Zvonkin's transform and the regularity of solutions to double divergence form elliptic equations

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Abstract. We study qualitative properties of solutions to double divergence form elliptic equations (or stationary Kolmogorov equations) on $\mathbb{R}^d$. It is shown that the Harnack inequality holds for nonnegative solutions if the diffusion matrix A is nondegenerate and satisfies the Dini mean oscillation condition and the drift coefficient b is locally integrable to a power $p>d$. We establish new estimates for the L^p-norms of solutions and obtain a generalization of the known theorem of Hasminskii on the existence of a probability solution to the stationary Kolmogorov equation to the case where the matrix A satisfies Dini's condition or belongs to the class VMO. These results are based on a new analytic version of Zvonkin's transform of the drift coefficient.

Keywords: double divergence form elliptic equation, Kolmogorov equation, Dini condition, class VMO, Zvonkin's transform

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1. Introduction

We study qualitative properties of solutions to the double divergence form elliptic equation (or the stationary Kolmogorov equation)

$$\partial_{x_i}\partial_{x_j}\bigl(a^{ij}\varrho\bigr)-\partial_{x_i}\bigl(b^i\varrho\bigr)=0 \tag{1.1}$$

on an open set $\Omega\subset\mathbb{R}^d$. The matrix $A=(a^{ij})$ is supposed to be symmetric and positive definite, a^{ij} and b^i are Borel functions. Set

$$L\varphi=a^{ij}\partial_{x_i}\partial_{x_j}\varphi+b^i\partial_{x_i}\varphi,\quad L^*\varphi=\partial_{x_i}\partial_{x_j}\bigl(a^{ij}\varphi\bigr)-\partial_{x_i}\bigl(b^i\varphi\bigr).$$

Then equation (1.1) can be written in a shorter form

$$L^*\varrho=0.$$

A function $\varrho\in L^1_{loc}(\Omega)$ is a solution to equation (1.1) if

$$a^{ij}\varrho, b^i\varrho\in L^1_{loc}(\Omega)$$

and for every function $\varphi\in C_0^\infty(\Omega)$ the equality

$$\int_\Omega L\varphi(x)\varrho(x)\,dx=0$$

is fulfilled. A nonnegative solution ϱ to the Kolmogorov equation (1.1) satisfying the condition

$$\int_\Omega \varrho(x)\,dx=1$$

is called a probability solution.

An important example of a double divergence form elliptic equation is delivered by the stationary Kolmogorov equation for invariant measures of a diffusion process. Various properties of solutions to such equations were studied by many authors. The principal questions are

1) the existence of solutions, especially, of probability solutions,

2) the existence of solution densities and their properties such as local boundedness, continuity and Sobolev differentiability,

3) local separation of densities from zero, that is, certain forms of the Harnack inequality.

In case of locally Lipschitz coefficients the existence of a probability solution is given by the classical theorem of Hasminskii [15] under the existence of a Lyapunov function. This theorem was generalized in [5] (see also further generalizations in [4, Chapter 2], [6], [22]), where either the diffusion coefficient is nondegenerate and locally Sobolev with the order of integrability higher than dimension along with the same local integrability of the drift coefficient or both the diffusion and drift coefficients are continuous. It was shown in [3] that the solution density is locally Sobolev in the first case and its continuous version is locally separated from zero. It was proved in [27] and [28] that in the case where the matrix $A = (a^{ij})$ is nondegenerate and satisfies Dini's condition and the coefficients b^i are bounded, the solution has a continuous version, and when the coefficients a^{ij} are Hölder continuous, then the solution has a Hölder continuous version. These results have been generalized in [7] to the case of integrable b^i. Analogous results have been obtained in [10] and [11] under the assumption that the matrix A satisfies the Dini mean oscillation condition, which is weaker than the classical Dini condition. Note also the paper [23], where some additional regularity of solutions along level sets has been established. In the papers [1], [2] some interesting counter-examples were constructed and the so-called renormalized solutions were studied, in particular, an example was constructed of a positive definite and continuous diffusion matrix A for which the equation $\partial_{x_i}\partial_{x_j}(a^{ij}\varrho) = 0$ has a locally unbounded solution. The Harnack inequality for double divergence form equations with the matrix A belonging to the Sobolev class with a sufficiently high integrability exponent is a corollary of the Harnack inequality for divergence form elliptic equations (see [4, Chapter 3]). However, in case of merely Hölder continuous matrix A the double divergence form equation cannot be reduced to a divergence form equation, moreover, the classical results about the regularity of solutions to divergent form elliptic equations are not true for solutions to double divergence form equations. In the case where the matrix A satisfies Dini's condition, the Harnack inequality was obtained in [24] for $b = 0$, and for any bounded drift b it was established in [7]. The proof consisted in obtaining certain inequalities for solutions generalizing classical mean value theorems and heavily used the boundedness of b. Another way of proving the Harnack inequality for $b = 0$ was suggested in [11], where the reasoning employs estimates for the modulus of continuity of the solution and some properties of renormalized solutions from the paper [12] considering double divergence form equations without first order terms. In [3] and [7] (see also [4, Chapter 1]) the integrability of solutions was investigated in some cases when the diffusion matrix does not satisfy Dini's condition. In particular, it was shown that if A belongs to the class VMO and the coefficient b is locally integrable to some power $p > d$, then the solution belongs to all $L^p_{loc}(\Omega)$. In spite of a considerable number of papers devoted to double divergence form elliptic equations, the answers to the following questions, certain specifications of general problems 1 — 3 mentioned above, have remained open so far:

• What are optimal conditions for the Harnack inequality for nonnegative solutions to double divergence form equations? In particular, does the Harnack inequality hold for A satisfying Dini's condition and an unbounded locally Lebesgue integrable drift b?

• What are optimal conditions for a high local integrability of solutions? In particular, the dependence of the integrability of the solution on the modulus of continuity of the matrix A has not been studied.

• What are optimal conditions for the existence and uniqueness of a probability solution to the stationary Kolmogorov equation?

Here we obtain new results related to these questions. (i) We prove that the Harnack inequality holds for nonnegative solutions on $\mathbb{R}^d$ if the matrix A is nondegenerate and satisfies the Dini mean oscillation condition and the coefficient b is locally integrable to a power $p > d$.

(ii) We establish new estimates for the L^p-norms of solutions and obtain sufficient conditions for the local exponential integrability. Note that it was asserted in [7] that in the case of a locally bounded coefficient b and a nondegenerate matrix A of class VMO the solution is locally exponentially integrable. However, the justification given there contains a gap, namely, a wrong dependence on p of the constant in a priori L^p-estimates of second derivatives of solutions to the equation $\mathrm{tr}(AD^2u) = f$. In the general case, the dependence of the constant on p is influenced by the modulus of continuity of A. In the present paper we derive an estimate that takes the modulus of continuity of A into account.

(iii) Finally, an important new result of our paper is a generalization of the known theorem of Hasminskii on the existence of a probability solution to the stationary Kolmogorov equation to the case where the matrix A satisfies Dini's condition or belongs to the class VMO. Results on existence of positive or probability solutions to the stationary Kolmogorov equation in case of irregular coefficients are useful for constructing diffusion processes (see [22]). We also discuss uniqueness of probability solutions and their probabilistic interpretation.

These results are obtained with the aid of a new approach to the study of regularity of solutions to double divergence form equations based on Zvonkin's transform, well known in the theory of diffusion processes, which applies for smoothing the drift coefficient (more precisely, we deal with its elliptic version, in the original paper [31] this transform was used for parabolic equations). In recent years Zvonkin's transform has been applied for the study of diffusion processes with generalized coefficients (see [13], [26], [30]). In this paper we apply Zvonkin's transform not to random processes, but to solutions of the Kolmogorov equation, moreover, we do not assume any connection of solutions with diffusion processes. It is shown below that with the aid of a suitable change of coordinates an integrable drift can be transformed into a continuously differentiable drift such that the new diffusion matrix enables us to apply known results about regularity of solutions. This leads to substantial generalizations of some results and simplification of proofs of other results on regularity of solutions to double divergence form elliptic equations. Note that change of coordinates has proved to be also useful in the study of uniqueness of solutions (see [18]).

In §2 we construct Zvonkin's transform in the analytic setting, in §3 we study the regularity of solutions, and in §4 we apply our results for proving existence and uniqueness of probability solutions.

2. Zvonkin's transform

We first illustrate our approach by example of a smooth change of coordinates.

Let $\Phi\colon \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism of class C^2 and $\Psi = \Phi^{-1}$. Set

$$q^{km}(y) = a^{ij}(\Psi(y))\partial_{x_i}\Phi^k(\Psi(y))\partial_{x_j}\Phi^m(\Psi(y)),$$

$$h^k(y) = a^{ij}(\Psi(y))\partial_{x_i}\partial_{x_j}\Phi^k(\Psi(y)) + b^i(\Psi(y))\partial_{x_i}\Phi^k(\Psi(y)),$$

$$\sigma(y) = \varrho(\Psi(y))|\det \nabla\Psi(y)|.$$

Then on the domain $\Omega' = \Phi(\Omega)$ the function σ satisfies the equation $\mathcal{L}^*\sigma = 0$ with the operator

$$\mathcal{L}\varphi(y) = q^{km}(y)\partial_{y_k}\partial_{y_m}\varphi(y) + h^k(y)\partial_{y_k}\varphi(y).$$

Observe that the double divergence structure of the equation does not change. One can construct a mapping Φ of the form $\Phi(x) = x + u(x)$, where $u = (u^1, \ldots, u^d)$, such that u^k is a solution to the elliptic equation

$$Lu^k - \lambda u^k = -b^k.$$

For λ sufficiently large, this equation possesses a solution for which Φ is a diffeomorphism. Using Φ to change variables we obtain a new drift coefficient $h(y) = \lambda u(\Psi(y))$. It turns out that under fairly general assumptions about the coefficients (see below) the vector field h is continuously differentiable and the regularity of the matrix (q^{km}) is not worse than that of the original matrix (a^{ij}). This circumstance enables us after the change of coordinates to apply the known results on the regularity of solutions and obtain the desired properties for the function σ, hence also for original solution ϱ. The main difficulty consists in constructing the mapping u. Here we employ some recent results of N.V. Krylov on the solvability of elliptic equations in Sobolev spaces (see [19] and [20]).

Under broad assumptions, we construct a diffeomorphism, which will be called Zvonkin's transform.

Throughout this section we assume that the following conditions are fulfilled.

$\mathbf{H_a}$ The coefficients a^{ij} are defined on all of $\mathbb{R}^d$ and for some constant $m > 0$ and all $x \in \mathbb{R}^d$ the following inequalities hold:

$$m \cdot \mathrm{I} \le A(x) \le m^{-1} \cdot \mathrm{I}.$$

VMO The coefficients a^{ij} belong to the class VMO, that is, there exists a continuous increasing function ω on $[0, +\infty)$ such that $\omega(0) = 0$ and

$$\sup_{z\in\mathbb{R}^d} r^{-2d} \int_{B(z,r)} \int_{B(z,r)} |a^{ij}(x) - a^{ij}(y)|\, dxdy \le \omega(r), \quad r > 0.$$

$\mathbf{H_b}$ $b \in L^{d+}_{loc}$, which means that for every ball B there is a number $p = p(B) > d$ such that the restriction of $|b|$ to B belongs to $L^p(B)$.

The assumption that the diffusion matrix satisfies the aforementioned conditions on the whole space $\mathbb{R}^d$ does not restrict the generality of our considerations, although the equation will be considered on a domain. Moreover, in many problems it is useful to have global changes of variables, but not local. For our purposes of proving local Harnack inequalities or the continuity of solutions it suffices to extend the coefficients on the whole space with preservation of the required conditions. The drift coefficient can be extended by zero outside a fixed ball B and the diffusion coefficient can be extended by the formula $\psi A + (1-\psi) I$ with a smooth function ψ that equals 1 on B and 0 outside a larger ball. Of course, it is important here that the equation holds on a ball, but not on the whole space.

Let $B(x_0, 4R) \subset \Omega$ and $\beta(x) = b(x)$ if $x \in B(x_0, 4R)$ and $\beta(x) = 0$ if $x \notin B(x_0, 4R)$. Then $\beta \in L^p(\mathbb{R}^d)$ and

$$\|\beta\|_{L^p(\mathbb{R}^d)} = \|b\|_{L^p(B(x_0,4R))}.$$

Let $1 \le k \le d$. Let us consider on $\mathbb{R}^d$ the elliptic equation

$$\mathrm{tr}(AD^2u) + \langle \beta, \nabla u\rangle - \lambda u = -\beta^k, \quad \lambda > 0. \tag{2.1}$$

Proposition 2.1. *For every $\delta > 0$ there exists $\lambda > 0$ such that for every $1 \le k \le d$ equation* (2.1) *has a solution $u \in C^1(\mathbb{R}^d) \cap W^{p,2}(\mathbb{R}^d)$ for which*

$$\sup_{x\in\mathbb{R}^d} |\nabla u(x)| \le \delta, \quad \|u\|_{W^{2,p}(\mathbb{R}^d)} \le M,$$

where the constant M depends only on d, ν, ω and $\|b\|_{L^p(B(x_0,4R))}$.

Proof. According to [20, Chapter 6, Section 4, Theorem 1], there exist numbers $\lambda_0 > 0$ and N_0 such that for all $\lambda > \lambda_0$ and every function $v \in W^{p,2}(\mathbb{R}^d)$ we have the inequality

$$\lambda \|v\|_{L^p(\mathbb{R}^d)} + \|v\|_{W^{p,2}(\mathbb{R}^d)} \le N_0 \|\mathrm{tr}(AD^2v) - \lambda v\|_{L^p(\mathbb{R}^d)}.$$

For every function $f \in L^p(\mathbb{R}^d)$ there exists a unique solution $v \in W^{2,p}(\mathbb{R}^d)$ of the equation

$$\mathrm{tr}(AD^2v) - \lambda v = f.$$

Since $p > d$, by the embedding theorem for every function $v \in W^{p,2}(\mathbb{R}^d)$ one has the estimate

$$\|\nabla v\|_{L^\infty(\mathbb{R}^d)} \le N_1 \|v\|_{W^{p,2}(\mathbb{R}^d)}.$$

By [20, Chapter 1, Section 5, Corollary 2] there exists a constant N_2 such that

$$\|\nabla v\|_{L^p(\mathbb{R}^d)} \le N_2 \|D^2 v\|_{L^p(\mathbb{R}^d)} + N_2 \|v\|_{L^p(\mathbb{R}^d)}.$$

Therefore,

$$\|\nabla v\|_{L^\infty(\mathbb{R}^d)} \le N_3 \|D^2 v\|_{L^p(\mathbb{R}^d)} + N_3 \|v\|_{L^p(\mathbb{R}^d)}.$$

Let $\varepsilon > 0$. By a standard reasoning, replacing x with εx, we obtain the inequality

$$\|\nabla v\|_{L^\infty(\mathbb{R}^d)} \le \varepsilon^{1-d/p} N_3 \|D^2 v\|_{L^p(\mathbb{R}^d)} + \varepsilon^{-1-d/p} N_3 \|v\|_{L^p(\mathbb{R}^d)}.$$

Thus, we can assume that for every $\varepsilon > 0$ there exists a constant $N_4 = N_4(\varepsilon)$ for which

$$\|\nabla v\|_{L^\infty(\mathbb{R}^d)} \le \varepsilon \|v\|_{W^{2,p}(\mathbb{R}^d)} + N_4 \|v\|_{L^p(\mathbb{R}^d)}.$$

Let us now estimate the expression

$$\|\langle\beta,\nabla v\rangle\|_{L^p(\mathbb{R}^d)}.$$

We have

$$\|\langle\beta,\nabla v\rangle\|_{L^p(\mathbb{R}^d)} \le N_4\|\beta\|_{L^p(\mathbb{R}^d)}\|v\|_{L^p(\mathbb{R}^d)} + \varepsilon\|\beta\|_{L^p(\mathbb{R}^d)}\|v\|_{W^{2,p}(\mathbb{R}^d)}.$$

Take $\varepsilon_0 > 0$ and $\lambda_1 \ge \lambda_0$ such that

$$\varepsilon_0 N_0\|\beta\|_{L^p(\mathbb{R}^d)} < 1/2, \quad N_4 N_0\|\beta\|_{L^p(\mathbb{R}^d)} < \lambda_1/2.$$

Then for every $\lambda > \lambda_1$ and $v \in W^{p,2}(\mathbb{R}^d)$ we have

$$\lambda\|v\|_{L^p(\mathbb{R}^d)} + \|v\|_{W^{p,2}(\mathbb{R}^d)} \le 2N_0\|\mathrm{tr}(AD^2v) + \langle\beta,\nabla v\rangle - \lambda v\|_{L^p(\mathbb{R}^d)}.$$

This estimate remains unchanged if we replace β by $t\beta$, where $t \in [0,1]$. Set

$$L_t v = t\Big(\mathrm{tr}(AD^2v) + \langle\beta,\nabla v\rangle - \lambda v\Big) + (1-t)\Big(\mathrm{tr}(AD^2v) - \lambda v\Big), \quad t \in [0,1].$$

The continuous operators L_t from $W^{2,p}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ satisfy the condition

$$\|L_t v\|_{L^p(\mathbb{R}^d)} \ge (2N_0)^{-1}\|v\|_{W^{2,p}(\mathbb{R}^d)}.$$

Hence the standard method of continuation with respect to a parameter ensures the existence of a solution u to equation (2.1). By the embedding theorem $u \in C^1(\mathbb{R}^d)$. In addition,

$$\|u\|_{L^p(\mathbb{R}^d)} \le 2\lambda^{-1}N_0\|\beta\|_{L^p(\mathbb{R}^d)}, \quad \|u\|_{W^{2,p}(\mathbb{R}^d)} \le 2N_0\|\beta\|_{L^p(\mathbb{R}^d)}.$$

It has been shown above that for every $\varepsilon > 0$ we have

$$\|\nabla u\|_{L^\infty(\mathbb{R}^d)} \le \varepsilon\|u\|_{W^{2,p}(\mathbb{R}^d)} + N_4\|u\|_{L^p(\mathbb{R}^d)},$$

where the right-hand side is estimated from above by

$$2\varepsilon N_0\|\beta\|_{L^p(\mathbb{R}^d)} + \lambda^{-1}2N_0N_4\|\beta\|_{L^p(\mathbb{R}^d)}.$$

Therefore, taking ε sufficiently small and λ sufficiently large we can obtain the desired estimate $\|\nabla u\|_{L^\infty(\mathbb{R}^d)} \le \delta$. □

Remark 2.2. Since $u \in W^{p,2}(\mathbb{R}^d)$ with $p > d$, by the embedding theorem $u \in C^{1+(1-d/p)}(\mathbb{R}^d)$, hence the function u is bounded and the derivatives u_{x_i} satisfy the Hölder condition of order $1-d/p$. Note also that our construction of u does not use a special form of β: only the condition $\beta \in L^p(\mathbb{R}^d)$ is needed.

Let $u = (u^1,\ldots,u^d)$, where each u^k is a solution to equation (2.1) from Proposition 2.1. Below u' and Φ' denote the Jacobi matrices of the mappings u and Φ. Let us take a number δ from the hypotheses of Proposition 2.1 such that for all $x \in \mathbb{R}^d$ the inequality

$$\|u'(x)\| \le \frac{1}{2} \quad \text{and} \quad \tfrac{1}{2} \le \det\big(I + u'(x)\big) \le 2$$

is fulfilled. Set

$$\Phi(x) = x + u(x).$$

We now establish some properties of the mapping Φ.

Proposition 2.3. (i) *The mapping Φ is a diffeomorphism of $\mathbb{R}^d$ of class C^1, moreover, the functions $\partial_{x_i}\Phi^k$ are locally Hölder continuous.*

(ii) *The inequalities*

$$\frac{1}{2}\|x-y\| \le \|\Phi(x)-\Phi(y)\| \le 2\|x-y\|$$

hold.

Proof. Since u is a contracting mapping with the Lipschitz constant $1/2$, the mapping Φ is a homeomorphism and the stated inequalities hold. The inclusion $u \in C^1(\mathbb{R}^d)$ yields that $\Phi \in C^1(\mathbb{R}^d)$ and the Jacobi matrix has the form $\Phi' = I + u'$. By the estimate $\|u'\| \le 1/2$ the matrix Φ' is invertible. Therefore, Φ is a diffeomorphism. □

We recall that $B(x_0, 4R) \subset \Omega$. Let $\Psi = \Phi^{-1}$ and $y_0 = \Phi(x_0)$ and consider the ball $B(y_0, 2R)$. According to the inequalities in (ii) of Proposition 2.3, we have the inclusions

$$B(x_0, R) \subset \Psi(B(y_0, 2R)) \subset B(x_0, 4R).$$

Proposition 2.4. *Let $\varrho \in L^1_{loc}(\Omega)$ be a solution to equation* (1.1). *Then the function*

$$\sigma(y) = |\det \Psi'(y)| \varrho(\Psi(y))$$

on $B(y_0, 2R)$ satisfies the equation $\mathcal{L}^*\sigma = 0$, where

$$\mathcal{L}f(y) = q^{km}(y)\partial_{y_k}\partial_{y_m} f(y) + h^k(y)\partial_{y_k} f(y)$$

and the coefficients have the form

$$q^{km}(y) = a^{ij}(\Psi(y))\partial_{x_i}\Phi^k(\Psi(y))\partial_{x_j}\Phi^m(\Psi(y)), \quad h^k(y) = \lambda u^k(\Psi(y)).$$

Proof. The function ϱ belongs to $L^r(B(x_0, 4R))$ for all $1 \le r < d/(d-1)$ (see the comments at the beginning of the proof of [7, Theorem 2.1]). In particular, for $p > d$ the value $p/(p-1)$ is less than $d/(d-1)$ and the inclusion $\varrho \in L^{p/(p-1)}(B(x_0, 4R))$ holds. Therefore, in the integral equality determining the solution, in place of test functions of class $C_0^\infty(\Omega)$ we can substitute test functions of class $W^{p,2}(\Omega)$ with compact support in $B(x_0, 4R)$. For every function $\varphi \in C_0^\infty(B(y_0, 2R))$ (outside the ball $B(y_0, 2R)$ we always extend φ by zero), the function $\varphi \circ \Phi$ belongs to $W^{2,p}(B(x_0, 4R))$, has compact support in $B(x_0, 4R)$ and satisfies the equality

$$\int_{B(x_0,4R)} \biggl(\Bigl[a^{ij}(x)\partial_{x_i}\Phi^k(x)\partial_{x_j}\Phi^m(x)\Bigr]\partial_{y_k}\partial_{y_m}\varphi(\Phi(x)) \\ + \Bigl[a^{ij}(x)\partial_{x_i}\partial_{x_j}\Phi^k(x) + b^i(x)\partial_{x_i}\Phi^k(x)\Bigr]\partial_{y_k}\varphi(\Phi(x)) \biggr) \varrho(x)\,dx = 0.$$

Since $b = \beta$ on $B(x_0, 4R)$ and $\Phi^k = x_k + u^k(x)$, we have

$$a^{ij}(x)\partial_{x_i}\partial_{x_j}\Phi^k(x) + b^i(x)\partial_{x_i}\Phi^k(x) = \lambda u^k(x).$$

Then

$$\int_{B(x_0,4R)} \Bigl(q^{km}(\Phi(x))\partial_{y_k}\partial_{y_m}\varphi(\Phi(x)) + \lambda u^k(x)\partial_{y_k}\varphi(\Phi(x))\Bigr)\varrho(x)\,dx = 0.$$

Using the change of variable $y = \Phi(x)$ and taking into account that the support of φ belongs to $B(y_0, 2R)$, we obtain

$$\int_{B(y_0,2R)} \Bigl(q^{km}(y)\partial_{y_k}\partial_{y_m}\varphi(y) + \lambda u^k(\Psi(y))\partial_{y_k}\varphi(y)\Bigr)\varrho(\Psi(y))|\det \Psi'(y)|\,dy = 0.$$

Since φ was arbitrary, we conclude that $\sigma(y) = \varrho(\Psi(y))|\det \Psi'(y)|$ is a solution to the equation $\mathcal{L}^*\sigma = 0$. □

Observe that the vector field $h(y) = \lambda u(\Psi(y))$ is continuously differentiable on the ball $B(y_0, 2R)$. In addition, the derivatives of Φ also satisfy the Hölder condition. Therefore, the function σ on $B(y_0, 2R)$ satisfies the equation $\mathcal{L}^*\sigma = 0$, in which the coefficients q^{mk} of the second order terms form a nondegenerate matrix and belong to the class VMO and the coefficients h^k are continuous on $B(y_0, 2R)$. This enables us to apply the results from the papers [7], [11], and [27], [28] to the function σ and then to transfer them to ϱ. Let us give an example demonstrating a simple derivation of the known result of [7, Theorem 3.1]) from the case of a nice drift.

We recall that a mapping satisfies Dini's condition if for its modulus of continuity ω we have

$$\int_0^1 \frac{\omega(t)}{t}\,dt < \infty.$$

Example 2.5. If conditions $\mathbf{H_a}$ and $\mathbf{H_b}$ are fulfilled and the matrix A satisfies Dini's condition, then every solution $\varrho \in L^1_{loc}(\Omega)$ to equation (1.1) has a continuous version.

Proof. Let us verify the existence of a continuous version of ϱ on the ball $B(x_0, R/2) \subset B(x_0, 4R) \subset \Omega$. Let Φ be the diffeomorphism constructed above. By Proposition 2.4 the function $\sigma(y) = \varrho(\Psi(y))|\det \Psi'(y)|$ satisfies on $B(y_0, 2R)$ an equation with some coefficients for which the hypotheses of [28, Theorem 1] are fulfilled, that is, the matrix (q^{mk}) is nondegenerate and the functions q^{mk}, h^k satisfy Dini's condition. Hence σ has a continuous version on $B(y_0, R)$. Since Φ is a diffeomorphism of class C^1, the mappings Φ and Ψ take sets of measure zero to sets of measure zero and a modification of the function σ on a set of measure zero yields a change of ϱ on a set of measure zero. Therefore, the function ϱ has a continuous version on $B(x_0, R/2)$. □

Note that on the ball $B(x_0, R/2) \subset B(x_0, 4R) \subset \Omega$ the modulus of continuity of the solution ϱ depends only on the quantities d, p, R, ω, ν, and $\|b\|_{L^p(B(x_0,4R))}$.

In a similar way, by using [27, Theorem 2] one can derive the Hölder continuity of the solution, provided that the functions a^{ij} are Hölder continuous. However, unlike [7, Theorem 3.1], this method does not ensure the Hölder order of the solution to be equal to the Hölder order of the matrix A, because the expression for the coefficients q^{mk} involves the derivatives of mapping Φ, but their Hölder order depends on d and p.

3. Regularity of solutions

In this section we apply Zvonkin's transform for establishing the regularity of solutions. We first discuss the case where the matrix A satisfies the classical Dini condition, then consider the Dini mean oscillation condition, and finally study the integrability of solutions without the assumption about Dini's condition.

The next assertion generalizes the Harnack inequality to the case where the diffusion matrix satisfies Dini's condition and the drift coefficient is locally unbounded (and is merely integrable to some power larger than the dimension). In the known results, the drift coefficient is either zero or locally bounded, which has been substantially used in the proofs.

Theorem 3.1. *If a^{ij}, b^i satisfy conditions $\mathbf{H_a}$ and $\mathbf{H_b}$ and the matrix A satisfies Dini's condition, then the continuous version of every nonnegative solution $\varrho \in L^1_{loc}(\Omega)$ to equation* (1.1) *satisfies the Harnack inequality, that is, for every ball $B(x_0, R/2) \subset B(x_0, 4R) \subset \Omega$ there exists a number C such that*

$$\sup_{x\in B(x_0,R/2)} \varrho(x) \le C \inf_{x\in B(x_0,R/2)} \varrho(x),$$

where C depends on R, ω, d, ν, p, and $\|b\|_{L^p(B(x_0,4R))}$, but does not depend on the solution ϱ.

Proof. Let $B(x_0, R/2) \subset B(x_0, 4R) \subset \Omega$ and let Φ be the diffeomorphism constructed above. As above, according to Proposition 2.4 the function

$$\sigma(y) = \varrho(\Psi(y))|\det \Psi'(y)|$$

on $B(y_0, 2R)$ satisfies the equation with coefficients for which the hypotheses of [7, Corollary 3.6] are fulfilled, i.e., the matrix (q^{mk}) is nondegenerate, the functions q^{mk} satisfy Dini's condition and the functions h^k are bounded. Therefore, there exists a number C depending on the objects listed above such that

$$\sup_{y\in B(y_0,R)} \sigma(y) \le C \inf_{y\in B(y_0,R)} \sigma(y).$$

Since $2^{-1} \le |\det \Psi'(y)| \le 2$, we have

$$\sup_{y\in B(y_0,R)} \varrho(\Psi(y)) \le 4C \inf_{y\in B(y_0,R)} \varrho(\Psi(y)).$$

By the inclusion $B(x_0, R/2) \subset \Psi(B(y_0, R))$ we have

$$\sup_{x\in B(x_0,R/2)} \varrho(x) \le \sup_{y\in B(y_0,R)} \varrho(\Psi(y)), \qquad \inf_{y\in B(y_0,R)} \varrho(\Psi(y)) \le \inf_{x\in B(x_0,R/2)} \varrho(x).$$

Therefore, $\sup_{x\in B(x_0,R/2)} \varrho(x) \le 4C \inf_{x\in B(x_0,R/2)} \varrho(x)$. □

Remark 3.2. Using the method suggested in [21] and [25], increasing the dimension, it is possible to add a potential term to the drift coefficient. Let ϱ be a solution in $\mathbb{R}^d$ to the equation

$$\partial_{x_i}\partial_{x_j}\bigl(a^{ij}\varrho\bigr) - \partial_{x_i}\bigl(b^i\varrho\bigr) + c\varrho = 0 \tag{3.1}$$

Then on $\mathbb{R}^{d+1} = \mathbb{R}^d_x \times \mathbb{R}^1_y$ the function ϱ satisfies the equation

$$\partial_{x_i}\partial_{x_j}\bigl(a^{ij}\varrho\bigr) + \partial_y^2\varrho - \partial_{x_i}\bigl(b^i\varrho\bigr) - \partial_y(-cy\varrho) = 0.$$

Using this approach, one can apply the result obtained above to the equation with the potential term $c\varrho$, but in this case it is necessary to assume a higher integrability of the coefficients: $b^i, c \in L^p_{loc}(\Omega)$ with $p > d+1$.

Note also that the equation with a nonzero drift coefficient b can be transformed in a similar way into an equation without the drift. Let ϱ be a solution to equation (1.1) on $\mathbb{R}^d$. Then on $\mathbb{R}^{d+1} = \mathbb{R}^d_x \times (0,1)$ the function ϱ satisfies the equation

$$\partial_{x_i}\partial_{x_j}\bigl(a^{ij}\varrho\bigr) + M\partial_y^2\varrho - \partial_y\partial_{x_i}\bigl((2+\tfrac{y}{2})b^i\varrho\bigr) - \partial_{x_i}\partial_y\bigl((2+\tfrac{y}{2})b^i\varrho\bigr) = 0.$$

The new matrix $\widetilde{A}$ has the form

$$\begin{pmatrix} a^{11} & a^{12} & \dots & (2+\frac{y}{2})b^1 \\ a^{21} & a^{22} & \dots & (2+\frac{y}{2})b^2 \\ \cdot & \cdot & \dots & \cdot \\ (2+\frac{y}{2})b^1 & (2+\frac{y}{2})b^2 & \dots & M \end{pmatrix}.$$

For every $\xi \in \mathbb{R}^{d+1}$ we have the equality

$$\langle \widetilde{A}\xi, \xi\rangle = \sum_{i,j=1}^d a^{ij}\xi_i\xi_j + \sum_{j=1}^d (4+y)b^j\xi_j\xi_{d+1} + M\xi_d^2.$$

If the functions b^i are bounded and $A \ge m\cdot I$, then for M sufficiently large the matrix $\widetilde{A}$ is positive definite. A certain drawback of such transformations is the necessity to impose on b the same restrictions as on a^{ij}, for example, to require the continuity and Dini's condition. In the next remark we shall show how one can accomplish smoothing of coefficients b and c with the aid of renormalization of solutions and Zvonkin's transform.

Remark 3.3. Let $\varrho \in L^1_{loc}(\Omega)$ be a solution to equation (3.1), where $c \in L^p_{loc}(\Omega)$ for some $p > d$, the coefficients a^{ij}, b^i satisfy conditions $\mathbf{H_a}$ and $\mathbf{H_b}$ and the matrix A is continuous. Let $B(x_0, 4R) \subset \Omega$ and let Φ be the diffeomorphism constructed before Proposition 2.3. Set $y_0 = \Phi(x_0)$ and $\Psi = \Phi^{-1}$. Similarly to Proposition 2.4, one verifies that the function

$$\sigma(y) = |\det\Psi'(y)|\varrho(\Psi(y))|$$

satisfies the equation $\mathcal{L}^*\sigma + g\sigma = 0$ on $B(y_0, 2R)$, where

$$\mathcal{L}f(y) = q^{km}(y)\partial_{y_k}\partial_{y_m}f(y) + h^k(y)\partial_{y_k}f(y)$$

and the coefficients have the form

$$q^{km}(y) = a^{ij}(\Psi(y))\partial_{x_i}\Phi^k(\Psi(y))\partial_{x_j}\Phi^m(\Psi(y)), \quad h^k(y) = \lambda u^k(\Psi(y)), \quad g(y) = c(\Psi(y)).$$

Let us observe that h^k is a continuously differentiable function on $B(y_0, 2R)$, $g \in L^p(B(y_0, 2R))$, $Q = (q^{km})$ satisfies condition $\mathbf{H_a}$, and the function q^{km} is continuous. Let $\gamma > 0$.

Let us consider the Dirichlet problem

$$\mathcal{L}u + (g - \gamma)u = 0 \quad \text{on} \quad B(y_0, R), \quad u = 1 \quad \text{on} \quad \partial B(y_0, R).$$

Since we do not assume that the coefficient g is bounded from above, for completeness we give a short justification of the existence of a positive solution under the condition that the number γ is sufficiently large. It is clear that it suffices to consider the Cauchy problem with zero boundary condition and some right-hand side. Set $B = B(y_0, R)$. By [14, Theorem 9.14 and Theorem 9.15] (see also [20, Chapter 8]) there exists $\gamma_0 > 0$ such that for every $\gamma > \gamma_0$ the Dirichlet problem $\mathcal{L}v - \gamma v = f$ on B with $v = 0$ on ∂B has a solution $v \in W^{p,2}(B) \cap W_0^{p,1}(B)$ for every function $f \in L^p(B)$. In addition, for all $\gamma > \gamma_0$ and $v \in W^{p,2}(B) \cap W_0^{p,1}(B)$ we have the estimate

$$\gamma \|u\|_{L^p(B)} + \sqrt{\gamma}\|\nabla u\|_{L^p(B)} + \|D^2 u\|_{L^p(B)} \le N_1 \|\mathcal{L}v - \gamma v\|_{L^p(B)}.$$

By the embedding theorem $\|v\|_{L^\infty(B)} \le N_2 \|Dv\|_{L^p(B)}$ and

$$\|gv\|_{L^p(B)} \le N_2 \|g\|_{L^p(B)} \|Dv\|_{L^p(B)}.$$

Therefore, for γ sufficiently large and every $v \in W^{p,2}(B) \cap W_0^{p,1}(B)$ we have the estimate

$$\|v\|_{W^{p,2}(B)} \le N_3 \|\mathcal{L}v + (g - \gamma)v\|_{L^p(B)}.$$

The method of continuation with respect to a parameter gives a solution $v \in W^{p,2}(B) \cap W_0^{p,1}(B)$ to the equation $\mathcal{L}v + (g - \gamma)v = f$ for every $f \in L^p(B)$, hence a solution u to the considered Dirichlet problem. We show that it is positive. For a function η let η^- and η^+ be the negative and positive parts of η. Let $w = -u$. If $w \le 0$, then everything is proved. If w is positive somewhere, then $\sup_B w = \sup_B w^+$. Since

$$\mathcal{L}w + (g - \gamma)^- w \ge -(g - \gamma)^+ w^+,$$

by the maximum principle (see [14, Theorem 9.1]) we have

$$\sup_B w^+ \le N \|(g - \gamma)^+ w^+\|_{L^d(B)} \le N \|(g - \gamma)^+\|_{L^d(B)} \sup_B w^+.$$

Take γ so large that $N\|(g - \gamma)^+\|_{L^d(B)} < 1$. Then $\sup_B w^+ \le 0$. Therefore, $u \ge 0$. The strict positivity follows from the Harnack inequality (see, e.g., [29]). Finally, we observe that by the Sobolev embedding theorem the function u has a continuously differentiable version, moreover, u_{x_i} belongs to some Hölder space. We shall work with this version. Substituting into the integral equality

$$\int \bigl[\mathcal{L}\varphi + g\varphi\bigr] \sigma \, dy = 0$$

in place of the function φ the function $u\psi$, where $\psi \in C_0^\infty(B(y_0, R))$, we arrive at the equality

$$\int \bigl[\mathcal{L}\psi + 2\langle u^{-1} Q \nabla u, \nabla \psi \rangle + \gamma \psi\bigr] u\sigma \, dy = 0.$$

Therefore, the function $u\sigma$ is a solution to the equation $\widetilde{\mathcal{L}}^*(u\sigma) + \gamma(u\sigma) = 0$ with the operator

$$\widetilde{\mathcal{L}} f = \mathcal{L} f(y) + 2\langle u^{-1}(y) Q(y) \nabla u(y), \nabla f(y) \rangle.$$

Thus, after these transformations we obtain the equation in which the coefficient c is constant and the coefficient b is continuous and even belongs to some Hölder class. Next, we apply the transform from Remark 3.2 and arrive at the equation with zero coefficients b and c. Moreover, if the original matrix A satisfies Dini's condition, then after these transformations it also satisfy this condition, in particular, Theorem 3.1 extends to equations with the zero order term $c\varrho$, provided that $c \in L^p_{loc}(\Omega)$.

As already noted above, in [11, Lemma 4.2] the Harnack inequality was established under the assumption that $b = 0$ and A satisfies the Dini mean oscillation condition. Using Zvonkin's transform and the methods of killing first and zero order terms explained above, we can generalize this assertion and obtain an analog of Theorem 3.1 for the matrix A that satisfies the Dini mean oscillation condition.

Following [10] and [11], we shall say that a measurable function f on $\mathbb{R}^d$ satisfies the Dini mean oscillation condition if for some $t_0 > 0$

$$\int_0^{t_0} \frac{w(r)}{r}\, dr < \infty,$$

where

$$w(r) = \sup_{x\in\Omega} \frac{1}{|\Omega(x,r)|} \int_{\Omega(x,r)} \Bigl| f(y) - f_\Omega(x,r) \Bigr|\, dy,$$

$$f_\Omega(x,r) = \frac{1}{|\Omega(x,r)|} \int_{\Omega(x,r)} f(y)\, dy, \quad \Omega(x,r) = \Omega \cap B(x,r).$$

Clearly, the classical Dini condition implies the Dini mean oscillation condition. If there exists a number N such that $|\Omega(x,r)| \ge N r^d$ for all $x \in \Omega$ and $0 < r < \operatorname{diam}\Omega$, then, according to [16, Lemma A1], any measurable function f satisfying the Dini mean oscillation condition, has a version uniformly continuous on Ω, moreover, its modulus of continuity is estimated by $\displaystyle\int_0^{|x-y|} \frac{w(r)}{r}\, dr$. We shall work with this continuous version. In addition, according to [11, Lemma 2.1], the product fg satisfies the Dini mean oscillation condition if g satisfies the classical Dini condition and f satisfies the Dini mean oscillation condition.

Suppose that a function f is defined on $\mathbb{R}^d$ and $|\Omega(x,r)| \ge N r^d$ for all points $x \in \Omega$ and $r \in (0, \operatorname{diam}\Omega)$. Then for all $x \in \Omega$ and some constant $C(d,N) > 0$ the estimate

$$\frac{1}{|\Omega(x,r)|} \int_{\Omega(x,r)} |f(y) - f_\Omega(x,r)|\, dy \le \frac{C(d,N)}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_B(x,r)|\, dy$$

holds, where

$$f_B(x,r) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy.$$

Therefore, in order to verify the Dini mean oscillation condition it suffices to show that

$$\int_0^{t_0} \frac{\widetilde{w}(r)}{r}\, dr < \infty, \quad \text{where} \quad \widetilde{w}(r) = \sup_{x\in\Omega} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_B(x,r)|\, dy.$$

We need the following observations.

Lemma 3.4. (i) *Suppose that a function f on $B(z,4R) \subset \mathbb{R}^d$ satisfies the Dini mean oscillation condition, $\Phi\colon \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism of class C^1, the functions $\partial_{x_i}\Phi^j$ are Hölder continuous and $2^{-1}\|x-y\| \le \|\Phi(x) - \Phi(y)\| \le 2\|x-y\|$. Then the function $f \circ \Phi$ satisfies the Dini mean oscillation condition on $B(z', R)$, where $z' = \Phi^{-1}(z)$.*

(ii) *Suppose that a function f on $B(z,4R) \subset \mathbb{R}^d$ satisfies the Dini mean oscillation condition. Then the function*

$$F(x_1, \ldots, x_d, x_{d+1}) = f(x_1, x_2, \ldots, x_d)$$

satisfies the Dini mean oscillation condition on $B((z, z_{d+1}), R)$ for every z_{d+1}.

Proof. Let $0 < r < 2R$ and

$$w(r) = \sup_{x\in B(z,2R)} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_B(x,r)|\, dy.$$

Let us prove (i). By assumption the functions $\partial_{x_i}\Phi^j$ are the Hölder continuous of some order $\gamma \in (0,1)$. It suffices to verify that the Dini mean oscillation condition is fulfilled for the function $g(x) = f(\Phi(x))|\det \Phi'(x)|$. Let $x \in B(z', 2R)$ and $0 < r < R$. We have

$$\frac{1}{|B(x,r)|}\int_{B(x,r)}\Big|g(v) - g_B(x,r)\Big|\,dv \le \frac{1}{|B(x,r)|}\int_{B(\Phi(x),2r)}\Big|f(y) - |\det \Phi'(\Phi^{-1}(y))|^{-1} g_B(x,r)\Big|\,dy.$$

Let us estimate the difference

$$|\det \Phi'(\Phi^{-1}(y))|^{-1} g_B(x,r) - f_B(\Phi(x), 2r).$$

Let replace $|\det \Phi'(\Phi^{-1}(y))|^{-1}$ by $|\det \Phi'(x)|^{-1}$. The difference between these two numbers is estimated by $N_1 r^\gamma$ with some constant $N_1 > 0$. Observe that

$$\begin{aligned} g_B(x,r) &= \frac{1}{|B(x,r)|}\int_{\Phi(B(x,r))} f(y)\,dy \\ &= \frac{1}{|B(x,r)|}\int_{\Phi(B(x,r))}\big(f(y) - f_B(\Phi(x),2r)\big)\,dy + f_B(\Phi(x),2r)\frac{|\Phi(B(x,r))|}{|B(x,r)|}, \end{aligned}$$

where

$$\begin{aligned} &\frac{1}{|B(x,r)|}\left|\int_{\Phi(B(x,r))}\big(f(y) - f_B(\Phi(x),2r)\big)\,dy\right| \\ &\qquad\le \frac{2^d}{|B(\Phi(x),2r)|}\int_{B(\Phi(x),2r)}\big|f(y) - f_B(\Phi(x),2r)\big|\,dy \le 2^d w(2r). \end{aligned}$$

In addition, for some constant $N_2 > 0$ we have the estimate

$$\left|1 - \frac{|\Phi(B(x,r))|}{|\det \Phi'(x)|\,|B(x,r)|}\right| \le N_2 r^\gamma.$$

Thus,

$$\Big||\det \Phi'(\Phi^{-1}(y))|^{-1} g_B(x,r) - f_B(\Phi(x),2r)\Big| \le (N_1 + N_2) r^\gamma + 2^d w(2r).$$

Therefore, we arrive at the inequality

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |g(v) - g_B(x,r)|\,dv \le N_3(w(2r) + r^\gamma),$$

which shows that g satisfies the Dini mean oscillation condition.

Let us prove assertion (ii). Let c_d denote the volume of the unit ball in $\mathbb{R}^d$.

Let $(x, x_{d+1}) \in B((z, z_{d+1}), R)$ and $0 < r < R$. We observe that by Fubini's theorem

$$\begin{aligned} F_B((x,x_{d+1}),r) &= \frac{2}{c_{d+1}}\int_{B(0,1)} f(x_1 + ry_1, \ldots, x_d + ry_d)\sqrt{1 - y_1^2 - \ldots - y_d^2}\,dy_1\ldots dy_d \\ = \frac{2}{c_{d+1}}\int_{B(0,1)}&\Big(f(x_1 + ry_1, \ldots, x_d + ry_d) - f_B(x,r)\Big)\sqrt{1 - y_1^2 - \ldots - y_d^2}\,dy_1\ldots dy_d + f_B(x,r), \end{aligned}$$

where the first term is estimated in absolute value by $Nw(r)$ with some constant $N > 0$. We have

$$\begin{aligned} &\frac{1}{|B((x,x_{d+1}),r)|}\int_{B((x,x_{d+1}),r)} |F(v) - F_B((x,x_{d+1}),r)|\,dv \\ &\qquad\le Nw(r) + \frac{1}{|B((x,x_{d+1}),r)|}\int_{B((x,x_{d+1}),r)} |F(v) - f_B(x,r)|\,dv. \end{aligned}$$

Applying Fubini's theorem to the second term, we estimate it by

$$\frac{2r}{|B((x,x_{d+1}),r)|}\int_{B(x,r)} |f(y) - f_B(x,r)|\,dy.$$

Thus, for some constant $C(d,N)>0$ we obtain

$$\frac{1}{|B((x,x_{d+1}),r)|}\int_{B((x,x_{d+1}),r)}|F(v)-F_B((x,x_{d+1}),r)|\,dv\le C(d,N)w(r),$$

which implies our claim. $\square$

Our next result generalizes Example 2.5 and Theorem 3.1 to the case where the diffusion matrix satisfies the Dini mean oscillation condition.

Theorem 3.5. *Suppose that condition $\mathbf{H_a}$ is fulfilled, on every ball the matrix A satisfies the Dini mean oscillation condition with some function ω, and $b^i, c\in L^{d+}_{loc}(\Omega)$. Suppose also that $\varrho\in L^1_{loc}(\Omega)$ is a solution to the equation*

$$\partial_{x_i}\partial_{x_j}\big(a^{ij}\varrho\big)-\partial_{x_i}\big(b^i\varrho\big)+c\varrho=0.$$

Then the function ϱ has a continuous version. Moreover, if $\varrho\ge 0$, then the continuous version of ϱ satisfies the Harnack inequality, i.e., for every ball $B(x_0,R/2)\subset B(x_0,4R)\subset\Omega$ there exists a number C such that

$$\sup_{x\in B(x_0,R/2)}\varrho(x)\le C\inf_{x\in B(x_0,R/2)}\varrho(x),$$

where C depends on R, w, d, ν, p, $\|c\|_{L^p(B(x_0,4R))}$, and $\|b\|_{L^p(B(x_0,4R))}$, and does not depend on ϱ. The modulus of continuity of ϱ on $B(x_0,R/2)$ depends on the same objects.

Proof. Justification is the same as in Example 2.5 and Theorem 3.1, but in place of results from [28] and [7] we apply [10, Theorem 1.10] and [11, Lemma 4.2], because Zvonkin's transform combined with Remark 3.3 and the transformation from Remark 3.2 enable us to reduce the proof to the case of $b=0$ and $c=0$, moreover, the elements of the new matrix A satisfy the Dini mean oscillation condition by Lemma 3.4. $\square$

We now discuss the integrability of solutions without the assumption about Dini's condition. Let us recall that, according to [7, Theorem 2.1], if a^{ij} and b^i satisfy conditions $\mathbf{H_a}$ and $\mathbf{H_b}$ and $a^{ij}\in VMO$, then every solution $\varrho\in L^1_{loc}(\Omega)$ is locally integrable to every power $p\ge 1$. If the functions a^{ij} satisfy the Dini mean oscillation condition, then the solution is locally bounded and even continuous. It is of interest to study integrability when A is slightly better than VMO, but does not satisfy Dini's condition.

Suppose that the coefficients a^{ij} and b^i satisfy conditions $\mathbf{H_a}$ and $\mathbf{H_b}$ and that

$$\|A(x)-A(y)\|\le\omega(\|x-y\|),$$

where ω is an increasing continuous function on $[0,+\infty)$ and $\omega(0)=0$. We also assume that for some $C_\omega>0$ and all $t\ge 0$ the inequality

$$\omega(t)\ge C_\omega t^{1-d/p}$$

holds. Set

$$\Lambda(t)=-\int_{\frac{1}{t}}^1\ln(\omega^{-1}(s))\,ds.$$

Integrating by parts, it is readily verified that

$$\Lambda(\tfrac{1}{t})=\int_{\omega^{-1}(t)}^{\omega^{-1}(1)}\frac{\omega(s)}{s}\,ds-\ln\omega^{-1}(1)+t\ln\omega^{-1}(t).$$

We observe that if Dini's condition is fulfilled, then the function $\Lambda(t)$ is bounded from above.

Theorem 3.6. *Let $\varrho\in L^1_{loc}(\Omega)$ satisfy equation* (1.1)*. Then, for every closed ball $B\subset\Omega$ and every $\delta>0$, there exists a constant $c=c(B,\delta)>0$ such that for all $q\ge 1$ one has*

$$\ln\|\varrho\|_{L^p(B)}\le c+c\Lambda(cq)+\delta\ln q.$$

We first consider the case of a bounded drift coefficient b and by Zvonkin's transform extend the result to the case of integrable b.

The next auxiliary assertion is standard, but in order to precise depends of constants on parameters we include a justification.

Lemma 3.7. *Let $\alpha = (\alpha^{ij})$ be a constant symmetric positive definite matrix such that*

$$\gamma I \le \alpha \le \gamma^{-1} I.$$

Let $f \in C_0^\infty(B(0,1))$ and

$$1 < r < d/(d-1), \quad 1 < t < \tau < d/(d-1).$$

Then there exists a function $u \in C^\infty(B(0,1))$ such that $\mathrm{tr}(\alpha D^2 u) = f$ *and*

$$\|u\|_{L^r(B(0,1))} + \|\nabla u\|_{L^r(B(0,1))} \le C(r,d,\gamma)\|f\|_{L^t(B(0,1))}, \tag{3.2}$$

$$\|D^2 u\|_{L^t(B(0,1))} \le C(d,\tau,\gamma) t' \|f\|_{L^t(B(0,1))}, \tag{3.3}$$

where the constants $C(d,\tau,\gamma)$ and $C(r,d,\gamma)$ do not depend on t.

Proof. Changing coordinates we reduce the problem to the case of the unit matrix α. Let u be the solution to the Dirichlet problem $\Delta u = f$ on $B(0,2)$, $u = 0$ on $\partial B(0,2)$. We estimate ∇u. Let $g^j \in C_0^\infty(B(0,2))$ and let v be the solution to the Dirichlet problem $\Delta v = \operatorname{div} g$ on $B(0,2)$, $v = 0$ on $\partial B(0,2)$. By Theorem [4] and the embedding theorem one has

$$\sup_{B(0,2)} |v(x)| \le C_1(r,d)\|g\|_{L^{r'}(B(0,2))}.$$

Then

$$\int \langle \nabla u, g\rangle \, dx = -\int v\Delta u \, dx \le C_1(r,d)\|g\|_{L^{r'}(B(0,2))}\|f\|_{L^1(B(0,2))}.$$

Since g was arbitrary, we obtain the desired estimate of the norm $\|\nabla u\|_{L^r(B(0,2))}$. Similarly an estimate of the norm $\|u\|_{L^r(B(0,2))}$ is obtained.

Let now $\zeta \in C_0^\infty(B(0,2))$ and $\zeta = 1$ on $B(0,1)$. Applying to ζu Theorems 9.8 and 9.9 from [14], we obtain the inequality

$$\|D^2(\zeta u)\|_{L^t(B(0,2))} \le C_1(d,\tau) t' \|\Delta(\zeta u)\|_{L^t(B(0,2))},$$

where the right-hand side is estimated by

$$C_2(d,\tau) t' \bigl(\|f\|_{L^t(B(0,2))} + \|\nabla u\|_{L^\tau(B(0,2))} + \|u\|_{L^\tau(B(0,2))}\bigr).$$

It remains to apply the estimate for u and ∇u obtained above. $\square$

Lemma 3.8. *Let ϱ be a solution to equation* (1.1) *with a bounded drift coefficient. Then, for every ball B and every $\delta > 0$, there exists $c = c(B,\delta) > 0$ such that for all $q \ge 1$ we have*

$$\ln \|\varrho\|_{L^q(B)} \le c + c\Lambda(cq) + \delta \ln q.$$

Proof. It suffices to obtain our estimate for large q. Without loss of generality we can assume that $B = B(0,1)$ and $B(0,4) \subset \Omega$. Let $x_0 \in B(0,1)$ and $0 < \lambda < 1$ be fixed. The function $\sigma(y) = \varrho(x_0 + \lambda y)$ on the ball $B(0,2)$ satisfies the equation

$$\partial_{y_i y_j}\bigl(q^{ij}\sigma\bigr) - \partial_{y_i}\bigl(h^i\sigma\bigr) = 0,$$

where $q^{ij}(y) = a^{ij}(x_0 + \lambda y)$ and $h^i(y) = \lambda b^i(x_0 + \lambda y)$. Let

$$Q_0 = (q_0^{ij}),\ q_0^{ij} = a^{ij}(x_0),\ \zeta \in C_0^\infty(B(0,1)),\ 0 \le \zeta \le 1,\ \zeta(y) = 1 \ \text{ if } \ y \in B(0,1/2).$$

By Lemma 3.7, for every function $f \in C_0^\infty(B(0,1))$, there exists a smooth solution u to the equation $\mathrm{tr}(Q_0 D^2 u) = f$ satisfying estimates (3.2) and (3.3). Then

$$\int f\zeta\sigma \, dy = \int \mathrm{tr}((Q_0 - Q)D^2 u)\zeta\sigma \, dy - \int \bigl[u\mathrm{tr}(QD^2\zeta) + 2\langle Q\nabla u, \nabla\zeta\rangle + \langle h, \nabla(\zeta u)\rangle\bigr]\sigma \, dy.$$

Let $1 < t < \tau < d/(d-1)$ and $1 < r < d/(d-1)$. Observe that

$$\int \mathrm{tr}((Q_0 - Q)D^2u)\zeta\sigma\, dy \le \omega(\lambda)\|D^2u\|_{L^t(B(0,1))}\|\zeta\sigma\|_{L^{t'}(B(0,1))}.$$

By Lemma 3.7 we have $\|D^2u\|_{L^t(B(0,1))} \le C_1 t'\|f\|_{L^t(B(0,1))}$, where C_1 does not depend on t. Therefore,

$$\int \mathrm{tr}((Q_0 - Q)D^2u)\zeta\sigma\, dy \le C_1 t'\omega(\lambda)\|f\|_{L^t(B(0,1))}\|\zeta\sigma\|_{L^{t'}(B(0,1))}.$$

The expression

$$-\int \bigl[u\mathrm{tr}(QD^2\zeta) + 2\langle Q\nabla u, \nabla\zeta\rangle + \langle h, \nabla(\zeta u)\rangle\bigr]\sigma\, dy \tag{3.4}$$

is estimated from above by

$$C(\zeta)\bigl[\sup_y \|Q(y)\| + \sup_y |h(y)|\bigr]\int_{B(0,1)} \bigl(|u(y)| + |\nabla u(y)|\bigr)|\sigma(y)|\, dy.$$

By Hölder's inequality we obtain

$$\int_{B(0,1)} \bigl(|u(y)| + |\nabla u(y)|\bigr)|\sigma(y)|\, dy \le \bigl(\|u\|_{L^r(B(0,1))} + \|\nabla u\|_{L^r(B(0,1))}\bigr)\|\sigma\|_{L^{r'}(B(0,1))}.$$

Applying again Lemma 3.7, we can estimate (3.4) by

$$C_2\|f\|_{L^t(B(0,1))}\|\sigma\|_{L^{r'}(B(0,1))},$$

where C_2 does not depend on t. Thus, we arrive at the estimate

$$\int f\zeta\sigma\, dy \le C_1 t'\omega(\lambda)\|f\|_{L^t}\|\zeta\sigma\|_{L^{t'}} + C_2\|f\|_{L^t}\|\sigma\|_{L^{r'}(B(0,1))}.$$

Set now $t' = q$. Let $C_1 q\omega(\lambda) = 1/2$, i.e., $\lambda = \omega^{-1}\bigl(\frac{1}{2C_1 q}\bigr)$, where the number q is so large that $\lambda < 1$. Recall that $\zeta = 1$ on $B(0, 1/2)$. Therefore, we have

$$\|\sigma\|_{L^q(B(0,1/2))} \le 2C_2\|\sigma\|_{L^{r'}(B(0,1))}.$$

Let $r' < s < q$. Applying Hölder's inequality, we arrive at the estimate

$$\|\sigma\|_{L^q(B(0,1/2))} \le C_3\|\sigma\|_{L^s(B(0,1))},$$

where C_3 does not depend on q, λ and s. Returning to the original coordinates, we obtain

$$\|\varrho\|_{L^q(B(x_0,\lambda/2))} \le \lambda^{\frac{d}{q}-\frac{d}{s}}C_3\|\varrho\|_{L^s(B(x_0,\lambda))}.$$

Let $1 < R < 2$. Covering the ball $B(0, R)$ by finitely many balls $B(x_i, \lambda/2)$, we arrive at the estimate

$$\|\varrho\|_{L^q(B(0,R))} \le C_4\lambda^{-d/s}\|\varrho\|_{L^s(B(0,R+\lambda))},$$

where C_4 does not depend on s, R and λ. Let

$$\beta > 1, \quad \frac{\ln C_4}{\ln\beta} < \delta.$$

Set $q_m = 2C_1\beta^m$, $s = 2C_1\beta^{m-1}$ and

$$\lambda_m = \omega^{-1}\bigl(\tfrac{1}{2C_1\beta^m}\bigr), \quad R = r_m = 2 - \sum_{k=k_0}^{m}\lambda_k.$$

Since $\omega^{-1}(s) \le \bigl(\frac{s}{C}\bigr)^{p/(p-d)}$, the series $\sum_k \lambda_k$ converges. Further we assume that k_0 is so large that $r_m > 1$ for all $m > k_0$ and $\beta^{-k_0+1} < 1$. Thus, for all $m > k_0$ the inequality

$$\|\varrho\|_{L^{q_m}(B(0,r_m))} \le e^{v_m}\|\varrho\|_{L^{q_{m-1}}(B(0,r_{m-1}))}, \quad v_m = \delta\ln\beta - d(2C_1)^{-1}\beta^{-m+1}\ln\lambda_m$$

holds. By iterations from k to k_0 we obtain the inequality

$$\ln \|\varrho\|_{L^{q_k}(B(0,r_k))} \le C_5 + \sum_{m=k_0}^{k} v_m,$$

where C_5 does not depend on k. Observe that

$$\sum_{m=k_0}^{k} v_k \le \delta \ln q_k - d(2C_1)^{-1} \sum_{m=k_0}^{k} \beta^{-m+1} \ln \lambda_m.$$

We have the estimate

$$-\sum_{m=k_0}^{k} \beta^{-m+1} \ln \omega^{-1}\big(\tfrac{1}{2C_1\beta^m}\big) \le -\frac{\beta}{\beta-1} \int_{\beta^{-k}}^{\beta^{-k_0+1}} \ln \omega^{-1}\big(\tfrac{t}{2C_1}\big)\, dt.$$

Thus,

$$\ln \|\varrho\|_{L^{q_k}(B(0,1))} \le C_5 + \delta \ln q_k + C_6 \Lambda(q_k).$$

If now $\beta^{-1} q_k \le q \le q_k$ and c is greater than C_5, C_6 and β, we obtain the resulting estimate $\ln \|\varrho\|_{L^q(B(0,1))} \le c + c\Lambda(cq) + \delta \ln q$. □

We now prove Theorem 3.6.

Proof. It suffices to show that given

$$B(x_0, R/2) \subset B(x_0, 4R) \subset \Omega$$

and $\delta > 0$, there exists a constant $c > 0$ for which

$$\ln \|\varrho\|_{L^q(B(x_0,R/2))} \le c + c\Lambda(cq) + \delta \ln q.$$

Let Φ be the diffeomorphism constructed before Proposition 2.3. According to Proposition 2.4, the function

$$\sigma(y) = \varrho(\Psi(y))|\det \Psi'(y)|$$

on the ball $B(y_0, 2R)$ centered at $y_0 = \Phi(x_0)$ is a solution to an equation whose coefficients satisfy the hypotheses of Lemma 3.8, that is, the matrix (q^{mk}) satisfies condition $\mathbf{H_a}$ and the drift coefficient h is bounded. In addition, Ψ is Lipschitz, the functions $\partial_{x_i}\Phi$ are Hölder of order $1 - d/p$ and $\omega(t) \ge Ct^{1-d/p}$. So there exists a number $N > 0$ such that

$$\|Q(y) - Q(z)\| \le N\omega(N\|y - z\|).$$

Observe that

$$\Lambda_N\big(\tfrac{1}{t}\big) = -\int_t^1 \ln N^{-1}\omega^{-1}(N^{-1}s)\, ds \le N\Lambda\big(\tfrac{N}{t}\big) + \ln N.$$

Therefore, for any closed ball $\overline{B}(y_0, R)$ in $B(y_0, 2R)$ and every $\delta > 0$ there exists $c > 0$ such that

$$\ln \|\sigma\|_{L^q(B(y_0,R))} \le c + c\ln N + cN\Lambda(cNq) + \delta \ln q.$$

Since Φ is a C^1-diffeomorphism, an analogous estimate holds for the original function ϱ. □

Corollary 3.9. *Under the hypotheses of Theorem* 3.6*, for every closed ball $B \subset \Omega$ the following assertions hold.*

(i) *If* $\lim_{t\to 0+} \omega(t)|\ln t| = 0$*, then*

$$\exp(\gamma_1|\varrho|^{\gamma_2}) \in L^1(B) \quad \textit{for all } \gamma_1, \gamma_2 > 0.$$

(ii) *If the function $\omega(t)|\ln t|$ is bounded on* $(0, 1]$*, then exist numbers $\gamma_1, \gamma_2 > 0$ such that*

$$\exp(\gamma_1|\varrho|^{\gamma_2}) \in L^1(B).$$

(iii) *If for some* $0<\beta<1$ *the function* $\omega(t)|\ln t|^\beta$ *is bounded on* $(0,1]$, *then for some* $\gamma>0$

$$\exp(\gamma\bigl|\ln(|\varrho|+1)\bigr|^{\frac{1}{1-\beta}})\in L^1(B).$$

Proof. Let us consider case (i), when $\omega(t)=o(|\ln t|^{-1})$. Let $\varepsilon>0$. There is $s_0\in(0,1)$ such that for all $s\in(0,s_0)$ one has $|\ln\omega^{-1}(s)|\le\varepsilon s^{-1}$, which follows from the estimate $\omega(t)\le\varepsilon|\ln t|^{-1}$. Therefore,

$$\Lambda(cq)\le C_1(\varepsilon)+\varepsilon\ln q,$$

so that

$$\int_B|\varrho|^q\,dx\le C_2^q q^{\varepsilon q}.$$

Let $q=\gamma_2 k$ and $\varepsilon\gamma_2=1/2$. We obtain

$$\int_B\frac{(|\varrho|^{\gamma_2})^k}{k!}\,dx\le\frac{C_3^k k^{k/2}}{k!}.$$

By Stirling's formula this yields convergence of the series

$$\int_B\sum_k\frac{(\gamma_1|\varrho|^{\gamma_2})^k}{k!}\,dx,$$

which implies the integrability of $\exp(\gamma_1|\varrho|^{\gamma_2})$. Assertion (ii) is proved similarly.

Let us prove assertion (iii). The boundedness of the function $\omega(t)|\ln t|^\beta$ implies the estimate $|\ln\omega^{-1}(s)|\le M_1 s^{-\frac{1}{\beta}}$ with some constant $M_1>0$. By Theorem 3.6, for all $q\ge1$ and some $M_2>0$ we obtain the estimate

$$\ln\|\varrho\|_{L^q(B)}\le M_2 q^{\frac{1-\beta}{\beta}}.$$

Chebyshev's inequality gives

$$\ln\bigl|\{x\in B\colon |\varrho(x)|>t\}\bigr|\le -q\ln t+M_2 q^{\frac{1}{\beta}}.$$

Let $q=C(\beta,M')|\ln t|^{\beta/(1-\beta)}$, where $C(\beta,M_2)=(\beta/M_2)^{\frac{\beta}{1-\beta}}$. There is a number t_0 such that for all $t\ge t_0$ one has $q\ge1$ and for some constant $M_3>0$ the inequality

$$\ln\bigl|\{x\in B\colon |\varrho(x)|>t\}\bigr|\le -M_3|\ln t|^{1/(1-\beta)}$$

holds. For any continuously differentiable function f with $f'>0$ we have

$$\int_B f(|\varrho(x)|)\,dx\le\int_0^{t_0}\bigl|\{x\in B\colon f(|\varrho(x)|)>t\}\bigr|\,dt+\int_{f(t_0)}^{+\infty}f'(s)\exp(-M_3|\ln s|^{1/(1-\beta)})\,ds.$$

It is readily verified that for $f(t)=\exp(\gamma|\ln(t+1)|^{1/(1-\beta)})$ with $\gamma<M_3$ the corresponding integral in the right-hand side is finite, which gives the desired integrability. $\square$

4. Existence and uniqueness of probability solutions

In this section we apply the results obtained above for constructing positive and probability solutions to the Kolmogorov equation.

Our next theorem generalizes assertion (i) in Theorem 2.4.1 in [4] to the case where the coefficients a^{ij} satisfy the Dini mean oscillation condition. It was assumed in the cited theorem that the functions a^{ij} belong locally to the Sobolev class $W^{p,1}$ with $p>d$.

Theorem 4.1. *Suppose that the coefficients a^{ij} and b^i are defined on all of $\mathbb{R}^d$, $b^i \in L^{d+}_{loc}$, and for every ball B we can find a number $\nu_B > 0$ and continuous nonnegative increasing function w_B on $[0,1]$ such that $w_B(0) = 0$, the integral $\displaystyle\int_0^1 \frac{w_B(t)}{t}\,dt$ converges and*

$$\nu_B \cdot I \leq A(x) \leq \nu_B^{-1} \cdot I, \quad \sup_{x\in B} \frac{1}{|B(x,r)|} \int_{B(x,r)} |a^{ij}(y) - a^{ij}_B(x,r)|\,dy \leq w_B(r), \quad r \in (0,1].$$

Then there exists a continuous and positive solution ϱ to equation (1.1) *on $\mathbb{R}^d$.*

Proof. Let $\zeta \in C_0^\infty(\mathbb{R}^d)$ have support in $B(0,1)$, $\zeta \geq 0$ and $\|\zeta\|_{L^1(\mathbb{R}^d)} = 1$. For every $k \in \mathbb{N}$ let ζ_k denote the function $k^d\zeta(kx)$ and let

$$a^{ij}_k = a^{ij} * \zeta_k, \quad b^i_k = b^i * \zeta_k.$$

The functions a^{ij}_k and b^i_k are infinitely differentiable and on every ball $B_R = B(0,R)$ we have

$$\nu_{B_{R+1}} \cdot I \leq A_k \leq \nu^{-1}_{B_{R+1}} \cdot I,$$
$$\|b_k\|_{L^{p_R}(B_R)} \leq \|b\|_{L^{p_R}(B_{R+1})}, \quad p_R = p(B_{R+1}).$$

Moreover,

$$\sup_{x\in B_R} \frac{1}{|B(x,r)|} \int_{B(x,r)} |a^{ij}_k(y) - a^{ij}_{k\,B}(x,r)|\,dy \leq w_{B_{R+1}}(r).$$

According to [4, Theorem 2.4.1], there exists a positive smooth solution ϱ_k to the equation

$$\partial_{x_i}\partial_{x_j}\big(a^{ij}_k\varrho_k\big) - \partial_{x_i}\big(b^i_k\varrho_k\big) = 0.$$

Multiplying by a constant, we can assume that $\varrho_k(0) = 1$. Applying the Harnack inequality from Theorem 3.5, for every ball we obtain the inequality

$$\sup_{B_R} \varrho_k(x) \leq C_R\varrho_k(0) = C_R,$$

where the constant C_R does not depend on k. By Theorem 3.5 the modulus of continuity of ϱ_k on every ball B_R is estimated by some function ω_R that does not depend on k, is continuous at zero and $\omega_R(0) = 0$. Thus, on every ball the sequence $\{\varrho_k\}$ is uniformly bounded and equicontinuous, hence, has a subsequence that converges to some nonnegative continuous function ϱ uniformly on every ball. In addition, on every ball B_R the sequence $\{a^{ij}_k\}$ converges uniformly to a^{ij} and the sequence $\{b^i_k\}$ converges to b^i in $L^{p_R}(B_R)$. Passing to the limit in the integral identities determining the solutions ϱ_k, we obtain an analogous integral equality for ϱ. Therefore, the function ϱ is a solution to equation (1.1). Since $\varrho(0) = 1$ and the function ϱ satisfies the Harnack inequality by Theorem 3.5, it is positive. □

As a corollary we obtain a generalization of the Hasminskii theorem, which follows from Theorem 4.1 and [4, Corollary 2.3.3].

Corollary 4.2. *If in addition to the hypotheses of Theorem* 4.1 *there is a function V of class $W^{d,2}_{loc}(\mathbb{R}^d)$ along with numbers $C > 0$ and $R > 0$ for which*

$$\lim_{|x|\to+\infty} V(x) = +\infty, \quad LV(x) \leq -C \quad \text{if} \quad |x| > R,$$

then there exists a continuous positive probability solution ϱ to equation (1.1) *on $\mathbb{R}^d$.*

Note that assertion (ii) in [4, Theorem 2.4.1] gives a nonzero nonnegative solution to equation (1.1) under the assumption that the matrix A is locally positive definite and bounded and the drift coefficient b is locally bounded. If, in addition, there is a Lyapunov function, then there exists a probability solution. In the results obtained above the condition on A is stronger and the condition on b is weaker. Let us show that the condition on A can be weakened to the inclusion in the class VMO if the drift b is dissipative (which is stronger than the existence of a Lyapunov function).

Theorem 4.3. *Suppose that the coefficients a^{ij} and b^i are defined on all of $\mathbb{R}^d$, $b^i \in L^{d+}_{loc}$, and there exist numbers $\nu > 0$ and $M > 0$ and an increasing continuous function ω on $[0, +\infty)$ such that $\omega(0) = 0$ and*

$$\nu \cdot I \le A(x) \le \nu^{-1} \cdot I, \quad \sup_{z \in \mathbb{R}^d} r^{-2d} \int_{B(z,r)} \int_{B(z,r)} |a^{ij}(x) - a^{ij}(y)| \, dxdy \le \omega(r).$$

Suppose also that

$$\lim_{|x| \to \infty} \langle b(x), x \rangle = -\infty.$$

Then there exists a probability solution ϱ to equation (1.1) on $\mathbb{R}^d$.

Proof. Let $\zeta \in C_0^\infty(\mathbb{R}^d)$ have support in $B(0,1)$, $\zeta \ge 0$ and $\|\zeta\|_{L^1(\mathbb{R}^d)} = 1$. For each $k \in \mathbb{N}$ let $\zeta_k(x) = k^d \zeta(kx)$ and $a^{ij}_k = a^{ij} * \zeta_k$. The functions a^{ij}_k are infinitely differentiable and

$$\nu \cdot I \le A_k(x) \le \nu^{-1} \cdot I, \quad \sup_{z \in \mathbb{R}^d} r^{-2d} \int_{B(z,r)} \int_{B(z,r)} |a^{ij}_k(x) - a^{ij}_k(y)| \, dxdy \le \omega(r).$$

In addition, for the function $V(x) = |x|^2/2$ we have

$$\mathrm{tr}\big(A_k(x) D^2 V(x)\big) + \langle b(x), \nabla V(x) \rangle \le \nu^{-1} d + \langle b(x), x \rangle \to -\infty.$$

Therefore, for every k there exists a probability solution ϱ_k of the equation

$$\partial_{x_i} \partial_{x_j} \big(a^{ij}_k \varrho_k\big) - \partial_{x_i} \big(b^i \varrho_k\big) = 0.$$

According to [7, Theorema 2.1], for every ball B and every $s > 1$ there exists a number $C(s, B) > 0$ independent of k such that $\|\varrho_k\|_{L^s(B)} \le C(s, B)$. In addition, according to [4, Theorem 2.3.2] one has

$$\int_{\mathbb{R}^d} |\langle b(x), x \rangle| \varrho_k(x) \, dx \le d\nu^{-1} + 2 \int_{|x| \le R} |\langle b(x), x \rangle| \varrho_k(x) \, dx,$$

where $R > 0$ is taken such that $\langle b(x), x \rangle < 0$ whenever $|x| > R$. Observe that

$$\int_{|x| \le R} |\langle b(x), x \rangle| \varrho_k(x) \, dx \le C(p', B_R) \|b\|_{L^p(B_R)}, \quad p' = p/(p-1), \quad B_R = B(0, R).$$

Therefore, the sequence of measures $\varrho_k \, dx$ contains a subsequence converging weakly to some probability measure μ. Since $\|\varrho_k\|_{L^s(B)} \le C(s, B)$, we can assume that this subsequence converges weakly in $L^{p'}(B)$ for every ball B. Hence the measure μ has a density $\varrho \in L^{p'}_{loc}(\mathbb{R}^d)$ and for every function $\varphi \in C_0^\infty(\mathbb{R}^d)$ we can pass to the limit as $k \to \infty$ in the equality

$$\int_{\mathbb{R}^d} \big[a^{ij}_k \partial_{x_i} \partial_{x_j} \varphi + b^i \partial_{x_i} \varphi\big] \varrho_k(x) \, dx = 0.$$

The function ϱ is a probability solution of equation (1.1). □

Zvonkin's transform enables us to obtain the following modification of the Hasminskii theorem generalizing [30, Theorem 4.10(iii)] to the matrix A of class VMO.

Corollary 4.4. *Suppose that the coefficients a^{ij} and b^i are defined on all of $\mathbb{R}^d$ and we can find a number $\nu > 0$ and an increasing continuous function ω on $[0, +\infty)$ such that $\omega(0) = 0$ and*

$$\nu \cdot I \le A(x) \le \nu^{-1} \cdot I, \quad \sup_{z \in \mathbb{R}^d} r^{-2d} \int_{B(z,r)} \int_{B(z,r)} |a^{ij}(x) - a^{ij}(y)| \, dxdy \le \omega(r).$$

Let also $b^i = b^i_1 + b^i_2$, where $b_2 \in L^p(\mathbb{R}^d)$ with $p > d$ and

$$\lim_{|x| \to \infty} \langle b_1(x), x \rangle \le C_1 - C_2 |x|^{\kappa+1}, \quad |b_1(x)| \le C_3 (1 + |x|^\kappa),$$

for some numbers $C_1, C_2, C_3 > 0$ and $\kappa > 0$. Then there exists a probability solution ϱ to equation (1.1) on $\mathbb{R}^d$.

Proof. Applying Proposition 2.1, for every $\delta > 0$ we construct a mapping $u = (u^1, \dots, u^d)$ such that u^k is a solution to the equation

$$\mathrm{tr}(AD^2u^k) + \langle b_2, \nabla u^k\rangle - \lambda u^k = -b_2^k,$$

$u^k \in L^\infty(\mathbb{R}^d)$ and $\sup_x |\nabla u^k(x)| < \delta$. According to Proposition 2.3, for $\delta > 0$ sufficiently small the mapping $\Phi(x) = x + u(x)$ is a diffeomorphism of $\mathbb{R}^d$. Let $\Psi = \Phi^{-1}$. It is readily seen that $|\Psi(y)| \to \infty$ as $|y| \to \infty$. Let us consider the equation

$$\partial_{y_k}\partial_{y_m}\bigl(q^{km}\sigma\bigr) - \partial_{y_k}\bigl(h^k\sigma\bigr) = 0, \tag{4.1}$$

where

$$q^{km}(y) = a^{ij}(\Psi(y))\partial_{x_i}\Phi^k(\Psi(y))\partial_{x_j}\Phi^m(\Psi(y)),$$
$$h(y) = \lambda u(\Psi(y)) + b_1^i(\Psi(y))\partial_{x_i}\Phi(\Psi(y)).$$

Since $\Phi(x) = x + u(x)$, we have

$$\langle h(y), y\rangle(\Phi(x)) = \lambda\langle u(x), x + u(x)\rangle + \langle b_1(x), x + u(x)\rangle + \langle b_1^i(x)\partial_{x_i}u(x), x + u(x)\rangle.$$

Let $|u(x)| \le M$ and $|\nabla u(x)| \le \delta$. Then

$$\langle h(y), y\rangle(\Phi(x)) \le \lambda M^2 + \lambda M|x| + M|b_1(x)| + \delta|b_1(x)||x| + \delta M|b_1(x)| + \langle b_1(x), x\rangle.$$

Using our condition on b_1, we can estimate the right side by

$$C_1 + \lambda M^2 + \lambda M|x| + (1+\delta)MC_3(1 + |x|^\kappa) + \delta C_3(1 + |x|^{\kappa+1}) - C_2|x|^{\kappa+1}.$$

Let $\delta C_3 < C_2$. Then $\langle h(y), y\rangle \to -\infty$ as $|y| \to \infty$. Therefore, the hypotheses of Theorem 4.3 are fulfilled and there exists a probability solution σ of equation (4.1). It is clear that $\varrho(x) = \sigma(\Phi(x))|\mathrm{det}\Phi'(x)|$ is a probability solution to equation (1.1). $\square$

In connection with the existence theorems proved above we discuss a probabilistic interpretation of probability solutions of the Kolmogorov equation. Let a^{ij} and b^i be Borel functions on $\mathbb{R}^d$ and let μ be a Borel probability measure satisfying the stationary Kolmogorov equation

$$\partial_{x_i}\partial_{x_j}\bigl(a^{ij}\mu\bigr) - \partial_{x_i}\bigl(b^i\mu\bigr) = 0,$$

understood in the sense of the integral equality

$$\int_{\mathbb{R}^d}\bigl[a^{ij}\partial_{x_i}\partial_{x_j}\varphi + b^i\partial_{x_i}\varphi\bigr]\,d\mu = 0 \quad \forall\varphi \in C_0^\infty(\mathbb{R}^d).$$

Theorem 4.5. *Suppose that $a^{ij}, b^i \in L^1_{loc}(\mu)$ and*

$$\int_{\mathbb{R}^d}\frac{\|A(x)\| + |\langle b(x), x\rangle|}{1 + |x|^2}\,\mu(dx) < \infty.$$

Then for every $T > 0$ one can find a probability space $(Q, \mathcal{F}, P)$, a filtration $\mathcal{F}_t$, and a continuous random process ξ_t and a Wiener process w_t adapted to the filtration $\mathcal{F}_t$ such that

$$d\xi_t = b(\xi_t)\,dt + \sqrt{2A(\xi_t)}dw_t$$

on $[0, T]$ and the distribution of the random variable $\omega \to \xi_t(\omega)$ equals μ for all t. Moreover, if it is known that for every probability measure σ on $\mathbb{R}^d$ there exists a unique weak solution ξ_t^σ of the indicated stochastic equation on $[0, +\infty)$ with initial distribution σ and P_σ is a distribution of ξ^σ, then the measure μ is invariant for the semigroup

$$T_tf(x) = \int_{C([0,+\infty),\mathbb{R}^d)} f(\xi_t)\,P_{\delta_x}(d\xi)$$

on the space of bounded continuous functions, that is, for every bounded continuous function f and every $t \ge 0$ we have the identity

$$\int_{\mathbb{R}^d} T_tf(x)\mu(dx) = \int_{\mathbb{R}^d} f(x)\mu(dx).$$

Proof. By the Ambrosio-Figalli-Trevisan superposition principle [9] there exists a Borel probability measure P_μ on $C([0,T],\mathbb{R}^d)$ satisfying the following conditions: (i) $P_\mu(\{\xi\colon\ \xi_t\in Q\})=\mu(Q)$ for every Borel set Q and every $t\in[0,T]$, (ii) for every function $f\in C_0^\infty(\mathbb{R}^d)$ the mapping

$$(\xi,t)\to f(\xi_t)-f(\xi_0)-\int_0^t\bigl[a^{ij}(\xi_s)\partial_{x_i}\partial_{x_j}f(\xi_s)+b^i(\xi_s)\partial_{x_i}f(\xi_s)\bigr]\,ds$$

is a martingale with respect to P_μ and the filtration $\sigma(\xi_s, s\le t)$. According to [17, Proposition 2.1, Ch. IV], one can find a probability space $(Q,\mathcal{F},P)$, a filtration $\mathcal{F}_t$, and a continuous random process ξ_t and a Wiener process w_t adapted to $\mathcal{F}_t$ such that

$$d\xi_t=b(\xi_t)\,dt+\sqrt{2A(\xi_t)}dw_t$$

on $[0,T]$ and the distribution of ξ_t coincides with P_μ, in particular, the one-dimensional distribution of the process ξ_t does not depend on t and equals μ.

Assume now that the given stochastic equation has a unique weak solution and P_{δ_x} is the distribution of the solution with initial condition δ_x. We observe that due to our assumption about the uniqueness of solutions the mapping $\sigma\mapsto P_\sigma$ is Borel measurable when the spaces of measures are equipped with their weak topologies (or with metrics generating them). Indeed, even without any assumptions about uniqueness, the set of all pairs (P,σ) of probability measures for which P is a measure on the space $C([0,+\infty),\mathbb{R}^d)$ of continuous paths and σ is a measure on $\mathbb{R}^d$ such that σ is the image P under the mapping $x\mapsto x(0)$ and the process

$$f(x(t))-f(x(0))-\int_0^t Lf(x(s))\,ds$$

is a martingale with respect to the measure P for all functions $f\in C_b^2(\mathbb{R}^d)$, is a Borel set in the product $\mathcal{P}(C([0,+\infty),\mathbb{R}^d))\times\mathcal{P}(\mathbb{R}^d)$. This is seen from the fact that this set can be described by a countable number of equalities of the form

$$\int\psi_i\,dP=0$$

with some countable collection of bounded Borel functions ψ_i on the path space along with the equality of the measure σ to the image of P with respect to the operator $x\mapsto x(0)$. A countable collection $\{\psi_i\}$ arises because for verification of the martingale property we can use a countable collection of functions f, moreover, the martingale property itself can be verified only for rational times, and the comparison of the corresponding conditional expectations also employs a countable collection of functions.

According to [17, Theorem 5.3, Ch. IV] measures P_{δ_x} form a Markov family and T_tf is a semigroup the space of bounded continuous functions. Note that for every cylindrical set C one has

$$P_\mu(C)=\int_{\mathbb{R}^d}P_{\delta_x}(C)\,\mu(dx),$$

which implies the equality

$$\int_{C([0,+\infty),\mathbb{R}^d)}f(\xi_t)\,P_\mu(d\xi)=\int_{\mathbb{R}^d}\int_{C([0,+\infty),\mathbb{R}^d)}f(\xi_t)\,P_x(d\xi)\,\mu(dx)$$

for every bounded continuous function f. It remains to observe that the left side of the last equality is the integral of f against the measure μ and the right side is the integral of T_tf against the measure μ. $\square$

In the general case the stationary Kolmogorov equation can have several different probability solutions (see [4, Chapter 4]). Sufficient conditions for uniqueness of probability solutions have been obtained in [8] in the case where the matrix A satisfies the classical Dini condition, but by Theorem 3.5 and the results in [11] their justification extends without changes to the case of the matrix A satisfying the Dini mean oscillation condition. Thus, the following assertion is true.

Theorem 4.6. *Let a^{ij} and b^i be defined on all of $\mathbb{R}^d$, $b^i \in L^{d+}_{loc}$, and for every ball B one can find a number $\nu_B > 0$ and a continuous nonnegative increasing function w_B on $[0,1]$ such that $w_B(0) = 0$, the integral $\int_0^1 \frac{w_B(t)}{t}\,dt$ converges, and*

$$\nu_B \cdot I \le A(x) \le \nu_B^{-1} \cdot I, \quad \sup_{x\in B} \frac{1}{|B(x,r)|}\int_{B(x,r)} |a^{ij}(y) - a^{ij}_B(x,r)|\,dy \le w_B(r), \quad r \in (0,1].$$

Suppose that ϱ is a probability solution to equation (1.1) such that at least one of following conditions is fulfilled:

(i) $(1+|x|)^{-2}a^{ij}, (1+|x|)^{-1}b^i \in L^1(\varrho\,dx)$,

(ii) *there exists $V \in C^2(\mathbb{R}^d)$ with $\lim\limits_{|x|\to\infty} V(x) = +\infty$ and $LV \le C_1 + C_2 V$.*

Then ϱ is a unique probability solution.

Applying Zvonkin's transform, we obtain the following sufficient condition for uniqueness that agrees with Corollary 4.4.

Corollary 4.7. *Suppose that the coefficients a^{ij} and b^i are defined on all of $\mathbb{R}^d$ and we can find a number $\nu > 0$ and an increasing continuous function ω on $[0,+\infty)$ such that*

$$\nu \cdot I \le A(x) \le \nu^{-1} \cdot I, \quad \sup_{x\in\mathbb{R}^d} \frac{1}{|B(x,r)|}\int_{B(x,r)} |a^{ij}(y) - a^{ij}_B(x,r)|\,dy \le \omega(r),$$

$\omega(0) = 0$ and the integral $\int_0^1 \frac{\omega(r)}{r}\,dr$ converges.

Let also $b^i = b^i_1 + b^i_2$, where $b_2 \in L^p(\mathbb{R}^d)$ with $p > d$ and

$$\lim_{|x|\to\infty} \langle b_1(x), x\rangle \le C_1 - C_2|x|^{\kappa+1}, \quad |b_1(x)| \le C_3(1+|x|^\kappa),$$

for some $C_1, C_2, C_3 > 0$ and $\kappa > 0$. Then there exists a unique probability solution ϱ to equation (1.1) on $\mathbb{R}^d$.

Proof. The existence is proved in Corollary 4.4. After the change of coordinates as in the proof of Corollary 4.4 the coefficients of the new equation satisfy condition (ii) in Theorem 4.6, hence a probability solution is unique. ☐

We do not know whether the continuity of the matrix A is sufficient for the uniqueness of a probability solution.

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References

[1] Bauman P. Positive solutions of elliptic equations in nondivergent form and their adjoints. Ark. Mat. 1984. V. 22. P. 153–173.

[2] Bauman P. Equivalence of the Green's functions for diffusion operators in R^n: a counterexample. Proc. Amer. Math. Soc. 1984. V. 91. P. 64–68.

[3] Bogachev V.I., Krylov N.V., Röckner M. On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions. Commun. Partial Differ. Equ. 2001. V. 26, N 11-12. P. 2037–2080.

[4] Bogachev V.I., Krylov N.V., Röckner M., Shaposhnikov S.V. Fokker–Planck–Kolmogorov equations, Amer. Math. Soc., Providence, Rhode Island, 2015.

[5] Bogachev V.I., Röckner M. A generalization of Khasminskii's theorem on the existence of invariant measures for locally integrable drifts. Teory of Probab. Appl. 2000. V. 45, N 3. P. 363–378.

[6] Bogachev V.I., Röckner M., Shaposhnikov S.V. On positive and probability solutions of the stationary Fokker–Planck–Kolmogorov equation. Doklady Math. 2012. V. 85, N 3. P. 350–354.

[7] Bogachev V.I., Shaposhnikov S.V. Integrability and continuity of solutions to double divergence form equations. Annali di Matematica. 2017. V. 196. P. 1609–1635.

[8] Bogachev V.I., Shaposhnikov S.V. Uniqueness of probability solutions to the Kolmogorov equation with a diffusion matrix satisfying the Dini condition. Doklady Math. 2021.

[9] Bogachev V.I., Röckner M., Shaposhnikov S.V. On the Ambrosio–Figalli–Trevisan superposition principle for probability solutions to Fokker–Planck–Kolmogorov equations. J. Dynamics Differ. Equ. 2021. V. 33. P. 715–739.

[10] Dong H., Kim S. On C^1, C^2, and weak type-$(1,1)$ estimates for linear elliptic operators. Comm. Partial Differ. Equ. 2017. V. 42, N 3. P. 417–435.

[11] Dong H., Escauriaza L., Kim S. On C^1, C^2, and weak type-$(1,1)$ estimates for linear elliptic operators: part II. Math. Ann. 2018. B. 370. S. 447–489.

[12] Escauriaza L. Bounds for the fundamental solution of elliptic and parabolic equations in nondivergence form. Comm. Partial Differ. Equ. 2000. V. 25, N 5-6. P. 821–845.

[13] Flandoli F., Issoglio E., Russo F. Multidimensional stochastic differential equations with distributional drift. Trans. Amer. Math. Soc. 2017. V. 369, N 3. P. 1665–1688.

[14] Gilbarg D., Trudinger N.S. Elliptic partial differential equations of second order. Springer-Verlag, Berlin – New York, 1977.

[15] Khasminskii R.Z. Stochastic stability of differential equations. 2nd ed. Springer, Heidelberg, 2012.

[16] Hwang S., Kim S. Green's function for second order elliptic equations in non-divergence form. Potential Analysis. 2020. V. 52. P. 27–39.

[17] Ikeda N., Watanabe S. Stochastic differential equations and diffusion processes. 2nd ed. North-Holland, Amsterdam, 1989.

[18] Krasovitskii T.I. Degenerate elliptic equations and nonuniqueness of solutions to the Kolmogorov equation. Doklady Math. 2019. V. 100, N 1. P. 354–357.

[19] Krylov N.V. Parabolic and elliptic equations with VMO coefficients. Commun. Partial Differ. Equ. 2007. V. 32. P. 453–475.

[20] Krylov N.V. Lectures on elliptic and parabolic equations in Sobolev spaces. Amer. Math. Soc., Providence, Rhode Island, 2008.

[21] Kruzkov S.N. A priori bounds and some properties of solutions of elliptic and parabolic equations. Matem. Sb. 1964. V. 65, N 4. P. 522–570 (1964) (in Russian); English transl.: Ten papers on differential equations and functional analysis, Am. Math. Soc. Trans. Ser. 2. 68, 169–220. American Mathematical Society, Providence, Rhode Island, 1968.

[22] Lee H., Trutnau G. Existence and regularity of infinitesimally invariant measures, transition functions and time-homogeneous Itô-SDEs. J. Evol. Equ. 2021. V. 21, N 1. P. 601–623.

[23] Leitao R., Pimentel E.A., Santos M.S. Geometric regularity for elliptic equations in double-divergence form. Analysis and PDE. 2020. V. 13, N 4. P. 1129–1144.

[24] Mamedov F.I. On the Harnack inequality for the equation formally adjoint to a linear elliptic differential equation. Sib. Matem. Zh. 1992. V. 33, N 5. P. 100–106 (in Russian); English transl.: Sib. Math. J. 1992. V. 33, N 5. P. 835–841.

[25] Oleinik O.A., Kruzkov S.N. Quasi–linear second–order parabolic equations with many independent variables. Uspehi Mat. Nauk. 1961. V. 16, N 5. P. 115–155 (in Russian); English transl.: Russ. Math. Surv. 1961. V. 16, N 5. P. 105–146.

[26] Röckner M., Zhang X. Well-posedness of distribution dependent SDEs with singular drifts. Bernoulli. 2021. V. 27, N 2. P. 1131–1158.

[27] Sjögren P. On the adjoint of an elliptic linear differential operator and its potential theory. Ark. Mat. 1973. V. 11. P. 153–165.

[28] Sjögren P. Harmonic spaces associated with adjoints of linear elliptic operators. Ann. Inst. Fourier. 1975. V. 25, N 3-4. P. 509–518.

[29] Trudinger N.S. Local Estimates for Subsolutions and Supersolutions of General Second Order Elliptic Quasilinear Equations. Inventiones math. 1980. V. 61 P. 67–79.

[30] Zhang X., Zhao G. Singular Brownian diffusion processes. Comm. Math. Statist. 2018. V. 6. P. 533–581.

[31] Zvonkin A.K. A transformation of the phase space of a diffusion process that will remove the drift. Mat. Sb. 1974. V. 93. P. 129–149 (in Russian);