.5}$$

We write

$$\begin{aligned}\hat{U}_\varepsilon(t) &= e^{-\mathrm{i}t|\xi|^2}\hat{u}_{0,\varepsilon} - \mathrm{i}\int_0^t e^{-\mathrm{i}(t-s)|\xi|^2}\mathcal{F}(v_\varepsilon u_\varepsilon)_{XL}(s)\,\mathrm{d}s - \mathrm{i}\int_0^t e^{-\mathrm{i}(t-s)|\xi|^2}\mathcal{F}(v_\varepsilon u_\varepsilon)_R(s)\,\mathrm{d}s \\ &\quad + \mathrm{i}\int_0^t e^{-\mathrm{i}(t-s)|\xi|^2}\mathcal{F}(b_\varepsilon\cdot\nabla u_\varepsilon + c_\varepsilon u_\varepsilon - (\mathcal{T}_s(W_2))_\varepsilon u_\varepsilon)(s)\,\mathrm{d}s \\ &= e^{-\mathrm{i}t|\xi|^2}\hat{u}_{0,\varepsilon} - \mathrm{i}\int_0^t e^{-\mathrm{i}(t-s)|\xi|^2}\mathcal{F}(V_\varepsilon U_\varepsilon)_{XL}(s)\,\mathrm{d}s + \mathrm{i}\int_0^t e^{-\mathrm{i}(t-s)|\xi|^2}\mathcal{F}(V_\varepsilon U_\varepsilon - v_\varepsilon u_\varepsilon)_{XL}(s)\,\mathrm{d}s \\ &\quad - \mathrm{i}\int_0^t e^{-\mathrm{i}(t-s)|\xi|^2}(v_\varepsilon u_\varepsilon)_R(s)\,\mathrm{d}s + \mathrm{i}\int_0^t e^{-\mathrm{i}(t-s)|\xi|^2}\mathcal{F}(b_\varepsilon\cdot\nabla u_\varepsilon + c_\varepsilon u_\varepsilon - (\mathcal{T}_s(W_2))_\varepsilon u_\varepsilon)(s)\,\mathrm{d}s \qquad \text{(B.6)}\end{aligned}$$

Because of the regularity of U_ε and V_ε, the integration-by-parts argument in (4.8) is rigorously justified here and we obtain

$$\begin{aligned}&- \mathrm{i}\int_0^t e^{-\mathrm{i}(t-s)|\xi|^2}\mathcal{F}(V_\varepsilon U_\varepsilon)_{XL}(s,\xi)\,\mathrm{d}s \\ &= -\mathcal{F}\Omega_{\mathrm{b}}(V_\varepsilon, U_\varepsilon)(t,\xi) + e^{-\mathrm{i}t|\xi|^2}\mathcal{F}\Omega_{\mathrm{b}}(v_{0,\varepsilon}, u_{0,\varepsilon})(\xi) + \mathrm{i}\mathcal{F}\Big(\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(|\nabla||u_\varepsilon|^2, U_\varepsilon)(s)\,\mathrm{d}s\Big)(\xi) \qquad \text{(B.7)} \\ &\quad + \mathrm{i}\mathcal{F}\Big(\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(V_\varepsilon, -v_\varepsilon u_\varepsilon + b_\varepsilon\cdot\nabla u_\varepsilon + c_\varepsilon u_\varepsilon - (\mathcal{T}.(W_2))_\varepsilon u_\varepsilon)(s)\,\mathrm{d}s\Big)(\xi).\end{aligned}$$

Inserting this formula back into (B.6), we thus arrive at

$$\begin{aligned}U_\varepsilon(t) &= e^{\mathrm{i}t\Delta}u_{0,\varepsilon} - \Omega_{\mathrm{b}}(V_\varepsilon, U_\varepsilon)(t) + e^{\mathrm{i}t\Delta}\Omega_{\mathrm{b}}(v_{0,\varepsilon}, u_{0,\varepsilon}) + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}(V_\varepsilon U_\varepsilon - v_\varepsilon u_\varepsilon)_{XL}(s)\,\mathrm{d}s \\ &\quad - \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}(v_\varepsilon u_\varepsilon)_R(s)\,\mathrm{d}s + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}(b_\varepsilon\cdot\nabla u_\varepsilon + c_\varepsilon u_\varepsilon - (\mathcal{T}_s(W_2))_\varepsilon u_\varepsilon)(s)\,\mathrm{d}s \\ &\quad + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(|\nabla||u_\varepsilon|^2, U_\varepsilon)(s)\,\mathrm{d}s \\ &\quad + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(V_\varepsilon, -v_\varepsilon u_\varepsilon + b_\varepsilon\cdot\nabla u_\varepsilon + c_\varepsilon u_\varepsilon - (\mathcal{T}.(W_2))_\varepsilon u_\varepsilon)(s)\,\mathrm{d}s. \qquad \text{(B.8)}\end{aligned}$$

In order to pass to the limit on the right-hand side above, arguing as in (5.14) we derive the boundary estimate in H^{-1}

$$\begin{aligned}\|\Omega_{\mathrm{b}}(w_1,w_2)\|_{H^{-1}} &\lesssim \Big(\sum_{N\in 2^{\mathbb{Z}}}\|\langle\nabla\rangle^{-1}\Omega_{\mathrm{b}}(P_N w_1, P_{\le K^{-1}N}w_2)\|_{L^2}^2\Big)^{\frac{1}{2}} \\ &\lesssim \Big(\sum_{N\in 2^{\mathbb{Z}}}\Big(\sum_{N_1\le K^{-1}N}\langle N\rangle^{-2}N^{-1}\||\nabla|\langle\nabla\rangle\Omega_{\mathrm{b}}(P_N w_1, P_{N_1}w_2)\|_{L^2}\Big)^2\Big)^{\frac{1}{2}} \\ &\lesssim \Big(\sum_{N\in 2^{\mathbb{Z}}}\Big(\sum_{N_1\le K^{-1}N}\langle N\rangle^{-2}N^{-1}\|P_N w_1\|_{L^2}\|P_{N_1}w_2\|_{L^\infty}\Big)^2\Big)^{\frac{1}{2}} \\ &\lesssim \|w_1\|_{H^{-\frac{1}{2}}}\Big(\sum_{N\in 2^{\mathbb{Z}}}\Big(\sum_{N_1\le K^{-1}N}\langle N\rangle^{-\frac{3}{2}}N^{-1}N_1^{\frac{3}{2}}\langle N_1\rangle\|\langle N_1\rangle^{-1}P_{N_1}w_2\|_{L^2}\Big)^2\Big)^{\frac{1}{2}} \\ &\lesssim \|w_1\|_{H^{-\frac{1}{2}}}\Big(\sum_{N\in 2^{\mathbb{Z}}}\Big(\sum_{N_1\le K^{-1}N}N^{-1}N_1\|\langle N_1\rangle^{-1}P_{N_1}w_2\|_{L^2}\Big)^2\Big)^{\frac{1}{2}} \lesssim K^{-1}\|w_1\|_{H^{-\frac{1}{2}}}\|w_2\|_{H^{-1}}.\end{aligned}$$

Consequently, employing (A.1), (A.2), (B.3), and (B.4), we first obtain $(V_\varepsilon U_\varepsilon - v_\varepsilon u_\varepsilon)_{XL} \to 0$ in $L^q(0,\tau;H^{-3})$ for any $q\in[1,2]$ and then

$$U_\varepsilon(t) \longrightarrow e^{\mathrm{i}t\Delta}u_0 - \Omega_{\mathrm{b}}(v,u)(t) + e^{\mathrm{i}t\Delta}\Omega_{\mathrm{b}}(v_0,u_0) - \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}(vu)_R(s)\,\mathrm{d}s$$

$$+\,\mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}(b\cdot\nabla u + cu - \mathcal{T}_{\cdot}(W_2)u)(s)\,\mathrm{d}s + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(|\nabla||u|^2, u)(s)\,\mathrm{d}s$$
$$+\,\mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(v, -vu + b\cdot\nabla u + cu - \mathcal{T}_{\cdot}(W_2)u)(s)\,\mathrm{d}s$$

in $L^q(0,\tau;H^{-3})$ for any $q\in[1,2]$ as $\varepsilon\to 0$. As U_ε also converges to u in $L^q(0,\tau;H^{-1})$ by (B.3), we conclude

$$\begin{aligned} u(t) &= e^{\mathrm{i}t\Delta}u_0 - \Omega_{\mathrm{b}}(v,u)(t) + e^{\mathrm{i}t\Delta}\Omega_{\mathrm{b}}(v_0,u_0) - \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}(vu)_R(s)\,\mathrm{d}s \\ &\quad + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}(b\cdot\nabla u + cu - \mathcal{T}_{\cdot}(W_2)u)(s)\,\mathrm{d}s + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(|\nabla||u|^2, u)(s)\,\mathrm{d}s \\ &\quad + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(v, -vu + b\cdot\nabla u + cu - \mathcal{T}_{\cdot}(W_2)u)(s)\,\mathrm{d}s, \end{aligned}$$

i.e. (u,v) solves the normal form formulation (4.9).

(ii) Let $(u,v)\in\mathbb{X}(0,\tau)\times\mathbb{Y}(0,\tau)$ be a solution of the normal form (4.9). As in part (i), we apply a mollifier in space to obtain smooth functions $u_\varepsilon, v_\varepsilon, b_\varepsilon, c_\varepsilon, (\mathcal{T}_{\cdot}(W_2))_\varepsilon \in C([0,\tau]\times H^k(\mathbb{R}^3))$ for all $k\in\mathbb{N}$, and the convergence results in (A.1) and (A.2) are still valid. Let $u_{0,\varepsilon}, v_{0,\varepsilon}\in H^k$, $k\in\mathbb{N}$, be the mollification of u_0 and v_0.

We keep using the notations U_ε and V_ε as in part (i) for the approximations of the mild solutions, i.e. U_ε and V_ε are defined by (B.1) and (B.2). Moreover, we denote by $\widetilde{U}_\varepsilon$ the approximation of the normal form solution

$$\begin{aligned} \widetilde{U}_\varepsilon(t) &:= e^{\mathrm{i}t\Delta}u_{0,\varepsilon} - \Omega_{\mathrm{b}}(v_\varepsilon,u_\varepsilon)(t) + e^{\mathrm{i}t\Delta}\Omega_{\mathrm{b}}(v_{0,\varepsilon},u_{0,\varepsilon}) - \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}(v_\varepsilon u_\varepsilon)_R(s)\,\mathrm{d}s \\ &\quad + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}(b_\varepsilon\cdot\nabla u_\varepsilon + c_\varepsilon u_\varepsilon - (\mathcal{T}_{\cdot}(W_2))_\varepsilon u_\varepsilon)(s)\,\mathrm{d}s + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(|\nabla||u_\varepsilon|^2, u_\varepsilon)(s)\,\mathrm{d}s \\ &\quad + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(v_\varepsilon, -v_\varepsilon u_\varepsilon + b_\varepsilon\cdot\nabla u_\varepsilon + c_\varepsilon u_\varepsilon - (\mathcal{T}_{\cdot}(W_2))_\varepsilon u_\varepsilon)(s)\,\mathrm{d}s. \end{aligned}$$

We write

$$\begin{aligned} &\mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(|\nabla||u_\varepsilon|^2, u_\varepsilon)(s)\,\mathrm{d}s + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(v_\varepsilon, -v_\varepsilon u_\varepsilon + b_\varepsilon\cdot\nabla u_\varepsilon + c_\varepsilon u_\varepsilon - (\mathcal{T}_{\cdot}(W_2))_\varepsilon u_\varepsilon)(s)\,\mathrm{d}s \\ &= \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(|\nabla||u_\varepsilon|^2, U_\varepsilon)(s)\,\mathrm{d}s + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(V_\varepsilon, -v_\varepsilon u_\varepsilon + b_\varepsilon\cdot\nabla u_\varepsilon + c_\varepsilon u_\varepsilon - (\mathcal{T}_{\cdot}(W_2))_\varepsilon u_\varepsilon)(s)\,\mathrm{d}s \\ &\quad + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(|\nabla||u_\varepsilon|^2, u_\varepsilon - U_\varepsilon)(s)\,\mathrm{d}s \\ &\quad + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(v_\varepsilon - V_\varepsilon, -v_\varepsilon u_\varepsilon + b_\varepsilon\cdot\nabla u_\varepsilon + c_\varepsilon u_\varepsilon - (\mathcal{T}_{\cdot}(W_2))_\varepsilon u_\varepsilon)(s)\,\mathrm{d}s \\ &=: I + II + III + IV. \end{aligned} \tag{B.9}$$

Note that, as in part (i), equation (B.5) is still valid. Hence, by (B.7) we have

$$\mathcal{F}(I+II) = \mathcal{F}\Omega_{\mathrm{b}}(V_\varepsilon,U_\varepsilon)(t) - \mathcal{F}(e^{\mathrm{i}t\Delta}\Omega_{\mathrm{b}}(v_{0,\varepsilon},u_{0,\varepsilon})) - \mathrm{i}\mathcal{F}\int_0^t e^{\mathrm{i}(t-s)\Delta}(V_\varepsilon U_\varepsilon)_{XL}(s)\,\mathrm{d}s.$$

Inserting this identity together with (B.9) into the definition of $\widetilde{U}_\varepsilon$, we obtain

$$\begin{aligned} \widetilde{U}_\varepsilon(t) &= e^{\mathrm{i}t\Delta}u_{0,\varepsilon} - (\Omega_{\mathrm{b}}(v_\varepsilon,u_\varepsilon)(t) - \Omega_{\mathrm{b}}(V_\varepsilon,U_\varepsilon)(t)) - \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}(v_\varepsilon u_\varepsilon)(s)\,\mathrm{d}s \\ &\quad + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}(v_\varepsilon u_\varepsilon - V_\varepsilon U_\varepsilon)_{XL}\,\mathrm{d}s + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}(b_\varepsilon\cdot\nabla u_\varepsilon + c_\varepsilon u_\varepsilon - (\mathcal{T}_{\cdot}(W_2))_\varepsilon u_\varepsilon)(s)\,\mathrm{d}s \\ &\quad + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(|\nabla||u_\varepsilon|^2, u_\varepsilon - U_\varepsilon)(s)\,\mathrm{d}s \end{aligned}$$

$$+\,\mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(v_\varepsilon - V_\varepsilon, -v_\varepsilon u_\varepsilon + b_\varepsilon\cdot\nabla u_\varepsilon + c_\varepsilon u_\varepsilon - (\mathcal{T}.(W_2))_\varepsilon u_\varepsilon)(s)\,\mathrm{d}s.$$

We thus get

$$\begin{aligned}\widetilde{U}_\varepsilon(t) - U_\varepsilon(t) = &-(\Omega_{\mathrm{b}}(v_\varepsilon,u_\varepsilon)(t) - \Omega_{\mathrm{b}}(V_\varepsilon,U_\varepsilon)(t)) + \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}(v_\varepsilon u_\varepsilon - V_\varepsilon U_\varepsilon)_{XL}\,\mathrm{d}s\\ &+\mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(|\nabla||u_\varepsilon|^2, u_\varepsilon - U_\varepsilon)(s)\,\mathrm{d}s\\ &+\mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\Omega_{\mathrm{b}}(v_\varepsilon - V_\varepsilon, -v_\varepsilon u_\varepsilon + b_\varepsilon\cdot\nabla u_\varepsilon + c_\varepsilon u_\varepsilon - (\mathcal{T}.(W_2))_\varepsilon u_\varepsilon)(s)\,\mathrm{d}s. \qquad \text{(B.10)}\end{aligned}$$

Below we will fix K' and τ' in such a way that $K' \ge \tilde{K}$ and $\tau' \le \tilde{\tau} \le 1$, where $\tilde{K}$ and $\tilde{\tau}$ are the dyadic integer and the time step constructed in Step 1 and Step 2 of the proof of Theorem 1.2. Consequently, (u,v) is the unique fixed point of the fixed point operator Φ from the proof of Theorem 1.2 and is an element of the ball $B_{\mathbb{X}(0,\tau)\times\mathbb{Y}(0,\tau)}(M)$ constructed there. In particular, there is a radius M (depending only on $\|u_0\|_{H^1}$ and $\|v_0\|_{L^2}$) such that $\|u\|_{\mathbb{X}(0,\tau)} + \|v\|_{\mathbb{Y}(0,\tau)} \le M$ on $[0,\tau']$. We thus also have $\|u_\varepsilon\|_{\mathbb{X}(0,\tau)} + \|v_\varepsilon\|_{\mathbb{Y}(0,\tau)} \le M$ and by Lemma 5.6 $\|V_\varepsilon\|_{\mathbb{Y}(0,\tau)} \lesssim_M 1$ on $[0,\tau']$.

We want to estimate the difference in (B.10) in the space

$$S(0,\tau) := L^\infty(0,\tau;L^2)\cap L^2(0,\tau;\dot{B}^0_{6,2}).$$

To that purpose, we first note that adapting the proofs of Lemma 5.4, estimates (5.18) and (5.30), we obtain in the same way the estimates

$$\begin{aligned}&\|\Omega_{\mathrm{b}}(w_1,w_2)\|_{S(0,\tau)} \lesssim (K^{-1}+\tau^{\frac18})\|w_1\|_{\mathbb{Y}(0,\tau)}\|w_2\|_{S(0,\tau)},\\ &\Big\|\int_0^\cdot e^{\mathrm{i}(\cdot-s)\Delta}(w_1w_2)_{XL}(s)\,\mathrm{d}s\Big\|_{S(0,\tau)} \lesssim \tau^{\frac14}\|w_1\|_{\mathbb{Y}(0,\tau)}\|w_2\|_{S(0,\tau)},\\ &\Big\|\int_0^\cdot e^{\mathrm{i}(\cdot-s)\Delta}\Omega_{\mathrm{b}}(|\nabla||w_1|^2,w_2)(s)\,\mathrm{d}s\Big\|_{S(0,\tau)} \lesssim \tau^{\frac58}\|w_1\|^2_{L^\infty(I;H^1_x)}\|w_2\|_{S(0,\tau)}.\end{aligned}$$

We thus infer

$$\begin{aligned}&\|\Omega_{\mathrm{b}}(v_\varepsilon,u_\varepsilon) - \Omega_{\mathrm{b}}(V_\varepsilon,U_\varepsilon)\|_{S(0,\tau)}\\ &\lesssim \|\Omega_{\mathrm{b}}(v_\varepsilon - V_\varepsilon,u_\varepsilon)\|_{S(0,\tau)} + \|\Omega_{\mathrm{b}}(V_\varepsilon,u_\varepsilon - \widetilde{U}_\varepsilon)\|_{S(0,\tau)} + \|\Omega_{\mathrm{b}}(V_\varepsilon,\widetilde{U}_\varepsilon - U_\varepsilon)\|_{S(0,\tau)}\\ &\lesssim_M \|v_\varepsilon - V_\varepsilon\|_{\mathbb{Y}(0,\tau)} + \|u_\varepsilon - \widetilde{U}_\varepsilon\|_{S(0,\tau)} + (K^{-1}+\tau^{\frac18})\|\widetilde{U}_\varepsilon - U_\varepsilon\|_{S(0,\tau)},\end{aligned}$$

exploiting that $\|u_\varepsilon\|_{\mathbb{X}(0,\tau)} + \|V_\varepsilon\|_{\mathbb{Y}(0,\tau)} \lesssim_M 1$. Analogously, we deduce

$$\begin{aligned}&\Big\|\int_0^\cdot e^{\mathrm{i}(\cdot-s)\Delta}(v_\varepsilon u_\varepsilon - V_\varepsilon U_\varepsilon)_{XL}\,\mathrm{d}s\Big\|_{S(0,\tau)}\\ &\lesssim \tau^{\frac14}(\|v_\varepsilon - V_\varepsilon\|_{\mathbb{Y}(0,\tau)}\|u_\varepsilon\|_{S(0,\tau)} + \|V_\varepsilon\|_{\mathbb{Y}(0,\tau)}\|u_\varepsilon - \widetilde{U}_\varepsilon\|_{S(0,\tau)} + \|V_\varepsilon\|_{\mathbb{Y}(0,\tau)}\|\widetilde{U}_\varepsilon - U_\varepsilon\|_{S(0,\tau)})\\ &\lesssim_M \|v_\varepsilon - V_\varepsilon\|_{\mathbb{Y}(0,\tau)} + \|u_\varepsilon - \widetilde{U}_\varepsilon\|_{S(0,\tau)} + \tau^{\frac14}\|\widetilde{U}_\varepsilon - U_\varepsilon\|_{S(0,\tau)},\end{aligned}$$

and

$$\begin{aligned}\Big\|\int_0^\cdot e^{\mathrm{i}(\cdot-s)\Delta}\Omega_{\mathrm{b}}(|\nabla||u_\varepsilon|^2,u_\varepsilon - U_\varepsilon)(s)\,\mathrm{d}s\Big\|_{S(0,\tau)} &\lesssim \tau^{\frac58}\|u_\varepsilon\|^2_{L^\infty_t H^1_x}\|u_\varepsilon - U_\varepsilon\|_{S(0,\tau)}\\ &\lesssim_M \|u_\varepsilon - \widetilde{U}_\varepsilon\|_{S(0,\tau)} + \tau^{\frac58}\|\widetilde{U}_\varepsilon - U_\varepsilon\|_{S(0,\tau)}.\end{aligned}$$

For the last term in (B.10) we apply Lemma 5.7 to obtain

$$\begin{aligned}&\Big\|\int_0^\cdot e^{\mathrm{i}(\cdot-s)\Delta}\Omega_{\mathrm{b}}(v_\varepsilon - V_\varepsilon, v_\varepsilon u_\varepsilon - b_\varepsilon\cdot\nabla u_\varepsilon - c_\varepsilon u_\varepsilon - (\mathcal{T}.(W_2))_\varepsilon u_\varepsilon)(s)\,\mathrm{d}s\Big\|_{S(0,\tau)}\\ &\lesssim \tau^{\frac18}\|v_\varepsilon - V_\varepsilon\|_{\mathbb{Y}(0,\tau)}(\|v_\varepsilon\|_{\mathbb{Y}(0,\tau)} + \|b_\varepsilon\|_{L^\infty(0,\tau;H^1_x)} + \|c_\varepsilon\|_{\mathbb{Y}(0,\tau)} + \|(\mathcal{T}.(W_2))_\varepsilon\|_{\mathbb{Y}(0,\tau)})\|u_\varepsilon\|_{\mathbb{X}(0,\tau)}\\ &\lesssim_M \|v_\varepsilon - V_\varepsilon\|_{\mathbb{Y}(0,\tau)},\end{aligned}$$

where we also used

$$\|b_\varepsilon\|_{L^\infty(0,\tau;H^1_x)} + \|c_\varepsilon\|_{\mathbb{Y}(0,\tau)} + \|(\mathcal{T}(W_2))_\varepsilon\|_{\mathbb{Y}(0,\tau)} \lesssim \|b\|_{L^\infty(0,\tau;H^1_x)} + \|c\|_{\mathbb{Y}(0,\tau)} + \|\mathcal{T}(W_2)\|_{\mathbb{Y}(0,\tau)} \lesssim 1.$$

Combining these estimates with (B.10), we arrive at

$$\|\widetilde{U}_\varepsilon - U_\varepsilon\|_{S(0,\tau)} \leq C_0(K^{-1} + \tau^{\frac{1}{8}})\|\widetilde{U}_\varepsilon - U_\varepsilon\|_{S(0,\tau)} + C_0(\|u_\varepsilon - \widetilde{U}_\varepsilon\|_{S(0,\tau)} + \|v_\varepsilon - V_\varepsilon\|_{\mathbb{Y}(0,\tau)}).$$

Now we fix $K' \in 2^{\mathbb{N}}$ and $\tau' > 0$ such that $K' \geq \tilde{K}$, $\tau' \leq \tilde{\tau}$, and

$$C_0(K'^{-1} + \tau'^{\frac{1}{8}}) \leq \frac{1}{2}.$$

Consequently, we arrive at

$$\|\widetilde{U}_\varepsilon - U_\varepsilon\|_{S(0,\tau)} \leq 2C_2(\|u_\varepsilon - \widetilde{U}_\varepsilon\|_{S(0,\tau)} + \|v_\varepsilon - V_\varepsilon\|_{\mathbb{Y}(0,\tau)}) \tag{B.11}$$

if $K \in 2^{\mathbb{N}}$ with $K \geq K'$ and $\tau \in (0, \tau']$.

Since $v_\varepsilon \to v$ in $\mathbb{Y}(0,\tau)$ and $V_\varepsilon \to v$ in $\mathbb{Y}(0,\tau)$ by Lemma 5.6, we have

$$\|v_\varepsilon - V_\varepsilon\|_{\mathbb{Y}(0,\tau)} \longrightarrow 0, \quad \text{as } \varepsilon \to 0. \tag{B.12}$$

Using that $b_\varepsilon \to b$ and $c_\varepsilon \to c$ in $L^\infty(0,\tau;H^1)$ as $b, c \in C([0,\tau];H^1)$ and that $u_\varepsilon \to u$ in $\mathbb{X}(0,\tau)$, we thus obtain

$$\widetilde{U}_\varepsilon \longrightarrow u \ \text{ in } S(0,\tau), \quad \text{as } \varepsilon \to 0, \tag{B.13}$$

which implies

$$u_\varepsilon - \widetilde{U}_\varepsilon \longrightarrow 0 \ \text{ in } S(0,\tau), \ \text{as } \varepsilon \to 0. \tag{B.14}$$

Hence, we obtain from (B.11), (B.12) and (B.14) that

$$\widetilde{U}_\varepsilon - U_\varepsilon \longrightarrow 0 \ \text{ in } S(0,\tau), \ \text{as } \varepsilon \to 0. \tag{B.15}$$

Using (B.3), (B.13), and (B.15), we thus conclude

$$u(t) = e^{\mathrm{i}t\Delta}u_0 - \mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}(vu - b\cdot\nabla u - cu + \mathcal{T}(W_2)u)(s)\,\mathrm{d}s$$

in H^{-1} for all $t \in [0,T]$, i.e., (u,v) is a mild solution of (4.4).

(iii) Part (iii) is now an immediate consequence of parts (i) and (ii). Therefore, the proof is complete. □

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