The ergodicity of nonlinear Fokker–Planck flows in $L^1(\mathbb{R}^d)$

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Abstract

One proves in this work that the nonlinear semigroup $S(t)$ in $L^1(\mathbb{R}^d)$, $d \geq 3$, associated with the nonlinear Fokker–Planck equation $u_t - \Delta\beta(u) + \mathrm{div}(Db(u)u)=0$, $u(0) = u_0$ in $(0,\infty) \times \mathbb{R}^d$, under suitable conditions on the coefficients $\beta : \mathbb{R} \to \mathbb{R}$, $D : \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathbb{R} \to \mathbb{R}$, is mean ergodic. In particular, this implies the mean ergodicity of the time marginal laws of the solutions to the corresponding McKean–Vlasov stochastic differential equation. This completes the results established in [7] on the nature of the corresponding omega-set $\omega(u_0)$ for $S(t)$ in the case where the flow $S(t)$ in $L^1(\mathbb{R}^d)$ has not a fixed point and so the corresponding stationary Fokker–Planck equation has no solutions.

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1 Introduction

Consider the nonlinear Fokker–Planck equation

$$\begin{aligned} &u_t - \Delta\beta(u) + \mathrm{div}(Db(u)u) = 0, \quad \text{in } (0,\infty) \times \mathbb{R}^d, \\ &u(0,x) = u_0(x), \ x \in \mathbb{R}^d, \end{aligned} \tag{1.1}$$

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where $\beta:\mathbb{R}\to\mathbb{R}$, $D:\mathbb{R}^d\to\mathbb{R}^d$, $d\ge 3$, and $b:\mathbb{R}\to\mathbb{R}$ are given functions to be made precise in the following.

This equation describes, in statistical physics and mean field theory, the dynamics of a set of interacting particles or of many body systems in disordered media (the so-called anomalous diffusion). (See, e.g., [15].) In such a situation, for each $t\ge 0$, $u=u(t,\cdot)$ is a probability density for each probability density u_0. Another source for equation (1.1) is the description of the dynamics of Itô stochastic processes $X(t)$ in terms of their probability densities $u=u(t,x)$. Namely, if $u\in L^1_{\rm loc}((0,\infty)\times\mathbb{R}^d)$ is a Schwartz distributional solution to (1.1) such that $t\to u(t,\cdot)dx$ is weakly continuous and $u(t,\cdot)$ is a probability density, then there is a probabilistically weak solution X to the McKean–Vlasov stochastic differential equation in $\mathbb{R}^d$

$$dX(t)=D(X(t))b\left(\frac{d\mathcal{L}_{X(t)}}{dx}\,(X(t))\right)dt+\sqrt{\frac{2\beta\left(\frac{d\mathcal{L}_{X(t)}}{dx}(X(t))\right)}{\frac{d\mathcal{L}_{X(t)}}{dx}(X(t))}}\;dW(t),\tag{1.2}$$
$$\mathcal{L}_{X(t)}(dx)=u(t,x)dx,\ \ t\ge 0,$$

on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with the natural filtration $(\mathcal{F}_t)_{t\ge0}$, where W is a d-dimensional $(\mathcal{F}_t)$-Brownian motion. Here, $\mathcal{L}_{X(t)}$ is the law of the process $X(t)$ under $\mathbb{P}$ and $\mathcal{L}_{X_0}=u_0dx$ (see [3] for details).

A function $u:[0,\infty)\times\mathbb{R}^d\to\mathbb{R}$ is called a *mild solution* to (1.1) if $u\in C([0,\infty);L^1(\mathbb{R}^d))$ and

$$u(t)=\lim_{h\to0}u_h(t)\ \text{strongy in}\ L^1(\mathbb{R}^d),\ \ \forall t\ge 0,\tag{1.3}$$

where $u_h:[0,\infty)\to L^1(\mathbb{R}^d)$ is the solution to the equation

$$\begin{array}{ll}\frac1h\,(u_h(t)-u_h(t-h))+A_0u_h(t)=0 & \text{for } t\ge 0,\\ u_h(t)=u_0 & \text{for } t<0,\end{array}\tag{1.4}$$

and A_0 is the operator in $L^1(\mathbb{R}^d)$ defined by

$$A_0(y)\ =\ -\Delta\beta(y)+{\rm div}(Db(y)y)\ \text{in}\ \mathcal{D}'(\mathbb{R}^d),\ \ \forall y\in D(A_0),\tag{1.5}$$
$$D(A_0)\ =\ \{y\in L^1(\mathbb{R}^d);\ \beta(y)\in L^1_{\rm loc}(\mathbb{R}^d),\ Db(y)y\in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d);\ -\Delta\beta(y)+{\rm div}(Db(y)y)\in L^1(\mathbb{R}^d)\}.\tag{1.6}$$

In particular, the mild solution is a distributional solution (in the sense of L. Schwartz) of equation (1.1). The existence of a mild solution u to (1.1) was studied under various hypotheses on β, d and b in the works [3]–[7]. The idea, previously used by M.G. Crandall in the existence theory of entropy solutions to a nonlinear conservation law equation [13], is to represent (1.1) as a Cauchy problem in $L^1(\mathbb{R}^d)$

$$\frac{du}{dt} + Au = 0,\ \forall\, t \geq 0; \quad u(0) = u_0, \tag{1.7}$$

where A is an m-accretive operator in $L^1(\mathbb{R}^d)$ such that $(I + \lambda A)^{-1} f \in (I+\lambda A_0)^{-1} f$, $\forall f \in^1 (\mathbb{R}^d)$, $\lambda > 0$. Then, by the Crandall & Liggett generation theorem (see, e.g., [1], [2]) there exists

$$S(t)u_0 = u(t, u_0) = \lim_{n\to\infty} \left(I + \frac{t}{n}\, A \right)^{-n} u_0, \quad \forall t \geq 0,\ u_0 \in \overline{D(A)}, \tag{1.8}$$

strongly in $L^1(\mathbb{R}^d)$, uniformly in t on bounded intervals. The function $u = u(t, u_0)$ is a mild solution to equation (1.1) in the sense of (1.3)–(1.4) and the mapping $S(t) : \overline{D(A)} \to \overline{D(A)}$, $t \geq 0$, is a continuous semigroup of contractions in $L^1(\mathbb{R}^d)$ on $\overline{D(A)}$ – the closure of the domain $D(A)$ of A in $L^1(\mathbb{R}^d)$. We call such a semigroup of contractions a *nonlinear Fokker–Planck flow*. It should be emphasized that, in general, this semigroup $S(t)$ is not unique because its generator A is constructed from A_0 by

$$A(y) = A_0(J_\lambda(y)),\ \forall y \in D(A) = \{u = J_\lambda(f);\ f \in L^1(\mathbb{R}^d),\ \lambda > 0\}, \tag{1.9}$$

where $J_\lambda : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is a family of contractions such that $J_\lambda(f) \in (I + \lambda A_0)^{-1} f$, $\forall f \in L^1(\mathbb{R}^d)$, $\lambda > 0$. Since $I + \lambda A_0$ is, in general, not one-to-one, hence $(I+\lambda A_0)^{-1}$ is multivalued, the family $\{J_\lambda\}_{\lambda>0}$ is not unique, hence so is the operator A. There is an alternative approach to existence theory for nonlinear Fokker–Planck equations developed by J.A. Carrillo [11], G.Q. Chen and B. Perthame [12] in the context of entropy and kinetic solutions, but we shall not pursue this approach in this paper. (In fact, a mild solution to (1.1) is a weaker concept of solution than that of entropy solutions.) The semigroups $S(t)$ represents a section in the class of mild solutions.

Here, we shall consider equation (1.1) under the following hypotheses:

(H1) $\beta \in C^1(\mathbb{R}),\ \beta'(r) > 0,\ \forall r \in \mathbb{R} \setminus \{0\},\ \beta(0) = 0$, and

$$\mu_1 \min\{|r|^\nu, |r|\} \leq |\beta(r)| \leq \mu_2 |r|,\ \forall r \in \mathbb{R}, \tag{1.10}$$

for $\mu_1, \mu_2 > 0$ and $\nu > \frac{d-1}{d},\ d \geq 3$.

(H2) $D \in L^\infty(\mathbb{R}^d;\mathbb{R}^d) \cap W^{1,1}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$, $\operatorname{div} D \in (L^2(\mathbb{R}^d)+L^\infty(\mathbb{R}^d))$, $D = -\nabla\Phi$, where $\Phi \in C(\mathbb{R}^d) \cap W^{1,1}_{\rm loc}(\mathbb{R}^d)$ and

$$\Phi(x) \geq 1,\ \forall x \in \mathbb{R}^d,\ \lim_{|x|\to\infty} \Phi(x) = +\infty,$$
$$\Phi^{-m} \in L^1(\mathbb{R}^d \ \text{ for some } m \geq 2,$$
$$\mu_2 \Delta\Phi(x) - b_0|\nabla\Phi(x)|^2 \leq 0,\ \text{ a.e. } x \in \mathbb{R}^d.$$

(H3) $b \in C^1(\mathbb{R}) \cap C_b(\mathbb{R})$, $b(r) \geq b_0 > 0$ for all $r \in [0,\infty)$.

We note that hypothesis (H1) does not preclude the degeneracy of the nonlinear diffusion function β. For instance, any continuous, increasing function $\beta : \mathbb{R} \to \mathbb{R}$ of the form

$$\beta(r) = \begin{cases} \mu_1 r|r|^{d-1} & \text{for } |r| \leq r_0, \\ \mu_2 h(r) & \text{for } |r| > r_0, \end{cases}$$

where $r_0 > 0$, $\mu_1, \mu_2 > 0$, $|h(r)| \leq L|r|$, $\forall r \in \mathbb{R}$, $L > 0$, satisfies (1.10). As regards Hypothesis (H2), an example of such a function Φ is the following

$$\Phi(x) = \begin{cases} |x|^2 \log|x| + \mu & \text{for } |x| \leq \delta = \exp\left(-\frac{d+2}{2d}\right), \\ \varphi(|x|) + \eta|x| + \mu & \text{for } |x| > \delta, \end{cases}$$

where $\mu, \eta > 0$ are sufficiently large and φ is as in [5, Appendix]. As a matter of fact, in this case (see [4]–[7]) the family $\{J_\lambda\}_{\lambda>0}$ of resolvents which defines the operator A is given by the viscosity approximation scheme

$$J_\lambda(f) = \lim_{\varepsilon\to 0} y_\varepsilon \ \text{ in } L^1(\mathbb{R}^d), \tag{1.11}$$

where y_ε is the solution to the equation

$$y_\varepsilon - \lambda\Delta(\beta_\varepsilon(y_\varepsilon) + \varepsilon y_\varepsilon) + \lambda \operatorname{div}(D_\varepsilon b_\varepsilon(y_\varepsilon) y_\varepsilon) = f \ \text{ in } \mathbb{R}^d, \tag{1.12}$$

while $\beta_\varepsilon, D_\varepsilon$ and b_ε are smooth approximations of β, D and b.

However, it turns out (see [6]) that if one further assumes that, for some $\alpha > 0$,

$$|b(r)r - b(\bar r)\bar r| \leq \alpha|\beta(r) - \beta(\bar r)|,\ \forall r, \bar r \in \mathbb{R}, \tag{1.13}$$

then $(I + \lambda A_0)^{-1}$ is single-valued and so A is uniquely defined, more precisely $A = A_0$. In the following, we shall consider the continuous semigroup $S(t)$ generated by A, which is given by (1.9), and we shall call it the nonlinear Fokker–Planck *flow*. This semigroup leaves invariant the set $\mathcal{P}$ of all the probability densities ρ on $\mathbb{R}^d$, that is,

$$\mathcal{P} = \left\{ \rho \in L^1(\mathbb{R}^d);\ \rho \geq 0,\ \text{a.e. in } \mathbb{R}^d;\ \int_{\mathbb{R}^d} \rho\, dx = 1 \right\}.$$

Now, consider the orbit $\gamma(u_0) = \{S(t)u_0,\ t \geq 0\}$ of $S(t)$, where $u_0 \in C := \overline{D(A)} = L^1(\mathbb{R}^d)$ if $\beta \in C^2(\mathbb{R})$ and we associate to u_0 the ω-limit set

$$\begin{aligned} \omega(u_0) &= \{u_\infty = \lim S(t_n)u_0 \text{ in } L^1 \text{ for some } \{t_n\} \to \infty\} \\ &= \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)u_0}. \end{aligned} \tag{1.14}$$

This set is an attractor for $S(t)$ and, in particular, if $\omega(u_0) \neq \emptyset$ and it consists of one element u_∞ only, then we have

$$\lim_{t \to \infty} S(t)u_0 = u_\infty \text{ in } L^1.$$

In [5], it was proved that, if β is not degenerate in the origin, that is,

$$0 < \gamma_0 \leq \beta'(r) \leq \gamma_1, \quad \forall\, r \in \mathbb{R}, \tag{1.15}$$

(which also implies that $C = L^1$), then, for each $u_0 \in \mathcal{P}$, such that

$$u_0 \ln(u_0) \in L^1(\mathbb{R}^d),\ \|u_0\| = \int_{\mathbb{R}^d} u_0(x)\Phi(x)dx < \infty, \tag{1.16}$$

one has $\omega(u_0) = \{u_\infty\}$, where u_∞ is an equilibrium state of the flow $S(t)$ and, as a matter of fact, it is the unique solution in $(L^1 \cap L^\infty)(\mathbb{R}^d)$ to the stationary Fokker–Planck equation

$$-\Delta\beta(u) + \operatorname{div}(Db(u)u) = 0 \ \text{ in } \mathcal{D}'(\mathbb{R}^d). \tag{1.17}$$

This is an H-theorem type result for the Fokker–Planck equation (1.1) (see, e.g., [15], [16] for physical significance and examples). In [7], the nondegeneracy condition was relaxed to (H1), which along with (H2)–(H3) leads to the conclusion that, if $u_0 \in \mathcal{P} \cap C$ and $\|u_0\| \leq \eta$ for some $\eta > 0$, then $\omega(u_0)$ is nonempty, invariant under $S(t)$, $t \geq 0$, and compact in $L^1(\mathbb{R}^d)$ which implies that it is an attractor for the trajectory $\gamma(u_0)$. Such a situation occurs if the stationary equation (1.17) has multiple solutions. If, in addition, there is a fixed point a of $S(t)$, that is, there is $a \in \mathcal{P} \cap C$ such that $\|a\| \leq \eta$ and $S(t)a = a$ for $t > 0$, then $\omega(u_0)$ lies on a ball $\{y \in L^1(\mathbb{R}^d);\ |y - a|_{L^1(\mathbb{R}^d)} = r\}$.

In [7], sufficient conditions on β and b for the existence of such a fixed point for $S(t)$ were given. For instance, this happens if

$$\lim_{r\to+\infty}\int_1^r \frac{\beta'(s)}{sb(s)}\,ds = +\infty \text{ if } \nu \in \left(1-\tfrac{1}{d}, 1\right]$$

and

$$\lim_{r\to 0}\int_1^r \frac{\beta'(s)}{sb(s)}\,ds = -\infty \text{ if } \nu > 1.$$

Here, no such condition will be imposed and so it is not clear whether the semigroup $S(t)$ has a fixed point a in $\mathcal{P}\cap C$, but the nature of the omega-limit set $\omega(u_0)$ will be made clear from the asymptotic properties of the semigroup $S(t)$. Namely, we shall prove that, under the above hypotheses, the flow $t \to S(t)$ is ergodic in $L^1(\mathbb{R}^d)$ (Theorem 2.1) and, as a consequence (see Corollary 2.3), the time marginal laws of the probabilistically weak solution X to the McKean–Vlasov SDEs (1.2) are mean ergodic, which by our knowledge is a new result in the theory of McKean–Vlasov equations. The proof of Theorem 2.1 relies on the property of the flow $S(t)$ to be a semigroup of nonlinear contractions in $L^1(\mathbb{R}^d)$ and on the existence of a unique Haar measure on compact set $\omega(u_0)$.

Notation. $L^p(\mathbb{R}^d) = L^p,\ 1 \le p \le \infty$, is the space of real-valued Lebesgue measurable, p-integrable functions on $\mathbb{R}^d$ with the norm $|\cdot|_p$. The space $L^p(\mathbb{R}^d;\mathbb{R}^d)$ is analogously defined and $W^{1,1}(\mathbb{R}^d;\mathbb{R}^d)$ is the Sobolev space $\{u \in L^1(\mathbb{R}^d;\mathbb{R}^d);\ D_iu_j \in L^1(\mathbb{R}^d),\ i=1,...,d; u=(u_j)_{j=1}^d\}$. By $W^{1,1}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ we denote the corresponding local space. Let $C_b(\mathbb{R})$ denote the space of continuous and bounded functions on $\mathbb{R}$ and $C^1(\mathbb{R})$ the space of continuously differentiable functions on $\mathbb{R}$.

We recall that an operator $A : \mathcal{X} \to \mathcal{X}$, where $\mathcal{X}$ is a Banach space, is called m-accretive if $R(I+\lambda A) = \mathcal{X}$, $\forall \lambda > 0$, and

$$\|u_1 - u_2 + \lambda(Au_1 - Au_2)\|_{\mathcal{X}} \ge \|u_1 - u_2\|_{\mathcal{X}},\ \forall\lambda > 0,\ u_1, u_2 \in D(A),$$

where $D(A)$ is the domain of A and $R(I+\lambda A)$ is the range of $I+\lambda A$. (See, e.g., [1], [2].) For each $\eta > 0$, we consider the set

$$\mathcal{M}_\eta := \left\{ u \in L^1; \|u\| = \int_{\mathbb{R}^d} |u(x)|\Phi(x)dx \le \eta \right\},$$

where Φ is the potential of D defined as in hypothesis (H2).

2 The main result

Let $S(t) : C \to C$, $C = \overline{D(A)}$, be the semigroup generated by the operator A given above by (1.9) and, for a given $\eta > 0$, let the set

$$\mathcal{K} := \mathcal{M}_\eta \cap C \cap \mathcal{P}.$$

Everywhere in the following, we shall assume that hypotheses (H1)–(H3) hold. Theorem 2.1 which follows is the main result.

Theorem 2.1. *Let $\mathcal{X}$ be a real Banach space and let $F : \mathcal{K} \to \mathcal{X}$ be a uniformly continuous mapping. Then, for each $u_0 \in \mathcal{K}$, the set $\omega(u_0)$ is compact in $L^1(\mathbb{R}^d)$ and*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(S(t)u_0)dt = \int_{\omega(u_0)} F(\xi)d\xi, \tag{2.1}$$

where $\omega(u_0)$ is endowed with its natural commutative group structure (recalled below in the proof of the theorem) and $d\xi$ is the normalized Haar measure on $\omega(u_0)$.

The right hand side of (2.1) is the integral of F with respect to the measure $d\xi$ on the topological group $\omega(u_0)$ (see [20]).

We recall that the Haar measure μ on a locally compact topological commutative group G is a nonzero Borel measure μ which is invariant on G, that is, $\mu(gS) = \mu(Sg) = \mu(S)$ for any Borel subset $S \subset G$.

A simple example covered by Theorem 2.1 is $\mathcal{X} = \mathbb{R}$ and $F : L^1 \to \mathbb{R}$ is defined by

$$F(u) = \int_{\mathbb{R}^d} g(x)u(x)dx, \quad \forall u \in L^1(\mathbb{R}^d), \tag{2.2}$$

where $g \in L^\infty(\mathbb{R}^d)$. Then, by Theorem 2.1, we obviously have

Corollary 2.2. *Under hypotheses* (H1)–(H3)*, for each $u_0 \in \mathcal{K}$ and $g \in L^\infty(\mathbb{R}^d)$, we have*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T dt \int_{\mathbb{R}^d} g(x)(S(t)u_0)(x)dx = \int_{\omega(u_0)} \int_{\mathbb{R}^d} g(x)\xi(x)\, dx\, d\xi. \tag{2.3}$$

Furthermore, the semigroup $S(t)$ is mean-ergodic, that is,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T S(t)u_0 dt = \int_{\omega(u_0)} \xi\, d\xi \ \text{ strongly in } L^1(\mathbb{R}^d), \tag{2.4}$$

where $d\xi$ is, as above, the normalized Haar measure on $\omega(u_0)$.

To get the latter equation, we apply Theorem 2.1 with $\mathcal{X} := L^1(\mathbb{R}^d)$ and $F =$ inclusion map.

By Corollary 2.2 it follows in particular that, under hypotheses (H1)–(H3) for the nonlinear Fokker–Planck flow $t \to S(t)u_0$ with $u_0 \in \mathcal{K}$ the classical *Boltzmann hypothesis* (see, e.g., [21], p. 389) is satisfied with time average $\int_{\omega(u_0)} \xi \, d\xi$ which is the mean of the Haar measure $d\xi$, on $\omega(u_0)$. (Such a result is related to the Birkhoff ergodic theorem [9], [18].)

Now, coming back to the McKean–Vlasov equation (1.2), we get by Corollary 2.2 the following ergodicity result for the solutions $X(t)$ to (1.2).

Corollary 2.3. *Let $u_0 \in \mathcal{K}$. Then, under hypotheses* (H1)–(H3) *there is a probabilistically weak solution X to* (1.2), *where $\mathcal{L}_{X_0} = u_0 dx$, such that*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{E}[g(X(t))]dt = \int_{\omega(u_0)} \int_{\mathbb{R}^d} g(x)\xi(x)dx \, d\xi, \ \forall g \in L^\infty, \tag{2.5}$$

and, in particular,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathcal{L}_{X(t)}(B)dt = \int_B \left(\int_{\omega(u_0)} \xi \, d\xi \right)(x)dx, \tag{2.6}$$

for any Borelian set $B \subset \mathbb{R}^d$.

Remark 2.4. Due to the degeneracy of the diffusion coefficienty β, the case $d = 2$ is singular for the semigroup approach of equation (1.1), namely for the existence of an m-accretive realization A of the operator A_0 and this is the principal motivation to avoid it here (see [7]). However, the case $d = 1$ could be treated in a similar way following [4], but we omit the details.

3 Proofs

Proof of Theorem 2.1. The main step of the proof is to show that for each $u_0 \in \mathcal{K}$ the set $\omega(u_0)$ is compact in $L^1(\mathbb{R}^d)$. For this, it suffices to prove that the orbit $\gamma(u_0) = \{S(t)u_0; t \geq 0\}$ of the semigroup $S(t)$ is, for $u_0 \in \mathcal{K}$, precompact in the space L^1. To this end, we shall mention first the following lemma (see [7]).

Lemma 3.1. *Let $\eta > 0$ arbitrary but fixed. We have*

$$\|(I+\lambda A)^{-1}y\| \le \|y\|,\ \forall\lambda > 0,\ y \in \mathcal{K}, \tag{3.1}$$

$$(I+\lambda A)^{-1}(\mathcal{K}) \subset \mathcal{K} \cap D(A),\ \forall\lambda > 0, \tag{3.2}$$

$$\|S(t)y\| \le \|y\|,\ \forall y \in C,\ t \ge 0, \tag{3.3}$$

$$S(t)(\mathcal{K}) \subset \mathcal{K},\ \forall t \ge 0. \tag{3.4}$$

Proof. Recalling (1.9) and (1.11) for (3.1), it suffices to show that

$$\|y_\varepsilon\| \le \|f\|,\ \forall\lambda > 0,\ \varepsilon > 0, \tag{3.5}$$

where y_ε is the solution to equation (1.12). As regards (3.5), it follows by (H2), via Lemma 3.2 in [5]. By (3.1), it follows also (3.2), while by the exponential formula (1.8) one gets (3.3) and (3.4).

Now, we consider the restriction A^* of the operator A to $\mathcal{K}$, that is, the operator

$$A^*(y) = A(y),\ \forall y \in D(A^*) = D(A) \cap \mathcal{K}. \tag{3.6}$$

It is easily seen that $\overline{D(A^*)} \subset \mathcal{K} \subset (I+\lambda A^*)(D(A^*)) = R(I+\lambda A^*)$, $\forall\lambda > 0$, and that $(I+\lambda A^*)^{-1} = (I+\lambda A)^{-1}$ on $R(I+\lambda A^*)$. By (3.6), it follows also that the operator A^* with the domain $D(A^*)$ is accretive in L^1 and, therefore, by (1.8) we have

$$S(t)u_0 = \lim_{n\to\infty}\left(I + \frac{t}{n}A^*\right)^{-n} u_0 \ \text{ in } L^1,$$

for all $u_0 \in \overline{D(A^*)}$ (the closure of the domain $D(A^*)$) and uniformly in t on bounded intervals of $(0,\infty)$. In other words, $S(t)$ is a continuous semigroup of contractions on $\overline{D(A^*)}$ generated by A^*. Then, to show that the trajectory $\gamma(u_0)$ is precompact in L^1, it suffices to check by Theorem 3.1 in [14] that

Lemma 3.2. *The operator $(I+\lambda A^*)^{-1}$ is compact on $\mathcal{K}$ for $\lambda \in (0,\lambda_0)$ where some $0 < \lambda_0 < \infty$ are small enough.*

Proof. This lemma is just Lemma 4.2 in [7], but since the proof given there was outlined only and this lemma is a crucial step in our proof, we shall prove it here in all details. To this end, we consider a sequence $\{f_n\} \subset \mathcal{K}$ such that $\sup_{n\in\mathbb{N}} |f_n|_1 < \infty$ and set

$$y_n = (I+\lambda A_n^*)^{-1} f_n = (I+\lambda_n A_n)^{-1} f_n.$$

We have, therefore (see (1.11)–(1.12)), $y_n = \lim_{\varepsilon\to 0} y^n_\varepsilon$ in L^1, where y^ε_n is the solution to the equation

$$y^n_\varepsilon - \lambda\Delta(\beta(y^n_\varepsilon) + \varepsilon y^n_\varepsilon) + \lambda \operatorname{div}(D_\varepsilon b_\varepsilon(y^n_\varepsilon) y^n_\varepsilon) = f_n \text{ in } \mathbb{R}^d. \tag{3.7}$$

Letting $\varepsilon \to 0$, we have therefore

$$y_n - \lambda\Delta\beta(y_n) + \lambda \operatorname{div}(Db(y_n)y_n) = f_n \text{ in } \mathcal{D}'(\mathbb{R}^d), \tag{3.8}$$

$$|y_n|_1 \le |f_n|_1, \ \forall n \in \mathbb{N}. \tag{3.9}$$

To get the compactness of the set $\{y_n\}$ in L^1, we need some apriori estimates in the Sobolev space $W^{1,p}$. Namely, we shall prove first that $\beta(y_n) \in W^{1,q}_{\rm loc}(\mathbb{R}^d)$ and, for all $R > 0$,

$$\|\beta(y_n)\|_{L^q(B_R)} + \|\nabla\beta(y_n)\|_{L^q(B_R)} \le C_R(1 + |f_n|_1), \ \forall n \in \mathbb{N}, \tag{3.10}$$

where $q \in \left[1, \frac{d}{d-1}\right)$. Here, $B_R = \{x; |x|_d < R\}$. To prove this, we shall use some argument from Lemma 2.4 in [6]. Namely, we have, for all $\varphi \in C^\infty_0(\mathbb{R}^d)$,

$$\Delta(\varphi\beta(y_n) = f_1 + \operatorname{div} f_2 \ \text{ in } \mathcal{D}'(\mathbb{R}^d), \tag{3.11}$$

where

$$\begin{aligned} f_1 &= \frac{1}{\lambda}(y_n - f)\varphi - \beta(y_n)\Delta\varphi - (D\cdot\nabla\varphi)b^*(y_n), \\ f_2 &= 2\beta(y_n)\nabla\varphi + D\varphi b^*(y_n). \end{aligned} \tag{3.12}$$

We set $u = \varphi\beta(y_n)$, $u_\varepsilon\psi_\varepsilon$, $f^\varepsilon_i = f_i * \psi_\varepsilon$, $i = 1, 2$, where ψ_ε is a standard mollifier, that is,

$$\psi_\varepsilon(x) = \frac{1}{\varepsilon^d}\,\psi\left(\frac{x}{\varepsilon}\right), \ \psi \in C^\infty_0(\mathbb{R}^d), \text{support}\,\psi \subset \{x;\ |x| \le 1\}, \int_{\mathbb{R}^d} \psi(x)dx = 1.$$

Let $\mathcal{O}, \mathcal{O}'$ be open balls in $\mathbb{R}^d$ centered at zero such that $\overline{\mathcal{O}'} \subset \mathcal{O}$ and choose $\varphi \in C^2_0(\mathbb{R}^d)$ such that $\varphi = 1$ on $\mathcal{O}'$ and $(\operatorname{supp}\varphi)_\varepsilon \subset \mathcal{O}$, $\varepsilon \in (0, 1]$, where $(\operatorname{supp}\varphi)_\varepsilon$ denotes the closed ε-neighbourhood of $\operatorname{supp}\varphi$. By (3.11), we have

$$\Delta u_\varepsilon = f^\varepsilon_1 + \operatorname{div} f^\varepsilon_2 \text{ in } \mathcal{O}, \quad u_\varepsilon \in C\infty_0(\mathcal{O}),$$

and $u_\varepsilon = u^1_\varepsilon + u^2_\varepsilon$, where $u^1_\varepsilon, u^2_\varepsilon \in C^\infty(\mathcal{O}) \cap C(\overline{\mathcal{O}})$ are the solutions to

$$\Delta u^1_\varepsilon = f^\varepsilon_1 \text{ in } \mathcal{O}, \qquad u^1_\varepsilon = 0 \text{ in } \partial\mathcal{O}, \tag{3.13}$$

$$\Delta u^2_\varepsilon = \operatorname{div} f^\varepsilon_2 \text{ in } \mathcal{O}, \qquad u^2_\varepsilon = 0 \text{ on } \partial\mathcal{O}. \tag{3.14}$$

Then, by standard elliptic estimates, we have

$$\|u_\varepsilon^1\|_{W_0^{1,q}(\mathcal{O})} \le C\|f_1^\varepsilon\|_{L^1(\mathcal{O})} \le C(|y_n|_1 + |f_n|_1),\ \forall \varepsilon > 0, \tag{3.15}$$

where $1 \le q < \frac{d}{d-1}$ and so, by the Sobolev–Galiardo–Nirenberg theorem it follows by (3.15) that we have

$$\|u_\varepsilon^1\|_p \le C(|f_n|_1 + |y_n|_1),\ \forall p \in \left[1, \tfrac{d}{d-2}\right) \text{ if } d > 2.$$

If $\psi \in L^m(\mathcal{O})$, $m > d$, and $\theta \in W^{2,m}(\mathcal{O}) \cap W_0^{1,m}(\mathcal{O})$ is the solution to the Dirichlet problem

$$-\Delta\theta = \psi \text{ in } \mathcal{O};\quad \theta = 0 \text{ on } \partial\mathcal{O},$$

we have that

$$\int_{\mathcal{O}} u_\varepsilon^2 \Delta\theta dx = -\int_{\mathcal{O}} f_2^\varepsilon \cdot \nabla dx \le |f_2^\varepsilon|_1 \|\nabla\theta\|_\infty \le C(|f|_1 + |y_n|_1)\|\psi\|_m.$$

Taking into account that $|y_n|_1 \le |y|_1$, $\forall n \in \mathbb{N}$, this yields

$$\left|\int_{\mathcal{O}} u_\varepsilon^2 \psi dx\right| \le C(|f|_1 + |y|_1)\|\psi\|_m,\ \forall \psi \in L^m(\mathcal{O}).$$

Then, if $\frac{1}{m'} = 1 - \frac{1}{m}$, it follows by duality that $u_\varepsilon^2 \in L^m(\mathcal{O}) \subset L^q(\mathcal{O})$ for all $q \in \left[1, \frac{d}{d-1}\right)$ and

$$\|u_\varepsilon^2\|_q \le C(|f|_1 + |y|_1),\ \forall \varepsilon > 0,$$

and so, by (3.15) it follows also that $u_\varepsilon^i \in L^q(\mathcal{O}),\ i = 1, 2$, and

$$\|u_\varepsilon^i\|_q \le C(|f|_1 + |y|_1),\ \forall q \in \left[1, \tfrac{d}{d-1}\right),\ i = 1, 2.$$

Hence,

$$\|u_\varepsilon\|_q \le C(|f_n|_1 + |y_n|_1) \le C(|f_n|_1 + 1),\ \forall \varepsilon > 0,\ q \in \left[1, \tfrac{d}{d-1}\right).$$

Finally, by letting $\varepsilon \to 0$ we get

$$\|\varphi\beta(y_n)\|_q \le C(|f_n|_1 + |y_n|_1) \le C,\ \forall q \in \left[1, \tfrac{d}{d-1}\right).$$

(Here and in the following, C is a positive constant independent of n.) Because φ and the corresponding ball $\mathcal{O}$ are arbitrary, we conclude that y_n, $\beta(y_n) \in L^q_{\rm loc}(\mathbb{R}^d)$ and that (for a possible larger C, still independent of ε)

$$\|\beta(y_n)\|_{L^q(B_R)} \le C(|f_n|_1 + |y_n|_1) \le C,\ \forall q \in \left[1, \tfrac{d}{d-1}\right).$$

In particular, this implies that

$$\|f_2\|_q \le C(|f_n|_1 + |y_n|_1) \le C < \infty$$

and, therefore,

$$\|f_2^\varepsilon\|_q \le C(f_n|_1 + |y_n|_1) \le C < \infty,\ q \in \left[1, \tfrac{d}{d-1}\right),\ n \in \mathbb{N}. \tag{3.16}$$

Now,we come back to (3.13)–(3.14) and note that $\varphi\beta(y) = u = u_1 + u_2$, where u_1, u_2 are solutions to the equations

$$\begin{aligned} \Delta u_1 &= f_1 \text{ in } \mathcal{D}'(\mathbb{R}^d), & (3.17)\\ \Delta u_2 &= \operatorname{div}(f_2) \text{ in } \mathcal{D}'(\mathbb{R}^d), & (3.18)\end{aligned}$$

where f_1, f_2 are defined by (3.15).

Since $f_1 \in L^1$, it follows that u_1 can be represented as

$$u_1 = -E * f_1 \text{ in } \mathbb{R}^d,$$

where $E(x) \equiv \frac{1}{(d-2)\omega_d |x|^{d-2}}$ is the fundamental solution to Δ.

Hence (see, e.g., [15]), $u_1 \in M^{\frac{d}{d-2}}(\mathbb{R}^d) \subset L^p_{\rm loc}(\mathbb{R}^d),\ \forall p \in \left[1, \frac{d}{d-1}\right)$ and $|\nabla u_1| = |\nabla E * f_1| \in M^{\frac{d}{d-1}}(\mathbb{R}^d) \subset L^p_{\rm loc}(\mathbb{R}^d)$, $\forall p \in \left[1, \frac{d}{d-1}\right)$ with

$$\|\nabla u_1\|_{L^p(B_R)} \le C(|y_1|_1 + |f|_1),\ \forall R > 0,\ p \in \left[1, \tfrac{d}{d-1}\right). \tag{3.19}$$

(Here, M^ℓ is the Marcinkievicz space of order ℓ.)

Similarly, we have

$$u_2 = -\nabla E * f_2 \ \text{ in } \mathbb{R}^d.$$

Taking into account that $|\nabla^2 E(x)| \le C|x|^{-d}$, $\forall x \ne 0$, and that $\nabla^2 E$ is homogeneous of order d, it follows by the Calderon–Zygmund theorem [10] and estimate (3.12) that

$$|\nabla u_\varepsilon^2|_p \le C|f_2^\varepsilon|_p \le C(|f_n|_1 + |y_n|_1) \le C,\ \forall p \in \left[1, \tfrac{d}{d-1}\right),$$

and, after letting $\varepsilon \to 0$, this yields

$$\|\nabla u\|_{L^p(B_R)} \le C_R(|f_n|_1 + |y_n|_1),\ \forall p \in \left[1, \tfrac{d}{d-1}\right),$$

and so (3.10) holds.

Then, by the Kolmogorov compactness theorem it follows that the sequence $\{\beta(y_n)\}$ is compact in $L^q_{\rm loc}(\mathbb{R}^d)$ and, therefore, on a subsequence $\beta(y_n) \to \eta$, a.e. on $\mathbb{R}^d$ as $n \to \infty$. Since $\beta \in C^1$ and $\beta' > 0$, it follows that $y_n \to \beta^{-1}(\eta)$, a.e. on $\mathbb{R}^d$. Since $\nu > \frac{d-1}{d}$, we may choose q close to $\frac{d}{d-1}$ such that $\nu q \geq 1$ and so, by (1.11),

$$\mu_1^q \min\{|r|^q, |r|^{1-q}\} \leq |\beta(f)|^q, \ \forall r \in \mathbb{R}.$$

This implies that $y_n \to \beta^{-1}(\eta)$ in every $L^1(K), \subset \mathbb{R}^d$. Hence, the set $\{y_n\}$ is compact in $L^1_{\rm loc}$ and so, by (3.1) and taking into account that $\{y_n\} \subset \mathcal{M}_\eta$, it follows that $\{y_n\}$ is compact in $L^1(\mathbb{R}^d)$, as claimed. $\square$

Proof of Theorem 2.1 (continued) As mentioned earlier, by Lemma 3.2 it follows that the corresponding ω-limit set $\omega(u_0)$ is compact in $L^1(\mathbb{R}^d)$. It is also known from the theory of infinite dimensional dynamical system that, for each $t \geq 0$, $\omega(u_0)$ is invariant under $S(t)$ which is an homeomorphism of $\omega(u_0)$ onto $\omega(u_0)$, that is, $S(t)$ is a group on $\omega(u_0)$. Hence, $\omega(u_0)$ can be endowed with a topological commutative group structure with the product $g_1 \circ g_2 = \lim\limits_{n\to\infty} S(t_1^n + t_2^n)u_0, \ g_1, g_2 \in \omega(u_0)$, where $g_1 = \lim\limits_{n\to\infty} S(t_1^n)u_0$, $g_2 = \lim\limits_{n\to\infty} S(t_2^n)u_0$ and $\lim\limits_{n\to\infty} t_i^n = +\infty$, $i = 1, 2$. Then, by the classical A. Weil theorem (see [20]), there is a unique normalized Haar measure on $\omega(u_0)$ and so, by Birkhoff's ergodic theorem (see [9] and Theorem 1 in [17]) it follows that (2.1) holds for each uniformly continuous mapping $F : \mathcal{K} \to \mathcal{X}$, and so one obtains Theorem 2.1 as a special case. $\square$

Proof of Corollary 2.3. The existence of a probabilistically weak solution to (1.2) follows by [3, Section 2], which in turn is based on the superposition principle for linear Fokker–Planck equations (see [19], Theorem 2.5). Furthermore, formula (2.5) follows then by (2.3) taking into account that

$$\mathbb{E}[g(X(t))] = \int_{\mathbb{R}^d} g(x)(S(t)u_0)(x)dx, \ \forall t \geq 0, \ g \in L^\infty.$$

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