AVERAGING PRINCIPLE FOR STOCHASTIC COMPLEX GINZBURG-LANDAU EQUATIONS

MENGYU CHENG, ZHENXIN LIU, AND MICHAEL RÖCKNER

Abstract. Averaging principle is an effective method for investigating dynamical systems with highly oscillating components. In this paper, we study three types of averaging principle for stochastic complex Ginzburg-Landau equations. Firstly, we prove that the solution of the original equation converges to that of the averaged equation on finite intervals as the time scale ε goes to zero when the initial data are the same. Secondly, we show that there exists a unique recurrent solution (in particular, periodic, almost periodic, almost automorphic, etc.) to the original equation in a neighborhood of the stationary solution of averaged equation when the time scale is small. Finally, we establish the global averaging principle in weak sense, i.e. we show that the attractor of original system tends to that of the averaged equation in probability measure space as ε goes to zero.

1. Introduction

A highly oscillating system may be "averaged" under some suitable conditions, and the resulting averaged system is easier to analysis and governs the evolution of the original system over long time scales. This is the basic idea of averaging principle. According to the connotation of approximations, there are three types of interpretation for averaging principle, i.e. the so-called *first Bogolyubov theorem*, *second Bogolyubov theorem* and *global averaging principle*.

More specifically, consider the following systems in $\mathbb{R}^n, n \in \mathbb{N}$

$$\dot{X}_\varepsilon(t) = F\left(\frac{t}{\varepsilon}, X_\varepsilon(t)\right) \tag{1.1}$$

and

$$\dot{\bar{X}}(t) = \bar{F}\left(\bar{X}(t)\right) \tag{1.2}$$

for small parameter $0 < \varepsilon \ll 1$, where $F \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $\bar{F}(x) := \lim_{T\to\infty} \frac{1}{T}\int_0^T F(t,x)\mathrm{d}t$ uniformly with respect to (in short, w.r.t.) x on any bounded subset of $\mathbb{R}^n$. Here f is called a *KBM-vector field* (KBM stands for Krylov, Bogolyubov and Mitropolsky); see e.g. [33].

The first Bogolyubov theorem requires that the solution of the original equation (1.1) converges, as $\varepsilon \to 0$, to that of the averaged equation (1.2) on finite time intervals when $X_\varepsilon(0) = \bar{X}(0)$. And the second Bogolyubov theorem requests that the approximation be valid on the entire real axis, that is to say, the stationary solution of (1.2) approximates the periodic solution of (1.1). So sometimes it is called *theorem for periodic solution by averaging.* In addition, the global averaging principle describes that the attractor of (1.2) approximates that of (1.1).

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In this paper, we investigate the above three types of averaging principle for the following stochastic complex Ginzburg-Landau (in short, CGL) equation on the d-torus $\mathbb{T}^d, d = 1, 2, 3$

$$\mathrm{d}u_\varepsilon(t) = \left[(1+i\alpha)\Delta u_\varepsilon(t) - (1+i\beta)|u_\varepsilon(t)|^2 u_\varepsilon(t) + f\left(\frac{t}{\varepsilon}, u_\varepsilon(t)\right)\right]\mathrm{d}t + g\left(\frac{t}{\varepsilon}, u_\varepsilon(t)\right)\mathrm{d}W(t), \tag{1.3}$$

where $\alpha \in \mathbb{R}$, $|\beta| \leq \sqrt{3}$, ε is a small parameter and f, g satisfy some suitable conditions.

The complex Ginzburg-Landau equation arises in physics. Therefore, it has very rich physical backgrounds and connotations. It can be used to describe problems of Bandard convection, Taylor-Couette flow, plane Poiseuille flow and chemical turbulence. It has also been applied in superfluidity and superconductivity theory (see e.g. [1] for more information).

As we know, some perturbations may be neglected in the derivation of the ideal model. When considering the perturbation of each microscopic unit of the model, which will lead to a very large complex system, people usually represent micro effects by random perturbations in the dynamics of macro observables. Thus, it is more realistic to consider stochastic CGL equations.

The theory of averaging has been applied in many fields, such as celestial mechanics, oscillation theory and radiophysics. And the idea of averaging dates from the perturbation theory which was developed by Clairaut, Laplace and Lagrange in the 18th century. Then fairly rigorous averaging method for nonlinear oscillations was established by Krylov, Bogolyubov and Mitropolsky [3, 26], which is called the Krylov-Bogolyubov method nowadays. After that, there are vast amount of works on averaging for finite and infinite dimensional deterministic systems which we will not mention here.

Meanwhile, Stratonovich firstly proposed the stochastic averaging method on the basis of physical considerations, which was later proved mathematically by Khasminskii. After that, extensive works concerning averaging principle for finite and infinite dimensional stochastic differential equations were conducted, following Khasminskii's pioneering work [24]; see e.g. [2, 4–6, 13–16, 29, 31, 32, 40–45] and references therein. Note that the above existing results focus on the first Bogolyubov theorem.

To the best of our knowledge, there are few works on averaging principle for stochastic CGL equations. As discussed in [18, 20, 27], averaging method was developed to describe the behavior of solutions for small oscillations in damped/driven Hamiltonian systems.

From the perspective of theoretical and practical value, we establish three types of averaging principle for the stochastic CGL equations with highly oscillating components in this paper. Firstly, under some suitable conditions, employing the classical technique of truncation which is used in [4–6, 29], we show that

$$\lim_{\varepsilon\to 0}\mathbb{E}\sup_{s\leq t\leq s+T}\|Y_\varepsilon(t) - \bar{Y}(t)\|^2 = 0$$

for all $s \in \mathbb{R}$ and $T > 0$ provided $\lim\limits_{\varepsilon\to 0}\mathbb{E}\|\zeta_s^\varepsilon - \zeta_s\|^2 = 0$, where Y_ε is the solution of (1.3) with the initial value $Y_\varepsilon(s) = \zeta_s^\varepsilon$ and $\bar{Y}$ is the solution of the following averaged equation

$$\mathrm{d}u(t) = \left[(1+i\alpha)\Delta u(t) - (1+i\beta)|u(t)|^2 u(t) + \bar{f}(u(t))\right]\mathrm{d}t + \bar{g}\left(u(t)\right)\mathrm{d}W(t) \tag{1.4}$$

with initial condition $\bar{Y}(s) = \zeta_s$, where $\bar{f}$ and $\bar{g}$ satisfy

$$\left\|\frac{1}{T}\int_t^{t+T} f(s,x)\mathrm{d}s - \bar{f}(x)\right\| \leq \delta_f(T)\left(1 + \|x\|\right)$$

and

$$\frac{1}{T}\int_t^{t+T}\|g(s,x) - \bar{g}(x)\|^2\,\mathrm{d}s \leq \delta_g(T)\left(1 + \|x\|^2\right)$$

for all $t \in \mathbb{R}$. Here $\delta_f(T) \to 0$ and $\delta_g(T) \to 0$ as $T \to \infty$. We write $L_2(U, L^2(\mathbb{T}^d))$ to mean the space of Hilbert-Schmidt operators from Hilbert space U into $L^2(\mathbb{T}^d)$. Notice that this is the first Bogolyubov theorem (see Theorem 3.5).

Note that the second Bogolyubov theorem is not an initial-value problem, and the recurrent solution that we consider is more general than the classical second Bogolyubov theorem which only treats the periodic case. Little work has been done on the second Bogolyubov theorem for stochastic differential equations. For this purpose, recall that the second Bogolyubov theorem for stochastic ordinary differential equations was studied in [8]. Comparing to [23], we consider stochastic CGL equations that admit polynomial growth terms. Despite that a general second Bogolyubov theorem was established in [10], it cannot be applied to stochastic CGL equations. Indeed, we cannot use the method in [10] to obtain the tightness of measures for stochastic CGL equations.

With the help of Theorem 3.5, we establish the second Bogolyubov theorem for stochastic CGL equations. To be specific, we firstly show that there exists a unique $L^2(\Omega, \mathbb{P}; L^2(\mathbb{T}^d))$-bounded solution $u_\varepsilon(t), t \in \mathbb{R}$ of (1.3) which inherits the recurrent properties (in particular, periodic, quasi-periodic, almost periodic, almost automorphic, Birkhoff recurrent, Levitan almost periodic, almost recurrent, pseudo-periodic, pseudo-recurrent, Poisson stable) of the coefficients in distribution sense for each $0 < \varepsilon \leq 1$. This result is interesting on its own rights. Because recurrence is an important concept in dynamical systems, which roughly means that a motion returns infinitely often to any small neighborhood of the initial position. And we also prove that the $L^2(\Omega, \mathbb{P}; L^2(\mathbb{T}^d))$-bounded solution $u_\varepsilon(t), t \in \mathbb{R}$ to (1.3) is globally asymptotically stable in square-mean sense. Then we obtain

$$\lim_{\varepsilon \to 0} \sup_{s \leq t \leq s+T} W_2(\mathcal{L}(u_\varepsilon(t)), \mathcal{L}(\bar{u}(0))) = 0 \tag{1.5}$$

for all $s \in \mathbb{R}$ and $T > 0$ (see Theorem 4.11 and Corollary 4.12), where $\bar{u}$ is the unique stationary solution of (1.4) and $\mathcal{L}(\bullet)$ is the distribution of $\bullet$. Here W_2 is the Wasserstein distance.

The global averaging principle was conducted for deterministic systems; see e.g. [19, 21, 22, 46] among others. But to our knowledge, there is only one work on global averaging principle for stochastic equations, i.e. [10]. However, the results in [10] cannot be applied to stochastic CGL equations. So, another major result in present paper is to establish the global averaging principle in weak sense for stochastic CGL equations.

We set $Pr_2(L^2(\mathbb{T}^d)) := \left\{\mu \in Pr(L^2(\mathbb{T}^d)) : \int_{L^2(\mathbb{T}^d)} \|z\|^2 \mu(\mathrm{d}z) < \infty\right\}$, where $Pr(L^2(\mathbb{T}^d))$ is the space of probability measures on $L^2(\mathbb{T}^d)$. With the transition probability $P_{F_\varepsilon}(s, x, t, \mathrm{d}y) := \mathbb{P} \circ (u_\varepsilon(t, s, x))^{-1}(\mathrm{d}y)$ to (1.3), we can associate a mapping $P_\varepsilon^*(t, F, \cdot) : Pr(L^2(\mathbb{T}^d)) \to Pr(L^2(\mathbb{T}^d))$ defined by $P_\varepsilon^*(t, F, \mu)(B) := \int_{L^2(\mathbb{T}^d)} P_{F_\varepsilon}(0, x, t, B)\mu(\mathrm{d}x)$ for all $\mu \in Pr(L^2(\mathbb{T}^d))$, $B \in \mathcal{B}(L^2(\mathbb{T}^d))$, $F_\varepsilon := (f_\varepsilon, g_\varepsilon)$ and $0 < \varepsilon \leq 1$. Then we prove that P_ε^* is a cocycle over $(\mathcal{H}(F_\varepsilon), \mathbb{R}, \sigma)$ with fiber $Pr_2(L^2(\mathbb{T}^d))$ for any $0 < \varepsilon \leq 1$, where $(\mathcal{H}(F), \mathbb{R}, \sigma)$ is the shift dynamical system (see Appendix A.3 for details). Finally, we show that P_ε^* has a uniform attractor $\mathcal{A}^\varepsilon$ in $Pr_2(L^2(\mathbb{T}^d))$ for any $0 < \varepsilon \leq 1$, and

$$\lim_{\varepsilon \to 0} \operatorname{dist}_{Pr_2(L^2(\mathbb{T}^d))} \left(\mathcal{A}^\varepsilon, \bar{\mathcal{A}}\right) = 0$$

provided $\mathcal{H}(F)$ is compact (see Theorem 5.6), where $\operatorname{dist}_{Pr_2(L^2(\mathbb{T}^d))}$ is the Hausdorff semi-metric and $\bar{\mathcal{A}} := \{\mathcal{L}(\bar{u}(0))\}$ is the attractor of $\bar{P}^*$ to the averaged equation (1.4). Note that $\mathcal{H}(F)$ is compact provided F is Birkhoff recurrent.

The remainder of this paper is organized as follows. In the next section, we introduce some notations, definitions and facts concerning dynamical systems. In section 3, we study the first Bogolyubov theorem for stochastic CGL equations. In the fourth section, firstly we prove that there exists a unique $L^2(\Omega, \mathbb{P}; L^2(\mathbb{T}^d))$-bounded solution which possesses the same recurrent properties as the coefficients in distribution sense and this bounded solution is globally asymptotically stable in square-mean sense. Then we establish the second Bogolyubov theorem for stochastic CGL equations. In section 5, we prove the global averaging principle for stochastic CGL equations. In the Appendix at the end, we recall Poisson stable (or recurrent) functions, Shcherbakovs comparability method by character of recurrence and some spaces.

2. Preliminaries

We write $L^2(\mathbb{T}^d) := L^2(\mathbb{T}^d;\mathbb{C})$ to mean the space of all Lebesgue square integrable complex-valued functions on $\mathbb{T}^d, d = 1,2,3$. The inner product on $L^2(\mathbb{T}^d;\mathbb{C})$ is $\langle u, v\rangle := \langle u, v\rangle_{L^2(\mathbb{T}^d;\mathbb{C})} = \mathcal{R}\int_{\mathbb{T}^d} u(\xi)\bar{v}(\xi)\mathrm{d}\xi$ and norm is $\|u\| = \langle u, u\rangle^{\frac{1}{2}}$. Here $\bar{v}$ is the conjugate of v and $\mathcal{R}v$ is the real part of v. Denote by $H^1 := H^{1,2}(\mathbb{T}^d;\mathbb{C})$ the Sobolev spaces of complex-valued functions on $\mathbb{T}^d$. Let λ_* be the first eigenvalue of $-\Delta$ on $L^2(\mathbb{T}^d)$.

Let H be a separable Hilbert space with the norm $\|\cdot\|_H$. And we will omit the index H if it does not cause confusion. Denote by $C_b(\mathbb{R}, H)$ the Banach space of all continuous and bounded mappings $\varphi : \mathbb{R} \to H$ equipped with the norm $\|\varphi\|_\infty := \sup\{\|\varphi(t)\| : t \in \mathbb{R}\}$.

Remark 2.1. If $f \in C_b(\mathbb{R}, H)$ and $\tilde{f} \in \mathcal{H}(f)$, then $\|\tilde{f}\|_\infty \le \|f\|_\infty$, where $\mathcal{H}(f) := \overline{\{f^\tau : \tau \in \mathbb{R}\}}$ and $f^\tau(t) = f(t+\tau)$ for all $t \in \mathbb{R}$. See Appendix A.1 for more details about $\mathcal{H}(f)$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. For any $p \ge 2$, we write $L^p(\Omega, \mathbb{P}; H)$ to mean the space of H-valued random variables X such that $\mathbb{E}\|X\|^p := \int_\Omega \|X\|^p \mathrm{d}\mathbb{P} < \infty$. Then $L^p(\Omega, \mathbb{P}; H)$ is a Banach space equipped with the norm $\|X\|_p := \left(\int_\Omega \|X\|^p \mathrm{d}\mathbb{P}\right)^{1/p}$. An H-valued stochastic process $X(t), t \in \mathbb{R}$ is called $L^p(\Omega, \mathbb{P}; H)$-*bounded*, if $\sup\limits_{t\in\mathbb{R}} \|X(t)\|_p < \infty$.

We employ $C_b(H)$ to denote the space of all bounded continuous real-valued functions on H. Let $Pr(H)$ be the space of all Borel probability measures on H. We write $Pr_2(H)$ to mean the space of probability measures $\mu \in Pr(H)$ such that $\int_H \|z\|^2 \mu(\mathrm{d}z) < \infty$. Then $Pr_2(H)$ is a separable complete metric space endowed with the following Wasserstein distance

$$W_2(\mu_1, \mu_2) := \inf_{\pi\in\mathcal{C}(\mu_1,\mu_2)} \left(\int_{H\times H} \|x - y\|^2 \pi(\mathrm{d}x, \mathrm{d}y)\right)^{\frac{1}{2}}$$

for all $\mu_1, \mu_2 \in Pr_2(H)$. Here $\mathcal{C}(\mu_1, \mu_2)$ is the set of all couplings for μ_1 and μ_2. Recall that a sequence $\{\mu_n\} \subset Pr(H)$ is said to *weakly converge* to μ if $\int f \mathrm{d}\mu_n \to \int f \mathrm{d}\mu$ for all $f \in C_b(H)$ and the Wasserstein distance W_2 metrizes weak convergence. We say that μ_n *converges weakly to* μ *in* $Pr_2(H)$ if $W_2(\mu_n, \mu) \to 0$ as $n \to \infty$. Throughout the paper, denote by $\mathcal{L}(\xi) \in Pr(H)$ the law or distribution of H-valued random variable ξ.

Now we recall some known definitions in dynamical systems (see e.g. [11, 25, 34] for more details). Let $(\mathcal{X}, \rho)$ and $(\mathcal{P}, d_\mathcal{P})$ be two metric spaces.

Definition 2.2. A *nonautonomous dynamical system* (σ, φ) (in short, φ) consists of two ingredients:

(i) A *dynamical system* σ on $\mathcal{P}$ with time set $T = \mathbb{Z}$ or $\mathbb{R}$, i.e.
  (1) $\sigma_0(\cdot) = Id_\mathcal{P}$,
  (2) $\sigma_{t+s}(p) = \sigma_t(\sigma_s(p))$ for all $t, s \in T$ and $p \in \mathcal{P}$,
  (3) the mapping $(t, p) \mapsto \sigma_t(p)$ is continuous.
  If $T = \mathbb{R}$, σ is called *flow* on $\mathcal{P}$; if $T = \mathbb{R}^+$, σ is called *semiflow* on $\mathcal{P}$.

(ii) A *cocycle* $\varphi : T^+ \times \mathcal{P} \times \mathcal{X} \to \mathcal{X}$ satisfies
  (1) $\varphi(0, p, x) = x$ for all $(p, x) \in \mathcal{P} \times \mathcal{X}$,
  (2) $\varphi(t + s, p, x) = \varphi(t, \sigma_s(p), \varphi(s, p, x))$ for all $s, t \in T^+$ and $(p, x) \in \mathcal{P} \times \mathcal{X}$,
  (3) the mapping $(t, p, x) \mapsto \varphi(t, p, x)$ is continuous.

Here $\mathcal{P}$ is called the *base* or *parameter space* and $\mathcal{X}$ is the *fiber* or *state space*. For convenience, we also write $\sigma_t(p)$ as $\sigma_t p$.

Definition 2.3. Let (σ, φ) be a nonautonomous dynamical system with base space $\mathcal{P}$ and state space $\mathcal{X}$. The *skew product semiflow* $\Pi : T^+ \times \mathcal{P} \times \mathcal{X} \to \mathcal{P} \times \mathcal{X}$ is a semiflow of the form:

$$\Pi(t,(p,x)) := (\sigma_t p, \varphi(t,p)x) .$$

Definition 2.4. Define $\mathfrak{X} := \mathcal{P} \times \mathcal{X}$. A nonempty compact subset $\mathfrak{A}$ of $\mathfrak{X}$ is called *global attractor* for skew product semiflow Π, if

(i) $\Pi(t, \mathfrak{A}) = \mathfrak{A}$ for all $t \in \mathbb{R}^+$,

(ii) $\lim\limits_{t\to+\infty} \mathrm{dist}_{\mathfrak{X}} (\Pi(t,D), \mathfrak{A}) = 0$ for every nonempty bounded subset D of $\mathfrak{X}$,

where $\mathrm{dist}_{\mathfrak{X}}(A,B)$ is the Hausdorff semi-metric between sets A and B, i.e. $\mathrm{dist}_{\mathfrak{X}}(A,B) := \sup\limits_{x\in A} d(x,B)$ with $d(x,B) := \inf\limits_{y\in B} d_{\mathfrak{X}}(x,y)$. Here $d_{\mathfrak{X}}(x,y) = d_{\mathcal{P}}(p_1,p_2) + \rho(x_1,x_2)$ for all $x := (p_1,x_1), y := (p_2,x_2) \in \mathcal{P} \times \mathcal{X}$.

Lemma 2.5 (see e.g. [11])**.** *Let $\{S(t)\}_{t\geq 0}$ be a semiflow in a complete metric space $\mathcal{X}$ having a compact attracting set $K \subset \mathcal{X}$, i.e.*

$$\lim_{t\to+\infty} \mathrm{dist}_{\mathcal{X}}(S(t)B, K) = 0$$

for all bounded set $B \subset \mathcal{X}$. Then $\{S(t)\}_{t\geq 0}$ has a global attractor $\mathcal{A} := \omega(K)$. Where $\omega(K)$ is the ω-limit set of K, i.e. $\omega(K) := \cap_{t\geq 0}\overline{\cup_{s\geq t} S(s)K}$.

Definition 2.6. We say that a compact set $\mathcal{A} \subset \mathcal{X}$ is the *uniform attractor (with respect to $p \in \mathcal{P}$)* of cocycle φ if the following conditions are fulfilled:

(i) The set $\mathcal{A}$ is uniformly attracting, i.e.

$$\lim_{t\to+\infty} \sup_{p\in\mathcal{P}} \mathrm{dist}_{\mathcal{X}} (\varphi(t,p,B), \mathcal{A}) = 0$$

for all bounded set $B \subset \mathcal{X}$.

(ii) If $\mathcal{A}_1$ is another closed uniformly attracting set, then $\mathcal{A} \subset \mathcal{A}_1$.

Let $W(t), t \in \mathbb{R}$ be a two-sided cylindrical Wiener process with the identity covariance operator defined on a separable Hilbert space $(U, \langle\ ,\ \rangle_U)$. We set $\mathcal{F}_t := \sigma\{W(u) - W(v) : u,v \leq t\}$. Denote by $L_2(U,H)$ the space of all Hilbert-Schmidt operators from U to H. Let us consider the following stochastic CGL equation on $\mathbb{T}^d, d = 1,2,3$

$$\mathrm{d}u(t) = \left[(1+i\alpha)\Delta u(t) - (1+i\beta)|u(t)|^2 u(t) + f(t,u(t))\right]\mathrm{d}t + g(t,u(t))\mathrm{d}W(t),\ t \in \mathbb{R}, \tag{2.1}$$

where $\alpha \in \mathbb{R}$, $|\beta| \leq \sqrt{3}$.

Definition 2.7. Fix $s \in \mathbb{R}$ and $T > 0$. An $L^2(\mathbb{T}^d)$-valued $\mathcal{F}_t$-adapted process $u(t), t \in [s, s+T]$ is said to be a *solution* of equation (2.1), if $u \in L^4\left([s,s+T] \times \Omega, \mathrm{d}t \otimes \mathbb{P}; L^4(\mathbb{T}^d)\right) \cap L^2\left([s,s+T] \times \Omega, \mathrm{d}t \otimes \mathbb{P}; H^1\right)$ and it satisfies the following stochastic integral equation $\mathbb{P}$-a.s.

$$\begin{aligned} u(t) = \zeta_s &+ \int_s^t \left[(1+i\alpha)\Delta u(\tau) - (1+i\beta)|u(\tau)|^2 u(\tau) + f(\tau,u(\tau))\right]\mathrm{d}\tau \\ &+ \int_s^t g(\tau,u(\tau))\mathrm{d}W(\tau), \quad t \in [s,s+T] \end{aligned}$$

for any $\zeta_s \in L^2\left(\Omega, \mathbb{P}; L^2(\mathbb{T}^d)\right)$.

Let us introduce the following conditions.

(H1) There exist constants $\lambda_f \in \mathbb{R}$, $K, L_f > 0$ such that for all $t \in \mathbb{R}$ and $x, y \in L^2(\mathbb{T}^d)$
$$\langle f(t,x) - f(t,y), x-y\rangle \le \lambda_f \|x-y\|^2, \quad \|f(t,0)\| \le K$$
and
$$\|f(t,x) - f(t,y)\| \le L_f\|x-y\|;$$

(H2) There exist constants $K, L_g > 0$ such that for all $t \in \mathbb{R}$ and $x, y \in L^2(\mathbb{T}^d)$
$$\|g(t,x) - g(t,y)\|_{L_2(U,L^2(\mathbb{T}^d))} \le L_g\|x-y\|, \quad \|g(t,0)\|_{L_2(U,L^2(\mathbb{T}^d))} \le K;$$

(H3) There exist constants $K, L_g > 0$ such that for all $t \in \mathbb{R}$ and $x, y \in H^1$
$$\|g(t,x) - g(t,y)\|_{L_2(U,H^1)} \le L_g\|x-y\|_{H^1}, \quad \|g(t,0)\|_{L_2(U,H^1)} \le K;$$

Remark 2.8. (i) We only need monotonicity of f and do not need Lipschitz continuity, as usual, when we consider estimates of solutions. Lipschitz continuity of f need to be assumed when we study averaging principle for stochastic CGL equations. And notice that condition $\lambda_* - L_f - \frac{L_g^2}{2} > 0$ is stronger than $\lambda_* - \lambda_f - \frac{L_g^2}{2} > 0$. For simplicity, we assume that both monotone and Lipschitz continuous conditions hold in (H1).

(ii) Note that
$$\|G\|_{L_2(U,L^2(\mathbb{T}^d))} \le \|G\|_{L_2(U,H^1)}$$
for all $G \in L_2(U,H^1)$, see Remark B.0.6 in [28] for more details.

3. The first Bogolyubov theorem

Consider the following stochastic CGL equations with highly oscillating components
$$\begin{aligned} \mathrm{d}u_\varepsilon(t) = &\left[(1+i\alpha)\Delta u_\varepsilon(t) - (1+i\beta)|u_\varepsilon(t)|^2 u_\varepsilon(t) + f\left(t/\varepsilon, u_\varepsilon(t)\right)\right]\mathrm{d}t \\ &+ g\left(t/\varepsilon, u_\varepsilon(t)\right)\mathrm{d}W(t), \quad t \in \mathbb{R} \end{aligned} \tag{3.1}$$
where $f \in C(\mathbb{R}\times L^2(\mathbb{T}^d), L^2(\mathbb{T}^d))$, $g \in C(\mathbb{R}\times L^2(\mathbb{T}^d), L_2(U, L^2(\mathbb{T}^d)))$ and $0 < \varepsilon \ll 1$. The well-posedness of (3.1) is shown in Theorem 3.3.

We employ Ψ to denote the space of all decreasing, positive bounded functions $\delta_1 : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to+\infty} \delta_1(t) = 0$. Below we need additional conditions.

(G1) There exist functions $\delta_f \in \Psi$ and $\bar{f} \in C(L^2(\mathbb{T}^d), L^2(\mathbb{T}^d))$ such that
$$\frac{1}{T}\left\|\int_t^{t+T} [f(s,x) - \bar{f}(x)]\mathrm{d}s\right\| \le \delta_f(T)(1+\|x\|)$$
for any $T > 0$, $x \in L^2(\mathbb{T}^d)$ and $t \in \mathbb{R}$;

(G2) There exist functions $\delta_g \in \Psi$ and $\bar{g} \in C(L^2(\mathbb{T}^d), L_2(U, L^2(\mathbb{T}^d)))$ such that
$$\frac{1}{T}\int_t^{t+T} \|g(s,x) - \bar{g}(x)\|^2_{L_2(U,L^2(\mathbb{T}^d))}\,\mathrm{d}s \le \delta_g(T)(1+\|x\|^2)$$
for any $T > 0$, $x \in L^2(\mathbb{T}^d)$ and $t \in \mathbb{R}$.

We set $f_\varepsilon(t,x) := f(\frac{t}{\varepsilon}, x)$ and $g_\varepsilon(t,x) := g(\frac{t}{\varepsilon}, x)$ for any $t \in \mathbb{R}$, $x \in L^2(\mathbb{T}^d)$ and $\varepsilon \in (0,1]$. Equation (3.1) can be written as
$$\mathrm{d}u_\varepsilon(t) = \left[(1+i\alpha)\Delta u_\varepsilon(t) - (1+i\beta)|u_\varepsilon(t)|^2 u_\varepsilon(t) + f_\varepsilon(t, u_\varepsilon(t))\right]\mathrm{d}t + g_\varepsilon(t,u_\varepsilon(t))\mathrm{d}W(t), \ t \in \mathbb{R}. \tag{3.2}$$
Along with equations (3.1)–(3.2) we consider the following averaged equation
$$\mathrm{d}u(t) = \left[(1+i\alpha)\Delta u(t) - (1+i\beta)|u(t)|^2 u(t) + \bar{f}(u(t))\right]\mathrm{d}t + \bar{g}(u(t))\mathrm{d}W(t), \ t \in \mathbb{R}. \tag{3.3}$$

In what follows, for simplicity we write C instead of $C_{\lambda_*,\lambda_f,L_f,L_g,K,\alpha,\beta}$ when C depends on some parameters of $\lambda_*, \lambda_f, L_f, L_g, K$ in (H1)–(H3) and α, β. But we write C_a explicitly when C depends on other constant a. Here C and C_a may change from line to line.

Lemma 3.1. *Suppose that* (H1)–(H2) *hold. Fix $s \in \mathbb{R}$. Let $u(t, s, \zeta_s), t \geq s$ be the solution of*

$$\begin{cases} \mathrm{d}u(t) = \left[(1 + i\alpha)\Delta u(t) - (1 + i\beta)|u(t)|^2 u(t) + f(t, u(t))\right] \mathrm{d}t + g(t, u(t))\mathrm{d}W(t) \\ u(s) = \zeta_s, \end{cases}$$

where $\zeta_s \in L^2(\Omega, \mathbb{P}; L^2(\mathbb{T}^d))$. Then there exists a constant C_T, depending on λ_*, L_f, L_g, K and T, such that

$$\begin{aligned} &\mathbb{E} \sup_{s\leq t\leq s+T} \|u(t, s, \zeta_s)\|^2 + \mathbb{E} \int_s^{s+T} \|u(t, s, \zeta_s)\|^2_{H^1} \mathrm{d}t + \mathbb{E} \int_s^{s+T} \|u(t, s, \zeta_s)\|^4_{L^4(\mathbb{T}^d)} \mathrm{d}t \\ &\leq C_T (1 + \mathbb{E}\|\zeta_s\|^2) \end{aligned} \tag{3.4}$$

for any $T > 0$.

Proof. By Itô's formula and (H1)–(H2), we get

$$\begin{aligned} &\|u(t, s, \zeta_s)\|^2 \\ &= \|\zeta_s\|^2 + \int_s^t \Big(2\langle(1 + i\alpha)\Delta u(\sigma, s, \zeta_s) - (1 + i\beta)|u(\sigma, s, \zeta_s)|^2 u(\sigma, s, \zeta_s), u(\sigma, s, \zeta_s)\rangle \\ &\quad + 2\langle f(\sigma, u(\sigma, s, \zeta_s)), u(\sigma, s, \zeta_s)\rangle + \|g(\sigma, u(\sigma, s, \zeta_s))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\Big) \mathrm{d}\sigma \\ &\quad + 2 \int_s^t \langle u(\sigma, s, \zeta_s), g(\sigma, u(\sigma, s, \zeta_s))\mathrm{d}W(\sigma)\rangle \\ &\leq \|\zeta_s\|^2 + \int_s^t \Big(-2\|u(\sigma, s, \zeta_s)\|^2_{H^1} - 2\|u(\sigma, s, \zeta_s)\|^4_{L^4(\mathbb{T}^d)} + \left(2L_f^2 + 2L_g^2 + 1\right) \|u(\sigma, s, \zeta_s)\|^2 + 4K^2\Big) \mathrm{d}\sigma \\ &\quad + 2 \int_s^t \langle u(\sigma, s, \zeta_s), g(\sigma, u(\sigma, s, \zeta_s))\mathrm{d}W(\sigma)\rangle. \end{aligned} \tag{3.5}$$

Dropping negative terms on the right of the above inequality, it follows from Burkholder-Davis-Gundy inequality and Young's inequality that

$$\begin{aligned} &\mathbb{E} \sup_{s\leq t\leq s+T} \|u(t, s, \zeta_s)\|^2 \\ &\leq \mathbb{E}\|\zeta_s\|^2 + \mathbb{E} \sup_{s\leq t\leq s+T} \int_s^t \left[\left(2L_f^2 + 2L_g^2 + 1\right) \|u(\sigma, s, \zeta_s)\|^2 + 4K^2\right] \mathrm{d}\sigma \\ &\quad + 6\mathbb{E} \left(\int_s^{s+T} \|u(\sigma, s, \zeta_s)\|^2 \|g(\sigma, u(\sigma, s, \zeta_s))\|^2_{L_2(U,L^2(\mathbb{T}^d))} \mathrm{d}\sigma\right)^{\frac{1}{2}} \\ &\leq \mathbb{E}\|\zeta_s\|^2 + \mathbb{E} \int_s^{s+T} \left[\left(2L_f^2 + 2L_g^2 + 1\right) \|u(\sigma, s, \zeta_s)\|^2 + 4K^2\right] \mathrm{d}\sigma \\ &\quad + \frac{1}{2}\mathbb{E} \sup_{s\leq t\leq s+T} \|u(\sigma, s, \zeta_s)\|^2 + C\mathbb{E} \int_s^{s+T} \|g(\sigma, u(\sigma, s, \zeta_s))\|^2_{L_2(U,L^2(\mathbb{T}^d))} \mathrm{d}\sigma \\ &\leq \mathbb{E}\|\zeta_s\|^2 + C_T + \frac{1}{2}\mathbb{E} \sup_{s\leq t\leq s+T} \|u(\sigma, s, \zeta_s)\|^2 + \mathbb{E} \int_s^{s+T} C\|u(\sigma, s, \zeta_s)\|^2 \mathrm{d}\sigma. \end{aligned}$$

Then we get

$$\mathbb{E}\sup_{s\le t\le s+T}\|u(t,s,\zeta_s)\|^2\le 2\mathbb{E}\|\zeta_s\|^2+C_T+\int_s^{s+T}C\mathbb{E}\sup_{s\le v\le\sigma}\|u(v,s,\zeta_s)\|^2\mathrm{d}\sigma.$$

In view of Gronwall's lemma, we obtain

$$\mathbb{E}\sup_{s\le t\le s+T}\|u(t,s,\zeta_s)\|^2\le\left(2\mathbb{E}\|\zeta_s\|^2+C_T\right)\mathrm{e}^{CT}. \tag{3.6}$$

Taking expectation on (3.5), with the help of (3.6) we have

$$\begin{aligned}
&\mathbb{E}\int_s^{s+T}\|u(t,s,\zeta_s)\|_{H^1}^2\mathrm{d}t+\mathbb{E}\int_s^{s+T}\|u(t,s,\zeta_s)\|_{L^4(\mathbb{T}^d)}^4\mathrm{d}t\\
&\le\frac{1}{2}\mathbb{E}\|\zeta_s\|^2+\frac{1}{2}\mathbb{E}\int_s^{s+T}\left[\left(2L_f^2+2L_g^2+1\right)\|u(t,s,\zeta_s)\|^2+4K^2\right]\mathrm{d}t\\
&\le C_T\left(\mathbb{E}\|\zeta_s\|^2+1\right).
\end{aligned}$$

The proof is complete. □

Remark 3.2. (i) It can be verified that (G1) (respectively, (G2)) implies

$$\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}f(s,x)\mathrm{d}s=\bar{f}(x)$$

(respectively, $\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}\|g(s,x)-\bar{g}(x)\|_{L_2(U,L^2(\mathbb{T}^d))}^2\mathrm{d}s=0$) uniformly w.r.t. $t\in\mathbb{R}$ and x in any bounded set.

(ii) If f and g satisfy (H1)–(H2) and (G1)–(G2), then $\bar{f}$ and $\bar{g}$ also satisfy (H1)–(H2) with the same constants. Therefore, under the same conditinos, (3.4) hold uniformly for $0<\varepsilon\le 1$, $\bar{f}$ and $\bar{g}$.

Now we give a theorem about the existence and uniqueness of solutions to (2.1). The proof is based on the classical Galerkin method, see e.g. [28]. Therefore, we put the proof in Appendix.

Theorem 3.3. *Assume that* (H1)–(H2) *hold. Let $s\in\mathbb{R}$. Then for any $\zeta_s\in L^2(\Omega,\mathbb{P};L^2(\mathbb{T}^d))$ there exists a unique solution $u(t,s,\zeta_s),t\ge s$ of* (2.1)*. Moreover, if* (H3) *holds and $\zeta_s\in L^2(\Omega,\mathbb{P};H^1)$. Then the unique solution also satisfy the following estimate*

$$\sup_{s\le t\le s+T}\mathbb{E}\|u(t,s,\zeta_s)\|_{H^1}^2\le C_T\left(1+\mathbb{E}\|\zeta_s\|_{H^1}^2\right)$$

for all $T>0$.

For given process ϕ, we define a step process $\tilde{\phi}$ such that $\tilde{\phi}(\sigma)=\phi(s+k\delta)$ for any $\sigma\in[s+k\delta,s+(k+1)\delta)$. Employing the technique of time discretization, we have the following estimates.

Lemma 3.4. *Assume that* (H1)–(H2) *and* (G1)–(G2) *hold. Let Y_ε be the solution of* (3.2) *with the initial value $Y_\varepsilon(s)=\zeta_s^\varepsilon$ and $\bar{Y}$ be the solution of* (3.3) *with the initial value $\bar{Y}(s)=\zeta_s$. Then we have*

$$\mathbb{E}\int_s^{s+T}\|Y_\varepsilon(\sigma)-\tilde{Y}_\varepsilon(\sigma)\|^2\mathrm{d}\sigma\le C_T(1+E\|\zeta_s^\varepsilon\|^2)\delta^{\frac{1}{2}} \tag{3.7}$$

and

$$\mathbb{E}\int_s^{s+T}\|\bar{Y}(\sigma)-\tilde{Y}(\sigma)\|^2\mathrm{d}\sigma\le C_T(1+E\|\zeta_s\|^2)\delta^{\frac{1}{2}} \tag{3.8}$$

for any $s\in\mathbb{R}$ and $T>0$, where $\tilde{Y}:=\tilde{\bar{Y}}$.

Proof. We set $T(\delta) := \left[\frac{T}{\delta}\right]$, where $\left[\frac{T}{\delta}\right]$ is the integer part of $\frac{T}{\delta}$. It follows from (3.4) and Remark 3.2 that

$$
\begin{aligned}
(3.9)\qquad & \mathbb{E}\int_s^{s+T} \|Y_\varepsilon(\sigma) - \tilde{Y}_\varepsilon(\sigma)\|^2 \mathrm{d}\sigma \\
&= \mathbb{E}\int_s^{s+\delta} \|Y_\varepsilon(\sigma) - \zeta_s^\varepsilon\|^2 \mathrm{d}\sigma + \mathbb{E}\sum_{k=1}^{T(\delta)-1}\int_{s+k\delta}^{s+(k+1)\delta} \|Y_\varepsilon(\sigma) - Y_\varepsilon(s+k\delta)\|^2 \mathrm{d}\sigma \\
&\quad + \mathbb{E}\int_{s+T(\delta)\delta}^{s+T} \|Y_\varepsilon(\sigma) - Y_\varepsilon(s+T(\delta)\delta)\|^2 \mathrm{d}\sigma \\
&\le C_T\left(1 + \mathbb{E}\|\zeta_s^\varepsilon\|^2\right)\delta + 2\mathbb{E}\sum_{k=1}^{T(\delta)-1}\int_{s+k\delta}^{s+(k+1)\delta} \|Y_\varepsilon(\sigma) - Y_\varepsilon(\sigma-\delta)\|^2 \mathrm{d}\sigma \\
&\quad + 2\mathbb{E}\sum_{k=1}^{T(\delta)-1}\int_{s+k\delta}^{s+(k+1)\delta} \|Y_\varepsilon(\sigma-\delta) - Y_\varepsilon(s+k\delta)\|^2 \mathrm{d}\sigma \\
&=: C_T\left(1 + \mathbb{E}\|\zeta_s^\varepsilon\|^2\right)\delta + 2\sum_{k=1}^{T(\delta)-1}\mathcal{I}_k + 2\sum_{k=1}^{T(\delta)-1}\mathcal{J}_k.
\end{aligned}
$$

Given $k \in [1, T(\delta)-1]$, for any $\sigma \in [s+k\delta, s+(k+1)\delta)$, by Itô's formula, (H1)–(H2) and Young's inequality we get

$$
\begin{aligned}
&\|Y_\varepsilon(\sigma) - Y_\varepsilon(\sigma-\delta)\|^2 \\
&= 2\int_{\sigma-\delta}^{\sigma} \langle (1+i\alpha)\Delta Y_\varepsilon(\tau) - (1+i\beta)|Y_\varepsilon(\tau)|^2 Y_\varepsilon(\tau), Y_\varepsilon(\tau) - Y_\varepsilon(\sigma-\delta)\rangle \mathrm{d}\tau \\
&\quad + 2\int_{\sigma-\delta}^{\sigma} \langle f_\varepsilon(\tau, Y_\varepsilon(\tau)), Y_\varepsilon(\tau) - Y_\varepsilon(\sigma-\delta)\rangle \mathrm{d}\tau + \int_{\sigma-\delta}^{\sigma} \|g_\varepsilon(\tau, Y_\varepsilon(\tau))\|^2_{L_2(U, L^2(\mathbb{T}^d))} \mathrm{d}\tau \\
&\quad + 2\int_{\sigma-\delta}^{\sigma} \langle Y_\varepsilon(\tau) - Y_\varepsilon(\sigma-\delta), g_\varepsilon(\tau, Y_\varepsilon(\tau))\mathrm{d}W(\tau)\rangle \\
&\le \int_{\sigma-\delta}^{\sigma} \Big(2\langle (1+i\alpha)\nabla Y_\varepsilon(\tau), \nabla Y_\varepsilon(\sigma-\delta)\rangle + 2\langle (1+i\beta)|Y_\varepsilon(\tau)|^2 Y_\varepsilon(\tau), Y_\varepsilon(\sigma-\delta)\rangle \\
&\qquad + 2\langle f_\varepsilon(\tau, Y_\varepsilon(\tau)), Y_\varepsilon(\tau) - Y_\varepsilon(\sigma-\delta)\rangle + 2L_g^2\|Y_\varepsilon(\tau)\|^2 + 2K^2 \Big) \mathrm{d}\tau \\
&\quad + 2\int_{\sigma-\delta}^{\sigma} \langle Y_\varepsilon(\tau) - Y_\varepsilon(\sigma-\delta), g_\varepsilon(\tau, Y_\varepsilon(\tau))\mathrm{d}W(\tau)\rangle \\
&\le \int_{\sigma-\delta}^{\sigma} \Big(C\|Y_\varepsilon(\tau)\|_{H^1}\|Y_\varepsilon(\sigma-\delta)\|_{H^1} + C\|Y_\varepsilon(\tau)\|^3_{L^4(\mathbb{T}^d)}\|Y_\varepsilon(\sigma-\delta)\|_{L^4(\mathbb{T}^d)} \\
&\qquad + 2\|f_\varepsilon(\tau, Y_\varepsilon(\tau))\|\|Y_\varepsilon(\tau) - Y_\varepsilon(\sigma-\delta)\| + 2L_g^2\|Y_\varepsilon(\tau)\|^2 + 2K^2 \Big) \mathrm{d}\tau \\
&\quad + 2\int_{\sigma-\delta}^{\sigma} \langle Y_\varepsilon(\tau) - Y_\varepsilon(\sigma-\delta), g_\varepsilon(\tau, Y_\varepsilon(\tau))\mathrm{d}W(\tau)\rangle \\
&\le \int_{\sigma-\delta}^{\sigma} C\Big(\|Y_\varepsilon(\tau)\|^2_{H^1} + \|Y_\varepsilon(\sigma-\delta)\|^2_{H^1} + \|Y_\varepsilon(\tau)\|^4_{L^4(\mathbb{T}^d)} \\
&\qquad + \|Y_\varepsilon(\sigma-\delta)\|^4_{L^4(\mathbb{T}^d)} + \|Y_\varepsilon(\tau)\|^2 + \|Y_\varepsilon(\sigma-\delta)\|^2 + 1 \Big) \mathrm{d}\tau
\end{aligned}
$$

$$+ 2\int_{\sigma-\delta}^{\sigma} \langle Y_\varepsilon(\tau) - Y_\varepsilon(\sigma-\delta), g_\varepsilon(\tau, Y_\varepsilon(\tau))\mathrm{d}W(\tau)\rangle.$$

Then we have

$$\begin{aligned}
\mathcal{I}_k &= \mathbb{E}\int_{s+k\delta}^{s+(k+1)\delta} \|Y_\varepsilon(\sigma) - Y_\varepsilon(\sigma-\delta)\|^2 \mathrm{d}\sigma \\
&\le \mathbb{E}\int_{s+k\delta}^{s+(k+1)\delta} \Bigg\{ \int_{\sigma-\delta}^{\sigma} C\Big(\|Y_\varepsilon(\tau)\|_{H^1}^2 + \|Y_\varepsilon(\sigma-\delta)\|_{H^1}^2 + \|Y_\varepsilon(\tau)\|_{L^4(\mathbb{T}^d)}^4 \\
&\qquad + \|Y_\varepsilon(\sigma-\delta)\|_{L^4(\mathbb{T}^d)}^4 + \|Y_\varepsilon(\tau)\|^2 + \|Y_\varepsilon(\sigma-\delta)\|^2 + 1\Big)\mathrm{d}\tau \\
&\qquad + 2\int_{\sigma-\delta}^{\sigma} \langle Y_\varepsilon(\tau) - Y_\varepsilon(\sigma-\delta), g_\varepsilon(\tau, Y_\varepsilon(\tau))\mathrm{d}W(\tau)\rangle \Bigg\}\mathrm{d}\sigma =: \mathcal{I}_k^1 + \mathcal{I}_k^2.
\end{aligned} \tag{3.10}$$

For $\mathcal{I}_k^1$, we have

$$\begin{aligned}
\mathcal{I}_k^1 &:= \mathbb{E}\int_{s+k\delta}^{s+(k+1)\delta} \Bigg\{ \int_{\sigma-\delta}^{\sigma} C\Big(\|Y_\varepsilon(\tau)\|_{H^1}^2 + \|Y_\varepsilon(\sigma-\delta)\|_{H^1}^2 + \|Y_\varepsilon(\tau)\|_{L^4(\mathbb{T}^d)}^4 \\
&\qquad + \|Y_\varepsilon(\sigma-\delta)\|_{L^4(\mathbb{T}^d)}^4 + \|Y_\varepsilon(\tau)\|^2 + \|Y_\varepsilon(\sigma-\delta)\|^2 + 1\Big)\mathrm{d}\tau \Bigg\}\mathrm{d}\sigma \\
&= \mathbb{E}\int_{s+k\delta}^{s+(k+1)\delta} \int_{\sigma-\delta}^{\sigma} C\left(\|Y_\varepsilon(\tau)\|_{H^1}^2 + \|Y_\varepsilon(\tau)\|_{L^4(\mathbb{T}^d)}^4 + \|Y_\varepsilon(\tau)\|^2 + 1\right)\mathrm{d}\tau\mathrm{d}\sigma \\
&\quad + \mathbb{E}\int_{s+k\delta}^{s+(k+1)\delta} \delta C\left(\|Y_\varepsilon(\sigma-\delta)\|_{H^1}^2 + \|Y_\varepsilon(\sigma-\delta)\|_{L^4(\mathbb{T}^d)}^4 + \|Y_\varepsilon(\sigma-\delta)\|^2\right)\mathrm{d}\sigma \\
&\le \delta C\mathbb{E}\int_{s+(k-1)\delta}^{s+(k+1)\delta} \left(\|Y_\varepsilon(\tau)\|_{H^1}^2 + \|Y_\varepsilon(\tau)\|_{L^4(\mathbb{T}^d)}^4 + \|Y_\varepsilon(\tau)\|^2 + 1\right)\mathrm{d}\tau.
\end{aligned} \tag{3.11}$$

Now we estimate $\mathcal{I}_k^2$. In view of Burkholder-Davis-Gundy inequality, (H2) and Young's inequality, we obtain

$$\begin{aligned}
\mathcal{I}_k^2 &:= 2\mathbb{E}\int_{s+k\delta}^{s+(k+1)\delta} \int_{\sigma-\delta}^{\sigma} \langle Y_\varepsilon(\tau) - Y_\varepsilon(\sigma-\delta), g_\varepsilon(\tau, Y_\varepsilon(\tau))\mathrm{d}W(\tau)\rangle\mathrm{d}\sigma \\
&\le 6\int_{s+k\delta}^{s+(k+1)\delta} \mathbb{E}\left(\int_{\sigma-\delta}^{\sigma} \|g_\varepsilon(\tau, Y_\varepsilon(\tau))\|_{L_2(U,L^2(\mathbb{T}^d))}^2 \|Y_\varepsilon(\tau) - Y_\varepsilon(\sigma-\delta)\|^2\mathrm{d}\tau\right)^{\frac{1}{2}}\mathrm{d}\sigma \\
&\le 6\int_{s+k\delta}^{s+(k+1)\delta} \mathbb{E}\left(\int_{\sigma-\delta}^{\sigma} \left(2L_g^2\|Y_\varepsilon(\tau)\|^2 + 2K^2\right) \|Y_\varepsilon(\tau) - Y_\varepsilon(\sigma-\delta)\|^2\mathrm{d}\tau\right)^{\frac{1}{2}}\mathrm{d}\sigma \\
&\le \delta^{\frac{1}{2}} C\left[\int_{s+k\delta}^{s+(k+1)\delta} \mathbb{E}\int_{\sigma-\delta}^{\sigma} \left(\|Y_\varepsilon(\tau)\|^4 + \|Y_\varepsilon(\sigma-\delta)\|^4 + 1\right)\mathrm{d}\tau\mathrm{d}\sigma\right]^{\frac{1}{2}} \\
&\le \delta^{\frac{1}{2}} C\left(\mathbb{E}\int_{s+(k-1)\delta}^{s+(k+1)\delta} \delta\|Y_\varepsilon(\tau)\|^4\mathrm{d}\tau + \delta^2\right)^{\frac{1}{2}} \\
&\le \delta C\left(\mathbb{E}\int_{s+(k-1)\delta}^{s+(k+1)\delta} \|Y_\varepsilon(\tau)\|_{L^4(\mathbb{T}^d)}^4\mathrm{d}\tau\right)^{\frac{1}{2}} + C\delta^{\frac{3}{2}}.
\end{aligned} \tag{3.12}$$

Therefore (3.10)–(3.12) yield

$$\mathcal{I}_k \leq \delta C \mathbb{E} \int_{s+(k-1)\delta}^{s+(k+1)\delta} \left(\|Y_\varepsilon(\tau)\|_{H^1}^2 + \|Y_\varepsilon(\tau)\|_{L^4(\mathbb{T}^d)}^4 + \|Y_\varepsilon(\tau)\|^2 + 1 \right) \mathrm{d}\tau$$
$$+ \delta C \left(\mathbb{E} \int_{s+(k-1)\delta}^{s+(k+1)\delta} \|Y_\varepsilon(\tau)\|_{L^4(\mathbb{T}^d)}^4 \mathrm{d}\tau \right)^{\frac{1}{2}} + C\delta^{\frac{3}{2}}.$$

By Lemma 3.1 and Remark 3.2, we get

(3.13)

$$\begin{aligned}
2\sum_{k=1}^{T(\delta)-1} \mathcal{I}_k &\leq \delta C \mathbb{E} \int_s^{s+T} \left(\|Y_\varepsilon(\tau)\|_{H^1}^2 + \|Y_\varepsilon(\tau)\|_{L^4(\mathbb{T}^d)}^4 + \|Y_\varepsilon(\tau)\|^2 + 1 \right) \mathrm{d}\tau \\
&\quad + \delta C \sum_{k=1}^{T(\delta)-1} \left(\mathbb{E} \int_{s+(k-1)\delta}^{s+(k+1)\delta} \|Y_\varepsilon(\tau)\|_{L^4(\mathbb{T}^d)}^4 \mathrm{d}\tau \right)^{\frac{1}{2}} + C_T \delta^{\frac{1}{2}} \\
&\leq C_T \left(1 + \mathbb{E}\|\zeta_s^\varepsilon\|^2\right) \delta^{\frac{1}{2}} + \delta C \left(T(\delta)\right)^{\frac{1}{2}} \left(\sum_{k=1}^{T(\delta)-1} \mathbb{E} \int_{s+(k-1)\delta}^{s+(k+1)\delta} \|Y_\varepsilon(\tau)\|_{L^4(\mathbb{T}^d)}^4 \mathrm{d}\tau \right)^{\frac{1}{2}} \\
&\leq C_T \left(1 + \mathbb{E}\|\zeta_s^\varepsilon\|^2\right) \delta^{\frac{1}{2}}.
\end{aligned}$$

Similarly, we have

$$2\sum_{k=1}^{T(\delta)-1} \mathcal{J}_k \leq C_T \left(1 + \mathbb{E}\|\zeta_s^\varepsilon\|^2\right) \delta^{\frac{1}{2}}. \tag{3.14}$$

Combining (3.9), (3.13) and (3.14), we obtain

$$\mathbb{E} \int_s^{s+T} \|Y_\varepsilon(\sigma) - \tilde{Y}_\varepsilon(\sigma)\|^2 \mathrm{d}\sigma \leq C_T (1 + \mathbb{E}\|\zeta_s^\varepsilon\|^2) \delta^{\frac{1}{2}}.$$

It follows from the same steps as in the proof of (3.7) that

$$\mathbb{E} \int_s^{s+T} \|\bar{Y}(\sigma) - \tilde{Y}(\sigma)\|^2 \mathrm{d}\sigma \leq C_T (1 + \mathbb{E}\|\zeta_s\|^2) \delta^{\frac{1}{2}}.$$

□

Now we establish the first Bogolyubov theorem for stochastic CGL equations.

Theorem 3.5. *Suppose that* (G1)–(G2) *and* (H1)–(H2) *hold. For any* $s \in \mathbb{R}$, *let* Y_ε *be the solution of*

$$\begin{cases} \mathrm{d}u(t) = \left[(1+i\alpha)\Delta u(t) - (1+i\beta)|u(t)|^2 u(t) + f_\varepsilon(t, u(t))\right] \mathrm{d}t + g_\varepsilon(t, u(t)) \mathrm{d}W(t) \\ u(s) = \zeta_s^\varepsilon, \end{cases}$$

and $\bar{Y}$ *be the solution of*

$$\begin{cases} \mathrm{d}u(t) = \left[(1+i\alpha)\Delta u(t) - (1+i\beta)|u(t)|^2 u(t) + \bar{f}(u(t))\right] \mathrm{d}t + \bar{g}(u(t)) \mathrm{d}W(t) \\ u(s) = \zeta_s. \end{cases}$$

Assume further that $\lim_{\varepsilon \to 0} \mathbb{E}\|\zeta_s^\varepsilon - \zeta_s\|^2 = 0$. *Then*

$$\lim_{\varepsilon \to 0} \mathbb{E} \sup_{s \leq t \leq s+T} \|Y_\varepsilon(t) - \bar{Y}(t)\|^2 = 0$$

for any $T > 0$.

Proof. Note that
$$\left\langle -(1+i\beta)\left(|u|^2u-|v|^2v\right),u-v\right\rangle \le 0$$
for all $u,v\in L^2(\mathbb{T}^d)$ provided $|\beta|\le\sqrt{3}$. In view of Itô's formula, we have
$$\begin{aligned}
&\|Y_\varepsilon(t)-\bar{Y}(t)\|^2\\
&=\|\zeta_s^\varepsilon-\zeta_s\|^2+\int_s^t\Big(2\langle(1+i\alpha)\Delta(Y_\varepsilon(\sigma)-\bar{Y}(\sigma)),Y_\varepsilon(\sigma)-\bar{Y}(\sigma)\rangle\\
&\quad-2\langle(1+i\beta)\left(|Y_\varepsilon(\sigma)|^2Y_\varepsilon(\sigma)-|\bar{Y}(\sigma)|^2\bar{Y}(\sigma)\right),Y_\varepsilon(\sigma)-\bar{Y}(\sigma)\rangle\\
&\quad+2\langle f_\varepsilon(\sigma,Y_\varepsilon(\sigma))-\bar{f}(\bar{Y}(\sigma)),Y_\varepsilon(\sigma)-\bar{Y}(\sigma)\rangle+\|g_\varepsilon(\sigma,Y_\varepsilon(\sigma))-\bar{g}(\bar{Y}(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\Big)\mathrm{d}\sigma\\
&\quad+2\int_s^t\langle Y_\varepsilon(\sigma)-\bar{Y}(\sigma),\left(g_\varepsilon(\sigma,Y_\varepsilon(\sigma))-\bar{g}(\bar{Y}(\sigma))\right)\mathrm{d}W(\sigma)\rangle\\
&\le\|\zeta_s^\varepsilon-\zeta_s\|^2+2\int_s^t\langle Y_\varepsilon(\sigma)-\bar{Y}(\sigma),\left(g_\varepsilon(\sigma,Y_\varepsilon(\sigma))-\bar{g}(\bar{Y}(\sigma))\right)\mathrm{d}W(\sigma)\rangle\\
&\quad+\int_s^t\Big(2\langle f_\varepsilon(\sigma,Y_\varepsilon(\sigma))-\bar{f}(\bar{Y}(\sigma)),Y_\varepsilon(\sigma)-\bar{Y}(\sigma)\rangle+\|g_\varepsilon(\sigma,Y_\varepsilon(\sigma))-\bar{g}(\bar{Y}(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\Big)\mathrm{d}\sigma.
\end{aligned}$$
Therefore, by Burkholder-Davis-Gundy inequality and Young's inequality we get
$$\begin{aligned}
&\mathbb{E}\sup_{s\le t\le s+T}\|Y_\varepsilon(t)-\bar{Y}(t)\|^2\\
&\le\mathbb{E}\|\zeta_s^\varepsilon-\zeta_s\|^2+\mathbb{E}\sup_{s\le t\le s+T}\int_s^t2\langle f_\varepsilon(\sigma,Y_\varepsilon(\sigma))-\bar{f}(\bar{Y}(\sigma)),Y_\varepsilon(\sigma)-\bar{Y}(\sigma)\rangle\mathrm{d}\sigma\\
&\quad+\mathbb{E}\int_s^{s+T}\|g_\varepsilon(\sigma,Y_\varepsilon(\sigma))-\bar{g}(\bar{Y}(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\mathrm{d}\sigma\\
&\quad+6\mathbb{E}\left(\int_s^{s+T}\|g_\varepsilon(\sigma,Y_\varepsilon(\sigma))-\bar{g}(\bar{Y}(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\|Y_\varepsilon(\sigma)-\bar{Y}(\sigma)\|^2\mathrm{d}\sigma\right)^{\frac12}\\
&\le\mathbb{E}\|\zeta_s^\varepsilon-\zeta_s\|^2+\mathbb{E}\sup_{s\le t\le s+T}\int_s^t2\langle f_\varepsilon(\sigma,Y_\varepsilon(\sigma))-\bar{f}(\bar{Y}(\sigma)),Y_\varepsilon(\sigma)-\bar{Y}(\sigma)\rangle\mathrm{d}\sigma\\
&\quad+\frac12E\sup_{s\le t\le s+T}\|Y_\varepsilon(t)-\bar{Y}(t)\|^2+C\mathbb{E}\int_s^{s+T}\|g_\varepsilon(\sigma,Y_\varepsilon(\sigma))-\bar{g}(\bar{Y}(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\mathrm{d}\sigma.
\end{aligned}$$
Then we obtain
$$\begin{aligned}
&\mathbb{E}\sup_{s\le t\le s+T}\|Y_\varepsilon(t)-\bar{Y}(t)\|^2\\
&\le2\mathbb{E}\|\zeta_s^\varepsilon-\zeta_s\|^2+4\mathbb{E}\sup_{s\le t\le s+T}\int_s^t\langle f_\varepsilon(\sigma,Y_\varepsilon(\sigma))-\bar{f}(\bar{Y}(\sigma)),Y_\varepsilon(\sigma)-\bar{Y}(\sigma)\rangle\mathrm{d}\sigma\\
&\quad+C\mathbb{E}\int_s^{s+T}\|g_\varepsilon(\sigma,Y_\varepsilon(\sigma))-\bar{g}(\bar{Y}(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\mathrm{d}\sigma\\
&=:2\mathbb{E}\|\zeta_s^\varepsilon-\zeta_s\|^2+\mathcal{J}_1+\mathcal{J}_2.
\end{aligned}\tag{3.15}$$

Now we estimate $\mathcal{J}_1:=4\mathbb{E}\sup_{s\le t\le s+T}\int_s^t\langle f_\varepsilon(\sigma,Y_\varepsilon(\sigma))-\bar{f}(\bar{Y}(\sigma)),Y_\varepsilon(\sigma)-\bar{Y}(\sigma)\rangle\mathrm{d}\sigma$. It follows from (H1) that

$$
\begin{aligned}
\mathcal{J}_1 \le\ & 4\mathbb{E}\int_s^{s+T} \|f_\varepsilon(\sigma, Y_\varepsilon(\sigma)) - f_\varepsilon(\sigma, \bar{Y}(\sigma))\|\|Y_\varepsilon(\sigma) - \bar{Y}(\sigma)\|\mathrm{d}\sigma \\
& + 4\mathbb{E}\sup_{s\le t\le s+T}\int_s^t \langle f_\varepsilon(\sigma, \bar{Y}(\sigma)) - \bar{f}(\bar{Y}(\sigma)), Y_\varepsilon(\sigma) - \bar{Y}(\sigma)\rangle\mathrm{d}\sigma \\
\le\ & 4\mathbb{E}\int_s^{s+T} L_f\|Y_\varepsilon(\sigma) - \bar{Y}(\sigma)\|^2\mathrm{d}\sigma \\
& + 4\mathbb{E}\sup_{s\le t\le s+T}\int_s^t \langle f_\varepsilon(\sigma, \bar{Y}(\sigma)) - \bar{f}(\bar{Y}(\sigma)), Y_\varepsilon(\sigma) - \tilde{Y}_\varepsilon(\sigma)\rangle\mathrm{d}\sigma \\
& + 4\mathbb{E}\sup_{s\le t\le s+T}\int_s^t \langle f_\varepsilon(\sigma, \bar{Y}(\sigma)) - \bar{f}(\bar{Y}(\sigma)), \tilde{Y}_\varepsilon(\sigma) - \tilde{Y}(\sigma)\rangle\mathrm{d}\sigma \\
& + 4\mathbb{E}\sup_{s\le t\le s+T}\int_s^t \langle f_\varepsilon(\sigma, \bar{Y}(\sigma)) - \bar{f}(\bar{Y}(\sigma)), \tilde{Y}(\sigma) - \bar{Y}(\sigma)\rangle\mathrm{d}\sigma \\
=:\ & 4\mathbb{E}\int_s^{s+T} L_f\|Y_\varepsilon(\sigma) - \bar{Y}(\sigma)\|^2\mathrm{d}\sigma + \mathcal{J}_1^2 + \mathcal{J}_1^3 + \mathcal{J}_1^4.
\end{aligned} \tag{3.16}
$$

For $\mathcal{J}_1^2$, by (H1), Hölder's inequality, (3.4) and (3.7) we have

$$
\begin{aligned}
\mathcal{J}_1^2 :=\ & 4\mathbb{E}\sup_{s\le t\le s+T}\int_s^t \langle f_\varepsilon(\sigma, \bar{Y}(\sigma)) - \bar{f}(\bar{Y}(\sigma)), Y_\varepsilon(\sigma) - \tilde{Y}_\varepsilon(\sigma)\rangle\mathrm{d}\sigma \\
\le\ & 4\mathbb{E}\int_s^{s+T} \|f_\varepsilon(\sigma, \bar{Y}(\sigma)) - \bar{f}(\bar{Y}(\sigma))\|\|Y_\varepsilon(\sigma) - \tilde{Y}_\varepsilon(\sigma)\|\mathrm{d}\sigma \\
\le\ & 4\left[\mathbb{E}\int_s^{s+T} \left(2L_f\|\bar{Y}(\sigma)\| + 2K\right)^2\mathrm{d}\sigma\right]^{\frac{1}{2}} \left(\mathbb{E}\int_s^{s+T} \|Y_\varepsilon(\sigma) - \tilde{Y}_\varepsilon(\sigma)\|^2\mathrm{d}\sigma\right)^{\frac{1}{2}} \\
\le\ & C_T\left(1 + \mathbb{E}\|\zeta_s\|^2\right)\delta^{\frac{1}{4}}.
\end{aligned} \tag{3.17}
$$

Similar to $\mathcal{J}_1^2$, we get

$$
\mathcal{J}_1^4 := 4\mathbb{E}\sup_{s\le t\le s+T}\int_s^t \langle f_\varepsilon(\sigma, \bar{Y}(\sigma)) - \bar{f}(\bar{Y}(\sigma)), \tilde{Y}(\sigma) - \bar{Y}(\sigma)\rangle\mathrm{d}\sigma \le C_T\left(1 + \mathbb{E}\|\zeta_s\|^2\right)\delta^{\frac{1}{4}}. \tag{3.18}
$$

For $\mathcal{J}_1^3$, by (H1), Hölder's inequality and (3.8) we have

$$
\begin{aligned}
\mathcal{J}_1^3 :=\ & 4\mathbb{E}\sup_{s\le t\le s+T}\int_s^t \langle f_\varepsilon(\sigma, \bar{Y}(\sigma)) - \bar{f}(\bar{Y}(\sigma)), \tilde{Y}_\varepsilon(\sigma) - \tilde{Y}(\sigma)\rangle\mathrm{d}\sigma \\
\le\ & 4\mathbb{E}\sup_{s\le t\le s+T}\int_s^t \langle f_\varepsilon(\sigma, \bar{Y}(\sigma)) - f_\varepsilon(\sigma, \tilde{Y}(\sigma)), \tilde{Y}_\varepsilon(\sigma) - \tilde{Y}(\sigma)\rangle\mathrm{d}\sigma \\
& + 4\mathbb{E}\sup_{s\le t\le s+T}\int_s^t \langle f_\varepsilon(\sigma, \tilde{Y}(\sigma)) - \bar{f}(\tilde{Y}(\sigma)), \tilde{Y}_\varepsilon(\sigma) - \tilde{Y}(\sigma)\rangle\mathrm{d}\sigma \\
& + 4\mathbb{E}\sup_{s\le t\le s+T}\int_s^t \langle \bar{f}(\tilde{Y}(\sigma)) - \bar{f}(\bar{Y}(\sigma)), \tilde{Y}_\varepsilon(\sigma) - \tilde{Y}(\sigma)\rangle\mathrm{d}\sigma \\
\le\ & 4\mathbb{E}\int_s^{s+T} \|f_\varepsilon(\sigma, \bar{Y}(\sigma)) - f_\varepsilon(\sigma, \tilde{Y}(\sigma))\|\|\tilde{Y}_\varepsilon(\sigma) - \tilde{Y}(\sigma)\|\mathrm{d}\sigma \\
& + 4\mathbb{E}\sup_{s\le t\le s+T}\int_s^t \langle f_\varepsilon(\sigma, \tilde{Y}(\sigma)) - \bar{f}(\tilde{Y}(\sigma)), \tilde{Y}_\varepsilon(\sigma) - \tilde{Y}(\sigma)\rangle\mathrm{d}\sigma
\end{aligned} \tag{3.19}
$$

$$
\begin{aligned}
&\quad + 4\mathbb{E}\int_s^{s+T} \|\bar{f}(\tilde{Y}(\sigma)) - \bar{f}(\bar{Y}(\sigma))\|\|\tilde{Y}_\varepsilon(\sigma) - \tilde{Y}(\sigma)\|\mathrm{d}\sigma \\
&\le 8\mathbb{E}\int_s^{s+T} L_f\|\tilde{Y}(\sigma) - \bar{Y}(\sigma)\|\|\tilde{Y}_\varepsilon(\sigma) - \tilde{Y}(\sigma)\|\mathrm{d}\sigma \\
&\quad + 4\mathbb{E}\sup_{s\le t\le s+T}\int_s^t \langle f_\varepsilon(\sigma, \tilde{Y}(\sigma)) - \bar{f}(\tilde{Y}(\sigma)), \tilde{Y}_\varepsilon(\sigma) - \tilde{Y}(\sigma)\rangle\mathrm{d}\sigma \\
&\le 8L_f\left(\mathbb{E}\int_s^{s+T}\|\tilde{Y}(\sigma) - \bar{Y}(\sigma)\|^2\mathrm{d}\sigma\right)^{\frac{1}{2}}\left(E\int_s^{s+T}\|\tilde{Y}_\varepsilon(\sigma) - \tilde{Y}(\sigma)\|^2\mathrm{d}\sigma\right)^{\frac{1}{2}} \\
&\quad + 4\mathbb{E}\sup_{s\le t\le s+T}\int_s^t \langle f_\varepsilon(\sigma, \tilde{Y}(\sigma)) - \bar{f}(\tilde{Y}(\sigma)), \tilde{Y}_\varepsilon(\sigma) - \tilde{Y}(\sigma)\rangle\mathrm{d}\sigma \\
&\le C_T\left(1 + \mathbb{E}\|\zeta_s\|^2\right)\delta^{\frac{1}{4}} + \mathcal{J}_1^{3,2}.
\end{aligned}
$$

We set $t(s,\delta) := s + \left[\frac{t-s}{\delta}\right]\delta$. Recall that $\left[\frac{t-s}{\delta}\right]$ is the integer part of $\frac{t-s}{\delta}$. Now we estimate $\mathcal{J}_1^{3,2} := 4\mathbb{E}\sup\limits_{s\le t\le s+T}\int_s^t\langle f_\varepsilon(\sigma, \tilde{Y}(\sigma)) - \bar{f}(\tilde{Y}(\sigma)), \tilde{Y}_\varepsilon(\sigma) - \tilde{Y}(\sigma)\rangle\mathrm{d}\sigma$. We have

(3.20)

$$
\begin{aligned}
&4\mathbb{E}\sup_{s\le t\le s+T}\int_s^t \langle f_\varepsilon(\sigma, \tilde{Y}(\sigma)) - \bar{f}(\tilde{Y}(\sigma)), \tilde{Y}_\varepsilon(\sigma) - \tilde{Y}(\sigma)\rangle\mathrm{d}\sigma \\
&= 4\mathbb{E}\sup_{s\le t\le s+T}\Bigg\{\sum_{k=0}^{\left[\frac{t-s}{\delta}\right]-1}\int_{s+k\delta}^{s+(k+1)\delta}\langle f_\varepsilon(\sigma, \bar{Y}(s+k\delta)) - \bar{f}(\bar{Y}(s+k\delta)), Y_\varepsilon(s+k\delta) - \bar{Y}(s+k\delta)\rangle\mathrm{d}\sigma \\
&\quad + \int_{t(s,\delta)}^t \langle f_\varepsilon(\sigma, \bar{Y}(t(s,\delta))) - \bar{f}(\bar{Y}(t(s,\delta))), Y_\varepsilon(t(s,\delta)) - \bar{Y}(t(s,\delta))\rangle\mathrm{d}\sigma\Bigg\} \\
&\le 4\mathbb{E}\sup_{s\le t\le s+T}\Bigg\{\sum_{k=0}^{\left[\frac{t-s}{\delta}\right]-1}\left\langle\int_{s+k\delta}^{s+(k+1)\delta}\left(f_\varepsilon(\sigma, \bar{Y}(s+k\delta)) - \bar{f}(\bar{Y}(s+k\delta))\right)\mathrm{d}\sigma, Y_\varepsilon(s+k\delta) - \bar{Y}(s+k\delta)\right\rangle \\
&\quad + \int_{t(s,\delta)}^t \|f_\varepsilon(\sigma, \bar{Y}(t(s,\delta))) - \bar{f}(\bar{Y}(t(s,\delta)))\|\|Y_\varepsilon(t(s,\delta)) - \bar{Y}(t(s,\delta))\|\mathrm{d}\sigma\Bigg\} \\
&\le 4\mathbb{E}\sup_{s\le t\le s+T}\Bigg\{\sum_{k=0}^{\left[\frac{t-s}{\delta}\right]-1}\left\|\int_{s+k\delta}^{s+(k+1)\delta}\left(f_\varepsilon(\sigma, \bar{Y}(s+k\delta)) - \bar{f}(\bar{Y}(s+k\delta))\right)\mathrm{d}\sigma\right\| \\
&\qquad \times\|Y_\varepsilon(s+k\delta) - \bar{Y}(s+k\delta)\|\Bigg\} + C_T\left(1 + \mathbb{E}\|\zeta_s\|^2\right)\delta \\
&\le \frac{4T}{\delta}\max_{0\le k\le T(\delta)-1}\left(\mathbb{E}\left\|\int_{s+k\delta}^{s+(k+1)\delta} f_\varepsilon(\sigma, \bar{Y}(s+k\delta)) - \bar{f}(\bar{Y}(s+k\delta))\mathrm{d}\sigma\right\|^2\right)^{\frac{1}{2}} C_T\left(1 + \mathbb{E}\|\zeta_s\|^2\right)^{\frac{1}{2}} \\
&\quad + C_T\left(1 + \mathbb{E}\|\zeta_s\|^2\right)\delta \\
&\le \frac{4T}{\delta}\max_{0\le k\le T(\delta)-1}\delta\delta_f\left(\frac{\delta}{\varepsilon}\right)\left(\mathbb{E}\left(1 + \|\bar{Y}(s+k\delta)\|\right)^2\right)^{\frac{1}{2}} C_T\left(1 + \mathbb{E}\|\zeta_s\|^2\right)^{\frac{1}{2}} + C_T\left(1 + \mathbb{E}\|\zeta_s\|^2\right)\delta \\
&\le C_T\left(1 + \mathbb{E}\|\zeta_s\|^2\right)\left(\delta_f\left(\frac{\delta}{\varepsilon}\right) + \delta\right).
\end{aligned}
$$

Combining (3.19) and (3.20), we deduce

$$\mathcal{J}_1^3 \leq C_T\left(1+\mathbb{E}\|\zeta_s\|^2\right)\left(\delta^{\frac{1}{4}}+\delta_f\left(\frac{\delta}{\varepsilon}\right)\right). \tag{3.21}$$

Therefore, (3.16)–(3.18) and (3.21) yield

$$\mathcal{J}_1 \leq 4L_f\int_s^{s+T}\mathbb{E}\sup_{s\leq\tau\leq\sigma}\|Y_\varepsilon(\tau)-\bar{Y}(\tau)\|^2\mathrm{d}\sigma + C_T\left(1+\mathbb{E}\|\zeta_s\|^2\right)\left(\delta^{\frac{1}{4}}+\delta_f\left(\frac{\delta}{\varepsilon}\right)\right). \tag{3.22}$$

Now we estimate $\mathcal{J}_2$.

$$\begin{aligned}
\mathcal{J}_2 &:= C\mathbb{E}\int_s^{s+T}\|g_\varepsilon(\sigma,Y_\varepsilon(\sigma))-\bar{g}(\bar{Y}(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\mathrm{d}\sigma\\
&\leq C\mathbb{E}\int_s^{s+T}\|g_\varepsilon(\sigma,Y_\varepsilon(\sigma))-g_\varepsilon(\sigma,\bar{Y}(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\mathrm{d}\sigma\\
&\quad + C\mathbb{E}\int_s^{s+T}\|g_\varepsilon(\sigma,\bar{Y}(\sigma))-\bar{g}(\bar{Y}(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\mathrm{d}\sigma\\
&\leq C\mathbb{E}\int_s^{s+T}\|Y_\varepsilon(\sigma)-\bar{Y}(\sigma)\|^2\mathrm{d}\sigma\\
&\quad + C\mathbb{E}\int_s^{s+T}\|g_\varepsilon(\sigma,\bar{Y}(\sigma))-\bar{g}(\bar{Y}(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\mathrm{d}\sigma\\
&\leq C\int_s^{s+T}\mathbb{E}\sup_{s\leq\tau\leq\sigma}\|Y_\varepsilon(\tau)-\bar{Y}(\tau)\|^2\mathrm{d}\sigma+\mathcal{J}_2^2.
\end{aligned}$$

Recall that $T(\delta):=\left[\frac{T}{\delta}\right]$. For $\mathcal{J}_2^2 := C\mathbb{E}\int_s^{s+T}\|g_\varepsilon(\sigma,\bar{Y}(\sigma))-\bar{g}(\bar{Y}(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\mathrm{d}\sigma$, it follows from (H2) and (G2) that

$$\begin{aligned}
&\mathbb{E}\int_s^{s+T}\|g_\varepsilon(\sigma,\bar{Y}(\sigma))-\bar{g}(\bar{Y}(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\mathrm{d}\sigma\\
&\leq CE\int_s^{s+T}\|g_\varepsilon(\sigma,\bar{Y}(\sigma))-g_\varepsilon(\sigma,\tilde{Y}(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\mathrm{d}\sigma\\
&\quad + C\mathbb{E}\int_s^{s+T}\|g_\varepsilon(\sigma,\tilde{Y}(\sigma))-\bar{g}(\tilde{Y}(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\mathrm{d}\sigma\\
&\quad + C\mathbb{E}\int_s^{s+T}\|\bar{g}(\tilde{Y}(\sigma))-\bar{g}(\bar{Y}(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\mathrm{d}\sigma\\
&\leq C\mathbb{E}\int_s^{s+T}L_g^2\|\bar{Y}(\sigma)-\tilde{Y}(\sigma)\|^2\mathrm{d}\sigma + C\mathbb{E}\int_s^{s+T}\|g_\varepsilon(\sigma,\tilde{Y}(\sigma))-\bar{g}(\tilde{Y}(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\mathrm{d}\sigma\\
&\leq C_T\left(1+\mathbb{E}\|\zeta_s\|^2\right)\delta^{\frac{1}{2}} + C\mathbb{E}\sum_{k=0}^{T(\delta)-1}\int_{s+k\delta}^{s+(k+1)\delta}\|g_\varepsilon(\sigma,\bar{Y}(s+k\delta))-\bar{g}(\bar{Y}(s+k\delta))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\mathrm{d}\sigma\\
&\quad + C\mathbb{E}\int_{s+T(\delta)\delta}^{s+T}\|g_\varepsilon(\sigma,\bar{Y}(s+T(\delta)\delta))-\bar{g}(\bar{Y}(s+T(\delta)\delta))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\mathrm{d}\sigma\\
&\leq C_T\left(1+\mathbb{E}\|\zeta_s\|^2\right)\delta^{\frac{1}{2}} + C\sum_{k=0}^{T(\delta)-1}\delta\delta_g\left(\frac{\delta}{\varepsilon}\right)\mathbb{E}\left(1+\|\bar{Y}(s+k\delta)\|^2\right)\\
&\leq C_T\left(1+\mathbb{E}\|\zeta_s\|^2\right)\left(\delta^{\frac{1}{2}}+\delta_g\left(\frac{\delta}{\varepsilon}\right)\right).
\end{aligned}$$

Therefore,

$$\mathcal{J}_2 \le C \int_s^{s+T} \mathbb{E} \sup_{s\le\tau\le\sigma} \|Y_\varepsilon(\tau) - \bar{Y}(\tau)\|^2 \mathrm{d}\sigma + C_T \left(1 + \mathbb{E}\|\zeta_s\|^2\right) \left(\delta^{\frac{1}{2}} + \delta_g\left(\frac{\delta}{\varepsilon}\right)\right). \tag{3.23}$$

Combining (3.15), (3.22) and (3.23), we get

$$\begin{aligned} &\mathbb{E} \sup_{s\le t\le s+T} \|Y_\varepsilon(t) - \bar{Y}(t)\|^2 \\ &\le 2\mathbb{E}\|\zeta_s^\varepsilon - \zeta_s\|^2 + C\int_s^{s+T} \mathbb{E} \sup_{s\le\tau\le\sigma} \|Y_\varepsilon(\tau) - \bar{Y}(\tau)\|^2 \mathrm{d}\sigma \\ &\quad + C_T \left(1 + \mathbb{E}\|\zeta_s\|^2\right) \left(\delta^{\frac{1}{4}} + \delta_f\left(\frac{\delta}{\varepsilon}\right) + \delta_g\left(\frac{\delta}{\varepsilon}\right)\right). \end{aligned}$$

It follows from Gronwall's lemma that

$$\begin{aligned} &\mathbb{E} \sup_{s\le t\le s+T} \|Y_\varepsilon(t) - \bar{Y}(t)\|^2 \\ &\le \left[2\mathbb{E}\|\zeta_s^\varepsilon - \zeta_s\|^2 + C_T\left(1 + \mathbb{E}\|\zeta_s\|^2\right)\left(\delta^{\frac{1}{4}} + \delta_f\left(\frac{\delta}{\varepsilon}\right) + \delta_g\left(\frac{\delta}{\varepsilon}\right)\right)\right] \exp\{CT\}. \end{aligned} \tag{3.24}$$

Taking $\delta = \sqrt{\varepsilon}$ and letting $\varepsilon \to 0$ in (3.24), we have

$$\lim_{\varepsilon\to 0} \mathbb{E} \sup_{s\le t\le s+T} \|Y_\varepsilon(t) - \bar{Y}(t)\|^2 = 0.$$

□

4. The second Bogolyubov theorem

In this section, we establish the second Bogolyubov theorem for stochastic CGL equations. Firstly, we show that there exists a unique $L^2(\Omega, \mathbb{P}; L^2(\mathbb{T}^d))$-bounded solution $u_\varepsilon(t), t \in \mathbb{R}$ of (3.2) which inherits the recurrent properties (in particular, periodic, quasi-periodic, almost periodic, almost automorphic, Birkhoff recurrent, Levitan almost periodic, almost recurrent, pseudo-periodic, pseudo-recurrent, Poisson stable) of the coefficients in distribution sense for any $0 < \varepsilon \le 1$. See Appendix A.1–A.2 for more details about these recurrent functions. This result is interesting on its own rights and has been studied extensively; see e.g. [7, 9, 12, 30] and references therein. Without loss of generality, the proof is only given when $\varepsilon = 1$.

Lemma 4.1. *Assume that* (H1)–(H2) *hold and* $\lambda_* - \lambda_f - \frac{L_g^2}{2} > 0$. *Fix* $s \in \mathbb{R}$. *Let* $\zeta_s \in L^2(\Omega, \mathbb{P}; L^2(\mathbb{T}^d))$ *and* $u(t, s, \zeta_s)$, $t \ge s$ *be the solution to*

$$\begin{cases} \mathrm{d}u(t) = \left[(1+i\alpha)\Delta u(t) - (1+i\beta)|u(t)|^2u(t) + f(t, u(t))\right]\mathrm{d}t + g(t, u(t))\mathrm{d}W(t) \\ u(s) = \zeta_s. \end{cases}$$

Then for any $\eta \in (0, 2\lambda_* - 2\lambda_f - L_g^2)$ *there exist constants* $p' > 1$ *and* $M_1 > 0$ *such that*

$$\mathbb{E}\|u(t, s, \zeta_s)\|^{2p} \le \mathrm{e}^{-\eta p(t-s)}\mathbb{E}\|\zeta_s\|^{2p} + M_1, \tag{4.1}$$

where $p \in [1, p']$ *is an arbitrary constant. Moreover, if* (H3) *hold and* $\zeta_s \in L^2(\Omega, \mathbb{P}; H^1)$, *then there exists a constant* $M_2 > 0$ *such that*

$$\mathbb{E}\|u(t, s, \zeta_s)\|_{H^1}^2 \le M_2\left(\mathrm{e}^{-\eta(t-s)}\mathbb{E}\|\zeta_s\|_{H^1}^2 + 1\right). \tag{4.2}$$

Here M_1 *and* M_2 *depend only on* η *and* K.

Proof. By the product rule, Itô's formula and Young's inequality, we have

$$
\begin{aligned}
&\mathbb{E}\left(\mathrm{e}^{\eta p(t-s)}\|u(t,s,\zeta_s)\|^{2p}\right)\\
&= \mathbb{E}\|\zeta_s\|^{2p} + \int_s^t \eta p \mathrm{e}^{\eta p(\sigma-s)}\mathbb{E}\|u(\sigma,s,\zeta_s)\|^{2p}\mathrm{d}\sigma\\
&\quad + p\mathbb{E}\int_s^t \|u(\sigma,s,\zeta_s)\|^{2p-2}\mathrm{e}^{\eta p(\sigma-s)}\Big(2\langle(1+i\alpha)\Delta u(\sigma,s,\zeta_s), u(\sigma,s,\zeta_s)\rangle\\
&\qquad - 2\langle(1+i\beta)|u(\sigma,s,\zeta_s)|^2 u(\sigma,s,\zeta_s), u(\sigma,s,\zeta_s)\rangle\\
&\qquad + 2\langle f(\sigma,u(\sigma,s,\zeta_s)), u(\sigma,s,\zeta_s)\rangle + \|g(\sigma,u(\sigma,s,\zeta_s))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\Big)\mathrm{d}\sigma\\
&\quad + 2p(p-1)\mathbb{E}\int_s^t \mathrm{e}^{\eta p(\sigma-s)}\|u(\sigma,s,\zeta_s)\|^{2p-4}\|\left(g(\sigma,u(\sigma,s,\zeta_s))\right)^* u(\sigma,s,\zeta_s)\|_U^2\mathrm{d}\sigma\\
&\le \mathbb{E}\|\zeta_s\|^{2p} + \int_s^t \eta p \mathrm{e}^{\eta p(\sigma-s)}\mathbb{E}\|u(\sigma,s,\zeta_s)\|^{2p}\mathrm{d}\sigma\\
&\quad + p\mathbb{E}\int_s^t \|u(\sigma,s,\zeta_s)\|^{2p-2}\mathrm{e}^{\eta p(\sigma-s)}\left(-(2\lambda_* - 2\lambda_f - L_g^2 - \varepsilon - \varepsilon L_g^2)\|u(\sigma,s,\zeta_s)\|^2 + C_\varepsilon\right)\mathrm{d}\sigma\\
&\quad + 2p(p-1)\mathbb{E}\int_s^t \mathrm{e}^{\eta p(\sigma-s)}\|u(\sigma,s,\zeta_s)\|^{2p-2}\left[(L_g^2+\varepsilon L_g^2)\|u(\sigma,s,\zeta_s)\|^2 + C_\varepsilon\right]\mathrm{d}\sigma.
\end{aligned}
$$

When ε is small enough, it follows from Young's inequality and Gronwall's lemma that there exist constants $p' > 1$ and $M_1 > 0$ such that

$$\mathbb{E}\|u(t,s,\zeta_s)\|^{2p} \le \mathrm{e}^{-\eta p(t-s)}\mathbb{E}\|\zeta_s\|^{2p} + M_1$$

for all $p \in [1, p']$.

Notice that

$$\langle(1+i\beta)|u|^2 u, \Delta u\rangle \le 0$$

for all $u \in H^1$. Let P_n be the projection mapping from $L^2(\mathbb{T}^d)$ to $H_n := \operatorname{span}\{e_1, ..., e_n\}$ and $u^n(t), t \ge s$ the solution of (A.2). Recall that $\{e_i, i \in \mathbb{N}\} \subset H^1$ is the eigenfunctions of $-\Delta$ forming an orthonormal basis of $L^2(\mathbb{T}^d)$. Then by Itô's formula, integration by parts and Young's inequality, for small ε we have

$$
\begin{aligned}
&\mathbb{E}\|u^n(t)\|^2_{H^1}\\
&= \mathbb{E}\|P_n\zeta_s\|^2_{H^1} + \mathbb{E}\int_s^t \Big(2\langle(1+i\alpha)\Delta u^n(\sigma) - (1+i\beta)P_n|u^n(\sigma)|^2 u^n(\sigma), u^n(\sigma)\rangle_{H^1}\\
&\qquad + 2\langle P_n f(\sigma,u^n(\sigma)), u^n(\sigma)\rangle_{H^1} + \|P_n g(\sigma,u^n(\sigma))\|^2_{L_2(U,H^1)}\Big)\mathrm{d}\sigma\\
&= \mathbb{E}\|P_n\zeta_s\|^2_{H^1} + \mathbb{E}\int_s^t \Big(2\langle-(1+i\alpha)\Delta u^n(\sigma) + (1+i\beta)P_n|u^n(\sigma)|^2 u^n(\sigma), \Delta u^n(\sigma)\rangle\\
&\qquad - 2\langle P_n f(\sigma,u^n(\sigma)), \Delta u^n(\sigma)\rangle + \|P_n g(\sigma,u^n(\sigma))\|^2_{L_2(U,H^1)}\Big)\mathrm{d}\sigma\\
&\le \mathbb{E}\|\zeta_s\|^2_{H^1} + \mathbb{E}\int_s^t \Big(-2\|\Delta u^n(\sigma)\|^2 - 2\langle f(\sigma,u^n(\sigma)), \Delta u^n(\sigma)\rangle - \|g(\sigma,0)\|^2_{L_2(U,H^1)}\\
&\qquad + \|g(\sigma,u^n(\sigma)) - g(\sigma,0)\|^2_{L_2(U,H^1)} + 2\langle g(\sigma,u^n(\sigma)), g(\sigma,0)\rangle_{L_2(U,H^1)}\Big)\mathrm{d}\sigma\\
&\le \mathbb{E}\|\zeta_s\|^2_{H^1} + \mathbb{E}\int_s^t \Big(-2\|\Delta u^n(\sigma)\|^2 + \varepsilon\|\Delta u^n(\sigma)\|^2 + C_\varepsilon\|f(\sigma,u^n(\sigma))\|^2\\
&\qquad + L_g^2\|u^n(\sigma)\|^2_{H^1} + \varepsilon\|g(\sigma,u^n(\sigma))\|^2_{L_2(U,H^1)} + C_\varepsilon K^2\Big)\mathrm{d}\sigma
\end{aligned}
$$

$$\leq \mathbb{E}\|\zeta_s\|_{H^1}^2 + \mathbb{E}\int_s^t \Big((-2+\varepsilon)\|\Delta u^n(\sigma)\|^2 + C_\varepsilon L_f^2 \|u^n(\sigma)\|^2 + \left(L_g^2 + \varepsilon L_g^2\right)\|u^n(\sigma)\|_{H^1}^2 + C_\varepsilon \Big) \mathrm{d}\sigma$$

$$\leq \mathbb{E}\|\zeta_s\|_{H^1}^2 + \mathbb{E}\int_s^t \left(-\left(2\lambda_* - \varepsilon\lambda_* - L_g^2 - \varepsilon L_g^2\right)\|u^n(\sigma)\|_{H^1}^2 + C_\varepsilon \|u^n(\sigma)\|^2 + C_\varepsilon \right) \mathrm{d}\sigma.$$

Set $\tilde{\eta} := 2\lambda_* - \varepsilon\lambda_* - L_g^2 - \varepsilon L_g^2$. Then we obtain

$$\mathbb{E}\|u^n(t)\|_{H^1}^2 \leq \mathrm{e}^{-\tilde{\eta}(t-s)}\mathbb{E}\|\zeta_s\|_{H^1}^2 + \int_s^t C_\varepsilon \mathrm{e}^{-\tilde{\eta}(t-\sigma)}\mathbb{E}\|u^n(\sigma)\|^2 \mathrm{d}\sigma + \frac{C_\varepsilon}{\tilde{\eta}} - \frac{C_\varepsilon}{\tilde{\eta}}\mathrm{e}^{-\tilde{\eta}(t-s)}. \tag{4.3}$$

By (A.7) we have

$$\mathbb{E}\|u(t,s,\zeta_s)\|_{H^1}^2 \leq \liminf_{n\to\infty} \mathbb{E}\|u^n(t)\|_{H^1}^2.$$

Therefore, taking ε small enough such that $\tilde{\eta} > \eta$, by Lebesgue's dominated convergence theorem and (4.1) we get

$$\begin{aligned}
\mathbb{E}\|u(t,s,\zeta_s)\|_{H^1}^2 &\leq \mathrm{e}^{-\tilde{\eta}(t-s)}\mathbb{E}\|\zeta_s\|_{H^1}^2 + \int_s^t C\mathrm{e}^{-\tilde{\eta}(t-\sigma)}\left(\mathbb{E}\|\zeta_s\|^2 \mathrm{e}^{-\eta(\sigma-s)} + 1\right)\mathrm{d}\sigma + \frac{C}{\tilde{\eta}} \\
&\leq \mathrm{e}^{-\tilde{\eta}(t-s)}\mathbb{E}\|\zeta_s\|_{H^1}^2 + C\mathbb{E}\|\zeta_s\|^2 \mathrm{e}^{-\tilde{\eta}t+\eta s}\int_s^t \mathrm{e}^{(\tilde{\eta}-\eta)\sigma}\mathrm{d}\sigma + \frac{C}{\tilde{\eta}} \\
&\leq \mathrm{e}^{-\tilde{\eta}(t-s)}\mathbb{E}\|\zeta_s\|_{H^1}^2 + C\mathbb{E}\|\zeta_s\|^2 \mathrm{e}^{-\tilde{\eta}t+\eta s}\frac{1}{\tilde{\eta}-\eta}\mathrm{e}^{(\tilde{\eta}-\eta)t} + \frac{C}{\tilde{\eta}} \\
&\leq M_2\left(\mathbb{E}\|\zeta_s\|_{H^1}^2 \mathrm{e}^{-\eta(t-s)} + 1\right).
\end{aligned}$$

□

Proposition 4.2. *Consider equation* (2.1). *Suppose that conditions* (H1)–(H2) *hold and* $\lambda_* - \lambda_f - \frac{L_g^2}{2} > 0$, *then there is a unique* $L^2(\Omega,\mathbb{P};L^2(\mathbb{T}^d))$*-bounded solution* $u(t)$, $t\in\mathbb{R}$ *to equation* (2.1). *Moreover, the mapping* $\widehat{\mu}:\mathbb{R}\to Pr_2(L^2(\mathbb{T}^d))$, *defined by* $\widehat{\mu}(t) := \mathbb{P}\circ[u(t)]^{-1}$, *is unique with flow property, i.e.* $\mu(t,s,\widehat{\mu}(s)) = \widehat{\mu}(t)$ *for all* $t\geq s$. *Moreover, if* (H3) *hold then*

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|u(t)\|_{H^1}^2 < \infty. \tag{4.4}$$

Here $\mu(t,s,\mu_0)$ *denotes the distribution of* $u(t,s,\zeta_s)$ *on* $L^2(\mathbb{T}^d)$, *with* $\mu_0 = \mathbb{P}\circ\zeta_s^{-1}$.

Proof. Let $u_n(t) := u(t,-n,0)$ for all $n\in\mathbb{N}^+$. For $t\geq -m\geq -n$, by (H1)–(H2), Itô's formula and the product rule, we obtain

$$\begin{aligned}
&\mathbb{E}\left(\mathrm{e}^{(2\lambda_*-2\lambda_f-L_g^2)(t+m)}\|u_n(t)-u_m(t)\|^2\right) \\
&= \mathbb{E}\int_{-m}^t (2\lambda_*-2\lambda_f-L_g^2)\mathrm{e}^{(2\lambda_*-2\lambda_f-L_g^2)(\sigma+m)}\|u_n(\sigma)-u_m(\sigma)\|^2\mathrm{d}\sigma \\
&\quad + \mathbb{E}\int_{-m}^t \mathrm{e}^{(2\lambda_*-2\lambda_f-L_g^2)(\sigma+m)}\Big[2\langle(1+i\alpha)\Delta\left(u_n(\sigma)-u_m(\sigma)\right), u_n(\sigma)-u_m(\sigma)\rangle \\
&\quad + 2\langle -(1+i\beta)\left(|u_n(\sigma)|^2 u_n(\sigma) - |u_m(\sigma)|^2 u_m(\sigma)\right), u_n(\sigma)-u_m(\sigma)\rangle \\
&\quad + 2\langle f(\sigma,u_n(\sigma)) - f(\sigma,u_m(\sigma)), u_n(\sigma)-u_m(\sigma)\rangle \\
&\quad + \|g(\sigma,u_n(\sigma)) - g(\sigma,u_m(\sigma))\|_{L_2(U,L^2(\mathbb{T}^d))}^2\Big]\mathrm{d}\sigma + \mathbb{E}\|u(-m,-n,0)\|^2 \\
&\leq \mathbb{E}\|u(-m,-n,0)\|^2.
\end{aligned}$$

In view of (4.1), we deduce

$$\mathbb{E}\|u_n(t)-u_m(t)\|^2 \le M_1 \mathrm{e}^{-(2\lambda_*-2\lambda_f-L_g^2)(t+m)}.$$

Letting $n>m$, $m\to\infty$, we have

$$\mathbb{E}\|u_n(t)-u_m(t)\|^2 \to 0.$$

Since $L^2(\Omega,\mathbb{P};L^2(\mathbb{T}^d))$ is complete, there exists a process $u(t)$, $t\in\mathbb{R}$ such that

$$u_n(t)\to u(t) \quad \text{in } L^2(\Omega,\mathbb{P};L^2(\mathbb{T}^d)) \tag{4.5}$$

for any $t\in\mathbb{R}$. And it follows from (4.1) that $\sup\limits_{t\in\mathbb{R}}\mathbb{E}\|u(t)\|^2\le M_1$. By (4.2), we have

$$\sup_{t\in\mathbb{R}}\mathbb{E}\|u_n(t)\|_{H^1}^2 \le M_2.$$

Then there exists a subsequence of $\{u_n(t)\}$ which we still denote by $\{u_n(t)\}$ such that $u(t,-n,0)\to u(t)$ weakly in $L^2(\Omega,\mathbb{P};H^1)$ for all $t\in\mathbb{R}$. Therefore, we have

$$\sup_{t\in\mathbb{R}}\mathbb{E}\|u(t)\|_{H^1}^2 \le M_2.$$

Similar to the proof of Theorem 3.3 in Appendix A.4, we can also prove that the limit process $u(\cdot)$ in (4.5) is a solution to equation (2.1).

Then we prove the uniqueness of $L^2(\Omega,\mathbb{P};L^2(\mathbb{T}^d))$-bounded solution. Suppose that $u^1(\cdot)$ and $u^2(\cdot)$ are two $L^2(\Omega,\mathbb{P};L^2(\mathbb{T}^d))$-bounded solutions to equation (2.1). By (H1), (H2) and $\lambda_*-\lambda_f-\frac{L_g^2}{2}>0$ we have

$$\mathbb{E}\|u^1(t)-u^2(t)\|^2 \le \mathrm{e}^{-(2\lambda_*-2\lambda_f-L_g^2)(t+n)}\mathbb{E}\|u^1(-n)-u^2(-n)\|^2 \to 0 \quad \text{as } n\to\infty.$$

Note that

$$\sup_{t\in\mathbb{R}}\int_{L^2(\mathbb{T}^d)}\|x\|^2\widehat{\mu}(t)(\mathrm{d}x)=\sup_{t\in\mathbb{R}}\mathbb{E}\|u(t)\|^2<\infty.$$

The goal next is to prove that $\widehat{\mu}\in Pr_2(L^2(\mathbb{T}^d))$ is unique with flow property. In view of the Chapman-Kolmogorov equation, we have

$$\mu(t,s,\mathcal{L}(u_n(s)))=\mathcal{L}(u_n(t)).$$

Then according to the Feller property, we get

$$\mu(t,s,\widehat{\mu}(s))=\widehat{\mu}(t).$$

Suppose that $\mu_1,\mu_2\in Pr_2(L^2(\mathbb{T}^d))$ satisfy flow property, let $\zeta_{n,1}$ and $\zeta_{n,2}$ be random variables with distributions $\mu_1(-n)$ and $\mu_2(-n)$ respectively. Then consider solutions $u(t,-n,\zeta_{n,1})$ and $u(t,-n,\zeta_{n,2})$ on $[-n,\infty)$, we have

$$\begin{aligned}
&W_2(\mu_1(t),\mu_2(t))\\
&=W_2(\mu(t,-n,\mu_1(-n)),\mu(t,-n,\mu_2(-n)))\\
&\le\left(\mathbb{E}\|u(t,-n,\zeta_{n,1})-u(t,-n,\zeta_{n,2})\|^2\right)^{1/2}\\
&\le \mathrm{e}^{-(\lambda_*-\lambda_f-\frac{L_g^2}{2})(t+n)}\left(\mathbb{E}\|\zeta_{n,1}-\zeta_{n,2}\|^2\right)^{1/2}\to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

Thus, $\mu_1(t)=\mu_2(t)$ for all $t\in\mathbb{R}$.

The proof is complete. □

Remark 4.3. Note that the above $L^2(\Omega,\mathbb{P};L^2(\mathbb{T}^d))$-bounded solution is T-periodic provided f and g are T-periodic. The proof is similar to Theorem 4.1 in [9].

Definition 4.4 (See [17])**.** Let $s \in \mathbb{R}$. We say that a solution $u(t), t \geq s$ of equation (2.1) is *stable in square-mean sense*, if for each $\epsilon > 0$, there exists $\delta > 0$ such that for all $t \geq s$

$$\mathbb{E}\|u(t,s,\zeta_s) - u(t)\|^2 < \epsilon,$$

whenever $\mathbb{E}\|\zeta_s - u(s)\|^2 < \delta$. The solution $u(t), t \geq s$ is said to be *asymptotically stable in square-mean sense* if it is stable in square-mean sense and

$$\lim_{t\to\infty} \mathbb{E}\|u(t,s,\zeta_s) - u(t)\|^2 = 0. \tag{4.6}$$

We say $u(t), t \geq s$ is *globally asymptotically stable in square-mean sense* provided (4.6) holds for any $\zeta_s \in L^2(\Omega,\mathbb{P};L^2(\mathbb{T}^d))$.

Now we prove that the $L^2(\Omega,\mathbb{P};L^2(\mathbb{T}^d))$-bounded solution of equation (2.1) are globally asymptotically stable in square-mean sense.

Proposition 4.5. *Consider equation* (2.1)*. Suppose that* (H1)–(H2) *hold. Assume further that* $\lambda_* - \lambda_f - \frac{L_g^2}{2} > 0$*. Then the unique* $L^2(\Omega,\mathbb{P};L^2(\mathbb{T}^d))$*-bounded solution* $u(\cdot)$ *of equation* (2.1) *is globally asymptotically stable in square-mean sense. Moreover, let* $s \in \mathbb{R}$*. Then for any* $t \geq s$ *and* $\zeta_s \in L^2(\Omega,\mathbb{P};L^2(\mathbb{T}^d))$ *we have*

$$\mathbb{E}\|u(t,s,\zeta_s) - u(t)\|^2 \leq \mathrm{e}^{-(2\lambda_* - 2\lambda_f - L_g^2)(t-s)}\mathbb{E}\|\zeta_s - u(s)\|^2. \tag{4.7}$$

Proof. In view of Itô's formula, the product rule and (H1)–(H2), we obtain

$$\begin{aligned}
&\mathbb{E}\left(\mathrm{e}^{(2\lambda_* - 2\lambda_f - L_g^2)(t-s)}\|u(t,s,\zeta_s) - u(t)\|^2\right)\\
&= \mathbb{E}\|\zeta_s - u(s)\|^2 + \int_s^t (2\lambda_* - 2\lambda_f - L_g^2)\mathrm{e}^{(2\lambda_* - 2\lambda_f - L_g^2)(\sigma-s)}\mathbb{E}\|u(\sigma,s,\zeta_s) - u(\sigma)\|^2\mathrm{d}\sigma\\
&\quad + \mathbb{E}\int_s^t \mathrm{e}^{(2\lambda_* - 2\lambda_f - L_g^2)(\sigma-s)}\Big(2\langle(1+i\alpha)\Delta\left(u(\sigma,s,\zeta_s) - u(\sigma)\right), u(\sigma,s,\zeta_s) - u(\sigma)\rangle\\
&\qquad + 2\langle -(1+i\beta)\left(|u(\sigma,s,\zeta_s)|^2 u(\sigma,s,\zeta_s) - |u(\sigma)|^2 u(\sigma)\right), u(\sigma,s,\zeta_s) - u(\sigma)\rangle\\
&\qquad + 2\langle f(\sigma,u(\sigma,s,\zeta_s)) - f(\sigma,u(\sigma)), u(\sigma,s,\zeta_s) - u(\sigma)\rangle\\
&\qquad + \|g(\sigma,u(\sigma,s,\zeta_s)) - g(\sigma,u(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\Big)\mathrm{d}\sigma\\
&\leq \mathbb{E}\|\zeta_s - u(s)\|^2.
\end{aligned}$$

It follows that

$$\mathbb{E}\|u(t,s,\zeta_s) - u(t)\|^2 \leq \mathrm{e}^{-(2\lambda_* - 2\lambda_f - L_g^2)(t-s)}\mathbb{E}\|\zeta_s - u(s)\|^2$$

for all $t \geq s$. □

Let $(\mathcal{X},\rho)$ and $(\mathcal{Y},\rho_1)$ be two complete metric spaces. For given $\phi \in C(\mathbb{R}\times\mathcal{X},\mathcal{Y})$. We write $\mathfrak{N}_\phi$ (respectively, $\mathfrak{M}_\phi$) to mean the space of all sequences $\{t_n\}_{n=1}^\infty$ such that $\phi(\cdot + t_n,\cdot)$ converges, as $n\to\infty$, to $\phi(\cdot,\cdot)$ (respectively, $\phi(\cdot + t_n,\cdot)$ converges) uniformly w.r.t. t in any compact interval and x in any bounded subset of $\mathcal{X}$.

Definition 4.6. Let $\varphi(t), t \in \mathbb{R}$ be a solution of equation (2.1). Then φ is called *compatible* (respectively, *strongly compatible*) *in distribution* if the following conditions are fulfilled:

(i) there exists a bounded closed subset $\mathcal{Q} \subset L^2(\Omega,\mathbb{P};L^2(\mathbb{T}^d))$ such that $\varphi(\mathbb{R}) \subseteq \mathcal{Q}$;

(ii) $\mathfrak{N}_{(f,g)} \subseteq \tilde{\mathfrak{N}}_\varphi$ (respectively, $\mathfrak{M}_{(f,g)} \subseteq \tilde{\mathfrak{M}}_\varphi$), where $\tilde{\mathfrak{N}}_\varphi$ (respectively, $\tilde{\mathfrak{M}}_\varphi$) means the set of all sequences $\{t_n\} \subset \mathbb{R}$ such that the sequence $\{\varphi(\cdot + t_n)\}$ converges to $\varphi(\cdot)$ (respectively, $\{\varphi(\cdot + t_n)\}$ converges) in distribution uniformly on any compact interval.

Then we show that the $L^2(\Omega,\mathbb{P};L^2(\mathbb{T}^d))$-bounded solution for equation (2.1) is strongly compatible in distribution. Therefore, we need the following condition.

(H4) f and g are continuous in t uniformly with respect to x on each bounded subset $Q\subset L^2(\mathbb{T}^d)$.

Remark 4.7. (i) If f and g satisfy (H1), (H2) and (H3), then every pair of functions $\left(\tilde{f},\tilde{g}\right)\in\mathcal{H}(f,g)$ possess the same property with the same constants, where

$$\mathcal{H}(f,g):=\overline{\{(f^\tau,g^\tau):\tau\in\mathbb{R}\}}.$$

Here f^τ is the τ-translation of f defined by $f^\tau(t,x):=f(t+\tau,x)$ for all $t\in\mathbb{R}$ and $x\in L^2(\mathbb{T}^d)$.

(ii) If f and g satisfy the conditions (H1)–(H2) and (H4), then $f\in BUC(\mathbb{R}\times L^2(\mathbb{T}^d),L^2(\mathbb{T}^d))$, $g\in BUC(\mathbb{R}\times L^2(\mathbb{T}^d),L_2(U,L^2(\mathbb{T}^d)))$ and $\mathcal{H}(f,g)\subset BUC(\mathbb{R}\times L^2(\mathbb{T}^d),L^2(\mathbb{T}^d))\times BUC(\mathbb{R}\times L^2(\mathbb{T}^d),L_2(U,L^2(\mathbb{T}^d)))$. See Appendix A.3 for more details about the space BUC.

Lemma 4.8. *Suppose that f_n, f, g_n, g satisfy* (H1)–(H2) *with the same constants. Fix $s\in\mathbb{R}$. Let u_n be the solution of*

$$\begin{cases} \mathrm{d}u(t)=\left[(1+i\alpha)\Delta u(t)-(1+i\beta)|u(t)|^2u(t)+f_n(t,u(t))\right]\mathrm{d}t+g_n(t,u(t))\mathrm{d}W(t)\\ u(s)=\zeta_n^s\end{cases}$$

and u be the solution to

$$\begin{cases} \mathrm{d}u(t)=\left[(1+i\alpha)\Delta u(t)-(1+i\beta)|u(t)|^2u(t)+f(t,u(t))\right]\mathrm{d}t+g(t,u(t))\mathrm{d}W(t)\\ u(s)=\zeta^s.\end{cases}$$

Assume further that

(i) $\lim_{n\to\infty}f_n(t,x)=f(t,x)$ *for all* $t\in\mathbb{R}$ *and* $x\in L^2(\mathbb{T}^d)$*;*

(ii) $\lim_{n\to\infty}g_n(t,x)=g(t,x)$ *for all* $t\in\mathbb{R}$ *and* $x\in L^2(\mathbb{T}^d)$.

Then we have the following conclusions:

(i) *If* $\lim_{n\to\infty}\mathbb{E}\|\zeta_n^s-\zeta^s\|^2=0$, *then* $\lim_{n\to\infty}\mathbb{E}\sup_{s\le t\le s+T}\|u_n(t)-u(t)\|^2=0$ *for any* $T>0$*;*

(ii) *If* $\lim_{n\to\infty}\zeta_n^s=\zeta^s$ *in probability, then*

$$\lim_{n\to\infty}\sup_{t\in[s,s+T]}\|u_n(t)-u(t)\|=0\quad\textit{in probability}$$

for all $T>0$*;*

(iii) *If* $\lim_{n\to\infty}W_2(\mathcal{L}(\zeta_n^s),\mathcal{L}(\zeta^s))=0$, *then*

$$\lim_{n\to\infty}\sup_{s\le t\le s+T}W_2(\mathcal{L}(u_n(t)),\mathcal{L}(u(t)))=0$$

for all $T>0$.

Proof. The proof is similar to the proof of Theorem 3.1 in [9]. □

With the help of the above estimates, we can prove the compatible solution in distribution, the second Bogolyubov theorem and global averaging principle for stochastic CGL equations; the idea is similar to [10]. For the reader's convenience, we give detailed proof in what follows.

Theorem 4.9. *Suppose that* (H1)–(H3) *hold,* $\lambda_*-\lambda_f-\frac{L_g^2}{2}>0$. *Then the unique* $L^2(\Omega,\mathbb{P};L^2(\mathbb{T}^d))$-*bounded solution* $u(\cdot)$ *of* (2.1) *is strongly compatible in distribution.*

Proof. According to Remark 4.7, we have $\mathcal{H}(f,g) \subset BUC(\mathbb{R}\times L^2(\mathbb{T}^d), L^2(\mathbb{T}^d)) \times BUC(\mathbb{R}\times L^2(\mathbb{T}^d), L_2(U, L^2(\mathbb{T}^d)))$. Take $\{t_n\} \in \mathfrak{M}_{(f,g)}$, then there exists $(\tilde f, \tilde g) \in \mathcal{H}(f,g)$ such that

$$\lim_{n\to\infty} \sup_{|t|\le l, \|x\|\le r} \|f(t+t_n,x) - \tilde f(t,x)\| = 0,$$

$$\lim_{n\to\infty} \sup_{|t|\le l, \|x\|\le r} \|g(t+t_n,x) - \tilde g(t,x)\|_{L_2(U,L^2(\mathbb{T}^d))} = 0,$$

for any $l>0$ and $r>0$. Let u_n be the unique $L^2(\Omega,\mathbb{P};L^2(\mathbb{T}^d))$-bounded solution of

$$\mathrm{d}u(t) = \left[(1+i\alpha)\Delta u(t) - (1+i\beta)|u(t)|^2 u(t) + f(t+t_n, u(t))\right]\mathrm{d}t + g(t+t_n,u(t))\mathrm{d}W(t)$$

and $\tilde u$ be the unique $L^2(\Omega,\mathbb{P};L^2(\mathbb{T}^d))$-bounded solution of

$$\mathrm{d}u(t) = \left[(1+i\alpha)\Delta u(t) - (1+i\beta)|u(t)|^2 u(t) + \tilde f(t, u(t))\right]\mathrm{d}t + \tilde g(t,u(t))\mathrm{d}W(t). \tag{4.8}$$

Note that $u(\cdot+t_n)$ and $u_n(\cdot)$ share the same distribution. We now show that for any $[a,b]\subset\mathbb{R}$, $\lim_{n\to\infty}\sup_{t\in[a,b]} W_2(\mathcal{L}(u_n(t)),\mathcal{L}(\tilde u(t))) = 0$. It follows from Lemma 4.8 that we only need to show that $\lim_{n\to\infty} W_2(\mathcal{L}(u_n(t)),\mathcal{L}(\tilde u(t))) = 0$ for every $t\in\mathbb{R}$.

Since $\sup_{t\in\mathbb{R}}\|u(t)\|_{H^1}^2 < \infty$ and the imbedding of $H^1\subset L^2$ is compact, $\{\mathcal{L}(u(t))\}_{t\in\mathbb{R}}$ is tight in $Pr(L^2(\mathbb{T}^d))$. Recall that we denote by $u(t,-n,0), t\ge -n$ the solution of (2.1) with initial value $u(-n,-n,0)=0$. In view of (4.1), for any $t\in\mathbb{R}$ there exists a subsequence of $\{u(t,-n,0)\}$ which we still denote by $\{u(t,-n,0)\}$ such that

$$u(t,-n,0)\to u(t)$$

weakly in $L^{2p}(\Omega,\mathbb{P};L^2(\mathbb{T}^d))$. Here $p>1$ is some constant. Note that $u(t), t\in\mathbb{R}$ is the unique $L^2(\Omega,\mathbb{P};L^2(\mathbb{T}^d))$-bounded solution of (2.1). And we have

$$\sup_{t\in\mathbb{R}}\mathbb{E}\|u(t)\|^{2p} < \infty. \tag{4.9}$$

Then given $r\ge 1$, for every sequence $\{\gamma_k'\} := \{\gamma_k'\}_{k=1}^\infty$ there exists a subsequence $\{\gamma_k\}\subset\{\gamma_k'\}$ such that $\mathcal{L}(u_{\gamma_k}(-r))$ converges weakly to some probability measure μ_r in $Pr_2(L^2(\mathbb{T}^d))$. Let ξ_r be a random variable with distribution μ_r. Define $Y_r(t) := \tilde u(t,-r,\xi_r)$, where $\tilde u(t,-r,\xi_r)$, $t\in[-r,+\infty)$ is the solution to

$$\begin{cases} \mathrm{d}u(t) = \left[(1+i\alpha)\Delta u(t) - (1+i\beta)|u(t)|^2 u(t) + \tilde f(t,u(t))\right]\mathrm{d}t + \tilde g(t,u(t))\mathrm{d}W(t) \\ u(-r) = \xi_r. \end{cases}$$

In view of Lemma 4.8, we have

$$\lim_{k\to\infty}\sup_{-r\le t\le -r+T} W_2(\mathcal{L}(u_{\gamma_k}(t)),\mathcal{L}(Y_r(t))) = 0$$

for all $T>0$. Since $\{\mathcal{L}(u_{\gamma_k}(-r-1))\}$ is also tight, going if necessary to a subsequence, by (4.9) we can assume that there exists some probability measure μ_{r+1} in $Pr(L^2(\mathbb{T}^d))$ such that

$$\lim_{k\to\infty} W_2\left(\mathcal{L}(u_{\gamma_k}(-r-1)),\mu_{r+1}\right) = 0.$$

Let ξ_{r+1} be a random variable with distribution μ_{r+1}. By Lemma 4.8, we have

$$\lim_{k\to\infty}\sup_{-r-1\le t\le -r-1+T} W_2(\mathcal{L}(u_{\gamma_k}(t)),\mathcal{L}(Y_{r+1}(t))) = 0$$

for all $T>0$, where $Y_{r+1}(t) := \tilde u(t,-r-1,\xi_{r+1})$, $t\in[-r-1,+\infty)$. Therefore, we have $\mathcal{L}(Y_r(t)) = \mathcal{L}(Y_{r+1}(t))$ for all $t\ge -r$.

Define $\nu(t) := \mathcal{L}(Y_r(t))$, $t \geq -r$. Employing Lemma 4.1, we obatin

$$\sup_{t\in\mathbb{R}} \int_{L^2(\mathbb{T}^d)} \|x\|^2 \nu(t)(\mathrm{d}x) < +\infty.$$

So there exists a subsequence which we still denote by $\{u_{\gamma_k}\}$ satisfying

$$\lim_{k\to\infty} W_2(\mathcal{L}(u_{\gamma_k}(t)), \nu(t)) = 0$$

for every $t \in \mathbb{R}$. And

$$\begin{aligned} Y_r(t) = Y_r(s) + \int_s^t \Big[(1+i\alpha)\Delta Y_r(\sigma) - (1+i\beta)|Y_r(\sigma)|^2 Y_r(\sigma) + \tilde{f}(\sigma, Y_r(\sigma))\Big]\mathrm{d}\sigma \\ + \int_s^t \tilde{g}(\sigma, Y_r(\sigma))\mathrm{d}W(\sigma) \quad \mathbb{P}-\text{a.s.} \end{aligned}$$

where $t \geq s \geq -r$. By the uniqueness in law of the solution for equation (4.8), we get $\mathcal{L}(Y_r(t)) = \mu(t, s, \mathcal{L}(Y_r(s)))$, $t \geq s \geq -r$, i.e. $\nu(t) = \mu(t, s, \nu(s))$, $t \geq s$. According to Proposition 4.2, we obtain $\nu = \mathcal{L}(\tilde{u})$. Therefore, we have

$$\lim_{k\to\infty} W_2(\mathcal{L}(u_{\gamma_k}(t)), \mathcal{L}(\tilde{u}(t))) = 0$$

for every $t \in \mathbb{R}$.

The proof is complete. □

Corollary 4.10. *Under the conditions of Theorem 4.9 the following statements hold:*

(i) *If $f \in C(\mathbb{R}\times L^2(\mathbb{T}^d), L^2(\mathbb{T}^d))$ and $g \in C(\mathbb{R}\times L^2(\mathbb{T}^d), L_2(U, L^2(\mathbb{T}^d)))$ are jointly stationary (respectively, T-periodic, quasi-periodic with the spectrum of frequencies $\nu_1, \ldots, \nu_k$, almost periodic, almost automorphic, Birkhoff recurrent, Lagrange stable, Levitan almost periodic, almost recurrent, Poisson stable) in $t \in \mathbb{R}$ uniformly w.r.t. x on each bounded subset, then so is the unique solution $u \in C_b(\mathbb{R}, L^2(\Omega, \mathbb{P}; L^2(\mathbb{T}^d)))$ of* (2.1) *in distribution;*

(ii) *If $f \in C(\mathbb{R}\times L^2(\mathbb{T}^d), L^2(\mathbb{T}^d))$ and $g \in C(\mathbb{R}\times L^2(\mathbb{T}^d), L_2(U, L^2(\mathbb{T}^d)))$ are Lagrange stable and jointly pseudo-periodic (respectively, pseudo-recurrent) in $t \in \mathbb{R}$ uniformly w.r.t. x on each bounded subset, then equation* (2.1) *has a unique solution $u \in C_b(\mathbb{R}, L^2(\Omega, \mathbb{P}; L^2(\mathbb{T}^d)))$ which is pseudo-periodic (respectively, pseudo-recurrent) in distribution.*

Proof. These statements follow from Proposition 4.2 and Theorems A.14, 4.9. □

Motivated by the proof of Theorem 4.9, with the help of Theorem 3.5 we are able to establish the following second Bogolyubov theorem.

Theorem 4.11. *Suppose that the following conditions hold:*

(i) *the functions f and g satisfy the conditions* (H1)–(H4) *and* (G1)–(G2)*;*

(ii) $\lambda_* - \lambda_f - \frac{L_g^2}{2} > 0$.

Then for any $0 < \varepsilon \leq 1$

(i) *equation* (3.2) *has a unique solution $u_\varepsilon \in C_b(\mathbb{R}, L^2(\Omega, \mathbb{P}; L^2(\mathbb{T}^d)))$;*

(ii) *the solution u_ε of* (3.2) *is strongly compatible in distribution, i.e. $\mathfrak{M}_{(f_\varepsilon, g_\varepsilon)} \subseteq \tilde{\mathfrak{M}}_{u_\varepsilon}$ and*

$$\lim_{\varepsilon\to 0} \sup_{s\leq t\leq s+T} W_2(\mathcal{L}(u_\varepsilon(t)), \mathcal{L}(\bar{u}(0))) = 0$$

for all $s \in \mathbb{R}$ and $T > 0$, where $\bar{u}$ is the unique stationary solution of averaged equation (3.3)*;*

Proof. (i) follows from Proposition 4.2.

(ii) By Theorem 4.9 the solution u_ε of equation (3.2) is strongly compatible in distribution, i.e. $\mathfrak{M}_{(f_\varepsilon, g_\varepsilon)} \subseteq \tilde{\mathfrak{M}}_{u_\varepsilon}$, for any $0 < \varepsilon \leq 1$.

Firstly we prove that $\lim_{\varepsilon\to 0} W_2(\mathcal{L}(u_\varepsilon(t)), \mathcal{L}(\bar{u}(t))) = 0$ in $Pr_2(L^2(\mathbb{T}^d))$ for any $t \in \mathbb{R}$. Take a sequence $\{\varepsilon_n\}_{n=1}^\infty \subset (0,1]$ such that $\varepsilon_n \to 0$ as $n \to \infty$. Note that (4.4) and (4.9) holds uniformly for all $0 < \varepsilon \leq 1$. By Chebyshev's inequality, $\{\mathcal{L}(u_{\varepsilon_n}(t))\}_{n=1}^\infty$ is tight in $Pr(L^2(\mathbb{T}^d))$ for all $t \in \mathbb{R}$. For every $r \geq 1$, according to the tightness of $\{\mathcal{L}(u_{\varepsilon_n}(-r))\}_{n=1}^\infty$, there exists a subsequence $\{\varepsilon_{n_k}\} \subset \{\varepsilon_n\}$ such that $\mathcal{L}\left(u_{\varepsilon_{n_k}}(-r)\right)$ converges weakly to μ_r in $Pr(L^2(\mathbb{T}^d))$. By the Skorohod representation theorem, there exist a sequence of random variables $\hat{\psi}^k(-r)$ and $\hat{\zeta}_r$ with laws of $\mathcal{L}\left(u_{\varepsilon_{n_k}}(-r)\right)$ and μ_r respectively, defined on another probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$, such that

$$\hat{\psi}^k(-r) \to \hat{\zeta}_r \qquad \hat{\mathbb{P}} - \text{a.s.}$$

In view of (4.9), we have

$$\hat{\mathbb{E}}\left\|\hat{\psi}^k(-r)\right\|^{2p} = \int_{L^2(\mathbb{T}^d)} \|x\|^{2p} \mathcal{L}(\hat{\psi}^k(-r))(\mathrm{d}x) = \int_{L^2(\mathbb{T}^d)} \|x\|^{2p} \mathcal{L}(u_{\varepsilon_{n_k}}(-r))(\mathrm{d}x) < \infty,$$

where $p > 1$. It follows from the Vitali L^P convergence criterion that

$$\lim_{k\to\infty} \hat{\mathbb{E}}\left\|\hat{\psi}^k(-r) - \hat{\zeta}_r\right\|^2 = 0.$$

Let $\hat{\psi}^k$ be the solution of

$$\begin{cases} \mathrm{d}u(t) = \left[(1+i\alpha)\Delta u(t) - (1+i\beta)|u(t)|^2 u(t) + f_{\varepsilon_{n_k}}(t, u(t))\right]\mathrm{d}t + g_{\varepsilon_{n_k}}(t, u(t))\mathrm{d}\hat{W}(t) \\ u(-r) = \hat{\psi}^k(-r) \end{cases}$$

and $\hat{Y}_r$ be the solution of

$$\begin{cases} \mathrm{d}u(t) = \left[(1+i\alpha)\Delta u(t) - (1+i\beta)|u(t)|^2 u(t) + \bar{f}(u(t))\right]\mathrm{d}t + \bar{g}(u(t))\mathrm{d}\hat{W}(t) \\ u(-r) = \hat{\zeta}_r, \end{cases}$$

where $\hat{W}$ is a cylindrical Wiener process with the identity covariance operator on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$. It follows from Theorem 3.5 that

$$\lim_{k\to\infty} \hat{\mathbb{E}} \sup_{-r\leq s\leq -r+T} \left\|\hat{\psi}^k(s) - \hat{Y}_r(s)\right\|^2 = 0$$

for any $T > 0$.

Let ζ_r be a random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathcal{L}(\zeta_r) = \mu_r$. Let Y_r be the solution of

$$\begin{cases} \mathrm{d}u(t) = \left[(1+i\alpha)\Delta u(t) - (1+i\beta)|u(t)|^2 u(t) + \bar{f}(u(t))\right]\mathrm{d}t + \bar{g}(u(t))\mathrm{d}W(t) \\ u(-r) = \zeta_r. \end{cases}$$

By the uniqueness in law of the solution for equation (3.2) (respectively, equation (3.3)), $\mathcal{L}\left(\hat{\psi}^k(t)\right) = \mathcal{L}\left(u_{\varepsilon_{n_k}}(t)\right)$ and $\mathcal{L}(\hat{Y}_r(t)) = \mathcal{L}(Y_r(t))$ in $Pr(L^2(\mathbb{T}^d))$ for all $t \geq -r$. Then we have

$$\lim_{k\to\infty} \sup_{-r\leq t\leq -r+T} W_2\left(\mathcal{L}(u_{\varepsilon_{n_k}}(t)), \mathcal{L}(Y_r(t))\right) = 0 \tag{4.10}$$

for any $T > 0$.

In view of the tightness of $\left\{\mathcal{L}\left(u_{\varepsilon_{n_k}}(-r-1)\right)\right\}$ in $Pr(L^2(\mathbb{T}^d))$, there exists a subsequence $\{\varepsilon_{n_{k_j}}\} \subset \{\varepsilon_{n_k}\}$ such that $\mathcal{L}\left(u_{\varepsilon_{n_{k_j}}}(-r-1)\right)$ weakly converges to μ_{r+1}. We can choose a random variable ζ_{r+1} on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathcal{L}(\zeta_{r+1}) = \mu_{r+1}$. Let Y_{r+1} be the solution of

$$\begin{cases} \mathrm{d}u(t) = \left[(1+i\alpha)\Delta u(t) - (1+i\beta)|u(t)|^2 u(t) + \bar{f}(u(t))\right] \mathrm{d}t + \bar{g}(u(t))\mathrm{d}W(t) \\ u(-r-1) = \zeta_{r+1}. \end{cases}$$

Similar to the procedure of calculating (4.10), we obtain

$$\lim_{j\to\infty} \sup_{-r-1\le t\le -r-1+T} W_2\left(\mathcal{L}(u_{\varepsilon_{n_{k_j}}}(t)), \mathcal{L}(Y_{r+1}(t))\right) = 0$$

for all $T > 0$. Therefore, $\mathcal{L}(Y_r(t)) = \mathcal{L}(Y_{r+1}(t))$ for all $t \ge -r$.

We set $\nu(t) := \mathcal{L}(Y_r(t))$, $t \ge -r$. In light of Proposition 4.2, we obtain that ν is the law of the $L^2(\Omega, \mathbb{P}; L^2(\mathbb{T}^d))$-bounded solution $\bar{u}$ of (3.3). And there exits a subsequence which we still denote by $\{u_{\varepsilon_{n_{k_j}}}\}$ such that $\lim\limits_{j\to\infty} W_2\left(\mathcal{L}(u_{\varepsilon_{n_{k_j}}}(t)), \nu(t)\right) = 0$ for every $t \in \mathbb{R}$. Therefore we get

$$\lim_{j\to\infty} W_2\left(\mathcal{L}(u_{\varepsilon_{n_{k_j}}}(t)), \mathcal{L}(\bar{u}(t))\right) = 0$$

for every $t \in \mathbb{R}$. By the arbitrariness of $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0, 1]$, we have

$$\lim_{\varepsilon\to 0} W_2\left(\mathcal{L}(u_\varepsilon(t)), \mathcal{L}(\bar{u}(t))\right) = 0.$$

For any $[a, b] \subset \mathbb{R}$, now we prove that $\lim\limits_{\varepsilon\to 0} \sup\limits_{a\le t\le b} W_2\left(\mathcal{L}(u_\varepsilon(t)), \mathcal{L}(\bar{u}(t))\right) = 0$. In fact, $\mathcal{L}(u_\varepsilon(a))$ converges weakly to $\mathcal{L}(u(a))$ in $Pr_2(L^2(\mathbb{T}^d))$. In view of the Skorohod representation theorem, there exist random variables $\hat{\hat{\psi}}_\varepsilon(a)$ and $\hat{\hat{\psi}}(a)$ defined on another probability space $(\hat{\hat{\Omega}}, \hat{\hat{\mathcal{F}}}, \hat{\hat{\mathbb{P}}})$ such that $\lim\limits_{\varepsilon\to 0} \hat{\hat{\psi}}_\varepsilon(a) = \hat{\hat{\psi}}(a)$ $\mathbb{P}$-a.s., where $\mathcal{L}\left(\hat{\hat{\psi}}_\varepsilon(a)\right) = \mathcal{L}\left(u_\varepsilon(a)\right)$ and $\mathcal{L}\left(\hat{\hat{\psi}}(a)\right) = \mathcal{L}\left(\bar{u}(a)\right)$. Similar to the procedure of calculating (4.10), we have

$$\lim_{\varepsilon\to 0} \sup_{a\le t\le b} W_2\left(\mathcal{L}(u_\varepsilon(t)), \mathcal{L}(\bar{u}(0))\right) = 0.$$

Here the proof is complete. □

Corollary 4.12. *Under the conditions of Theorem 4.11 the following statements hold:*

(i) *If $f \in C(\mathbb{R} \times L^2(\mathbb{T}^d), L^2(\mathbb{T}^d))$ and $g \in C(\mathbb{R} \times L^2(\mathbb{T}^d), L_2(U, L^2(\mathbb{T}^d)))$ are jointly stationary (respectively, T-periodic, quasi-periodic with the spectrum of frequencies $\nu_1, \dots, \nu_k$, almost periodic, almost automorphic, Birkhoff recurrent, Lagrange stable, Levitan almost periodic, almost recurrent, Poisson stable) in t uniformly w.r.t. x on each bounded subset, then so is the unique solution $u_\varepsilon \in C_b(\mathbb{R}, L^2(\Omega, \mathbb{P}; L^2(\mathbb{T}^d)))$ of* (3.2) *in distribution;*

(ii) *If $f \in C(\mathbb{R}\times L^2(\mathbb{T}^d), L^2(\mathbb{T}^d))$ and $g \in C(\mathbb{R}\times L^2(\mathbb{T}^d), L_2(U, L^2(\mathbb{T}^d)))$ are Lagrange stable and jointly pseudo-periodic (respectively, pseudo-recurrent) in t uniformly w.r.t. x on each bounded subset, then the unique $L^2(\Omega, \mathbb{P}; L^2(\mathbb{T}^d))$-bounded solution u_ε of* (3.2) *is pseudo-periodic (respectively, pseudo-recurrent) in distribution;*

(iii)
$$\lim_{\varepsilon\to 0} \sup_{s\le t\le s+T} W_2(\mathcal{L}(u_\varepsilon(t)), \mathcal{L}(\bar{u}(0))) = 0$$
for all $s \in \mathbb{R}$ and $T > 0$, where $\bar{u}$ is the unique stationary solution of averaged equation (3.3).

Proof. These statements follow from Theorems A.14 and 4.11. □

5. Global averaging principle in weak sense

Let $F := (f, g) \in BUC(\mathbb{R} \times L^2(\mathbb{T}^d), L^2(\mathbb{T}^d)) \times BUC(\mathbb{R} \times L^2(\mathbb{T}^d), L_2(U, L^2(\mathbb{T}^d)))$. Recall that $f^\tau(t,x) = f(t+\tau, x)$ for all $(t,x) \in \mathbb{R} \times L^2(\mathbb{T}^d)$,

$$\begin{aligned}\mathcal{H}(F) &= \overline{\{F^\tau = (f^\tau, g^\tau) : \tau \in \mathbb{R}\}} \\ &\subset BUC(\mathbb{R} \times L^2(\mathbb{T}^d), L^2(\mathbb{T}^d)) \times BUC(\mathbb{R} \times L^2(\mathbb{T}^d), L_2(U, L^2(\mathbb{T}^d))).\end{aligned}$$

Then $(\mathcal{H}(F), \mathbb{R}, \sigma)$ is a shift dynamical system. Here $\sigma : \mathbb{R} \times \mathcal{H}(F) \to \mathcal{H}(F), (\tau, F) \mapsto F^\tau$.

Let $u(t,s,x), t \geq s$ be the solution of equation (2.1) with initial value $u(s,s,x) = x$. Define $P_F(s,x,t,\mathrm{d}y) := \mathbb{P} \circ (u(t,s,x))^{-1}(\mathrm{d}y)$. Then we can associate a mapping $P^*(t,F,\cdot) : Pr(L^2(\mathbb{T}^d)) \to Pr(L^2(\mathbb{T}^d))$ defined by

$$P^*(t,F,\mu)(A) := \int_{L^2(\mathbb{T}^d)} P_F(0,x,t,A)\mu(\mathrm{d}x)$$

for all $\mu \in Pr(L^2(\mathbb{T}^d))$ and $A \in \mathcal{B}(L^2(\mathbb{T}^d))$. Denote by $Pr_2(L^2(\mathbb{T}^d))$ the space of probability measures $\mu \in Pr(L^2(\mathbb{T}^d))$ such that $\int_{L^2(\mathbb{T}^d)} \|z\|^2 \mu(\mathrm{d}z) < \infty$. Define

$$B_r := \left\{ \mu \in Pr_2(L^2(\mathbb{T}^d)) : \int_{L^2(\mathbb{T}^d)} \|z\|^2 \mu(\mathrm{d}z) \leq r^2 \right\}$$

for any $r > 0$. A subset $D \subset Pr_2(L^2(\mathbb{T}^d))$ is called *bounded* if there exists a constant $r > 0$ such that $D \subset B_r$. For any $\rho > 0$, define

$$\mathcal{O}_\rho(B) := \{\mu \in Pr_2(L^2(\mathbb{T}^d)) : W_2(\mu, B) < \rho\}.$$

Lemma 5.1. *Consider equation* (2.1)*. Assume that conditions* (H1)–(H2) *hold. Then P^* is a cocycle on $(\mathcal{H}(F), \mathbb{R}, \sigma)$ with fiber $Pr_2(L^2(\mathbb{T}^d))$.*

Proof. It follows from Lemma 4.8 that P^* is a continuous mapping from $\mathbb{R}^+ \times \mathcal{H}(F) \times Pr_2(L^2(\mathbb{T}^d))$ into $Pr_2(L^2(\mathbb{T}^d))$. For any $\mu \in Pr_2(L^2(\mathbb{T}^d))$, $t, \tau \in \mathbb{R}^+$ and $F \in \mathcal{H}(F)$, in view of the uniqueness in law of the solutions for equation (2.1), we have $P^*(t+\tau, F, \mu) = P^*(t, \sigma_\tau F, P^*(\tau, \mu, F))$. It follows from the definition of P^* that $P^*(0,F,\cdot) = Id_{Pr_2(L^2(\mathbb{T}^d))}$ for all $F \in \mathcal{H}(F)$. $\square$

Corollary 5.2. *Under conditions of Lemma 5.1, the mapping given by*

$$\begin{gathered}\Pi : \mathbb{R}^+ \times \mathcal{H}(F) \times Pr_2(L^2(\mathbb{T}^d)) \to \mathcal{H}(F) \times Pr_2(L^2(\mathbb{T}^d)), \\ \Pi(t, (F, \mu)) := (\sigma_t F, P^*(t, F, \mu))\end{gathered}$$

is a continuous skew-product semiflow.

For any given $\tilde{F} \in \mathcal{H}(F)$, under conditions of Proposition 4.2, equation (2.1) has a unique $L^2(\Omega, \mathbb{P}; L^2(\mathbb{T}^d))$-bounded solution $u_{\tilde{F}}$ with the distribution $\mathcal{L}(u_{\tilde{F}}(t)) =: \mu_{\tilde{F}}(t), t \in \mathbb{R}$.

Remark 5.3.
- (i) It follows from Remark 4.7 that (4.1)–(4.2), (4.4), (4.7) and (4.9) hold uniformly for all $\tilde{F} \in \mathcal{H}(F)$ and $\varepsilon \in (0,1]$.
- (ii) Assume that $F := (f,g)$ satisfy (G1)–(G2). If $\mathcal{H}(F)$ is compact, then for any $\tilde{F} \in \mathcal{H}(F)$, $\tilde{f}$ (respectively, $\tilde{g}$) satisfies (G1) (respectively, (G2)) with the same δ_f and $\bar{f}$ (respectively, δ_g and $\bar{g}$).

Proposition 5.4. *Consider equation* (2.1)*. Assume that conditions* (H1)–(H4) *hold, and $\lambda_* - \lambda_f - \frac{L_g^2}{2} > 0$. Then we have the following results.*

(i) *We set* $\mathfrak{A}_{\tilde{F}} := \overline{\{\mu_{\tilde{F}}(t) \in Pr_2(L^2(\mathbb{T}^d)) : t \in \mathbb{R}\}}$. *Then*

$$P^*(t, \tilde{F}, \mathfrak{A}_{\tilde{F}}) = \mathfrak{A}_{\sigma_t \tilde{F}}$$

for all $t \in \mathbb{R}^+$ *and* $\tilde{F} \in \mathcal{H}(F)$.

(ii) *If* $\mathcal{H}(F)$ *is compact, then the skew product semiflow* Π *admits a global attractor* $\mathfrak{A} := \omega\left(\mathcal{H}(F) \times \overline{\cup_{\tilde{F}\in\mathcal{H}(F)}\mathfrak{A}_{\tilde{F}}}\right)$. *Moreover,* $\Pi_2\mathfrak{A}$ *is the uniform attractor of cocycle* P^*. *Here* $\Pi_2(\tilde{F}, \mu) := \mu$ *for all* $(\tilde{F}, \mu) \in \mathcal{H}(F) \times Pr_2(L^2(\mathbb{T}^d))$.

Proof. (i) Given $t \in \mathbb{R}^+$ and $\tilde{F} \in \mathcal{H}(F)$, let $u_{\sigma_t \tilde{F}}$ be the unique $L^2(\Omega, \mathbb{P}; L^2(\mathbb{T}^d))$-bounded solution of equation

$$\mathrm{d}u(s) = \Big[(1+i\alpha)\Delta u(s) - (1+i\beta)|u(s)|^2 u(s) + \tilde{f}(s+t, u(s))\Big]\,\mathrm{d}s + \tilde{g}(s+t, u(s))\mathrm{d}W(s),\ s \in \mathbb{R}.$$

Note that $\mathcal{L}(u_{\tilde{F}}(s+t)) = \mathcal{L}(u_{\sigma_t\tilde{F}}(s))$ for all $s \in \mathbb{R}$. Consequently, $P^*(t, \tilde{F}, \mathfrak{A}_{\tilde{F}}) = \mathfrak{A}_{\sigma_t \tilde{F}}$.

(ii) In view of (4.4), (4.9) and Remark 5.3, there exists a constant $R > 0$ such that

$$\bigcup_{\tilde{F}\in\mathcal{H}(F)} \mathfrak{A}_{\tilde{F}} \subset \mathcal{M} := \left\{\mu \in Pr_2(L^2(\mathbb{T}^d)) : \int_{H^1} \|z\|_{H^1}^2 \mu(\mathrm{d}z) + \int_{L^2(\mathbb{T}^d)} \|z\|^{2p} \mu(\mathrm{d}z) < R^2\right\},$$

where $p > 1$. Then according to the Chebyshev's inequality and the compactness of the inclusion $H^1 \subset L^2(\mathbb{T}^d)$, the set $\overline{\cup_{\tilde{F}\in\mathcal{H}(F)}\mathfrak{A}_{\tilde{F}}}$ is compact in $Pr_2(L^2(\mathbb{T}^d))$.

Let $r > 0$ be an arbitrary constant. For any $\mu \in B_r$, take a random variable ξ such that $\mathcal{L}(\xi) = \mu$. Let $Y(t, \xi), t \geq 0$ satisfies

$$\begin{aligned} Y(t,\xi) = \xi + \int_0^t \Big[(1+i\alpha)\Delta Y(s,\xi) - (1+i\beta)|Y(s,\xi)|^2 Y(s,\xi) + \tilde{f}(s, Y(s,\xi))\Big]\,\mathrm{d}s \\ + \int_0^t \tilde{g}(s, Y(s,\xi))\mathrm{d}W(s). \end{aligned}$$

Employing Proposition 4.5, we have

$$\mathbb{E}\|Y(t,\xi) - u_{\tilde{F}}(t)\|^2 \leq \mathrm{e}^{-\left(2\lambda_* - 2\lambda_f - L_g^2\right)t}\mathbb{E}\|u_{\tilde{F}}(0) - \xi\|^2.$$

Therefore, $\lim_{t\to+\infty} \sup_{\tilde{F}\in\mathcal{H}(F)} \mathrm{dist}_{Pr_2(L^2(\mathbb{T}^d))}\left(P^*(t, \tilde{F}, \mu), \overline{\cup_{\tilde{F}\in\mathcal{H}(F)}\mathfrak{A}_{\tilde{F}}}\right) = 0$ uniformly w.r.t. $\mu \in B_r$, i.e. $\overline{\cup_{\tilde{F}\in\mathcal{H}(F)}\mathfrak{A}_{\tilde{F}}}$ is a compact uniformly attracting set. It is obvious that $\mathcal{H}(F) \times \overline{\cup_{\tilde{F}\in\mathcal{H}(F)}\mathfrak{A}_{\tilde{F}}}$ is a compact attracting set for Π. It follows from Lemma 2.5 that Π admits a global attractor $\mathfrak{A} := \omega\left(\mathcal{H}(F) \times \overline{\cup_{\tilde{F}\in\mathcal{H}(F)}\mathfrak{A}_{\tilde{F}}}\right)$.

Now we prove that $\Pi_2\mathfrak{A}$ is the uniform attractor of cocycle P^*. Let $B \subset Pr_2(L^2(\mathbb{T}^d))$ be bounded, then $\mathcal{H}(F) \times B$ is bounded in $\mathfrak{X} := \mathcal{H}(F) \times Pr_2(L^2(\mathbb{T}^d))$. Therefore,

$$\begin{aligned} \mathrm{dist}_{\mathrm{Pr}_2(\mathrm{L}^2(\mathbb{T}^d))}(P^*(t, \tilde{F}, B), \Pi_2\mathfrak{A}) &\leq \mathrm{dist}_{\mathfrak{X}}(\mathcal{H}(F) \times P^*(t, \tilde{F}, B), \mathfrak{A}) \\ &\leq \mathrm{dist}_{\mathfrak{X}}(\Pi(t, \mathcal{H}(F) \times B), \mathfrak{A}) \to 0 \quad \text{as} \quad t \to +\infty. \end{aligned}$$

Then we verify the minimality property. Set $\omega_{\mathcal{H}(F)}(B) := \cap_{t\geq 0}\overline{\cup_{\tilde{F}\in\mathcal{H}(F)} \cup_{s\geq t} P^*(s, \tilde{F}, B)}$. Therefore, $\mu \in \omega_{\mathcal{H}(F)}(B)$ if and only if there exist $\{\nu_n\} \subset B$, $\{F_n\} \subset \mathcal{H}(F)$ and $\{t_n\} \subset \mathbb{R}_+$ such that $t_n \to +\infty$ and $P^*(t_n, F_n, \nu_n) \to \mu$ as $n \to +\infty$. Let $\mathcal{A}_1$ be a closed uniformly attracting set. Next we show that $\omega_{\mathcal{H}(F)}(\Pi_2\mathfrak{A}) \subset \mathcal{A}_1$. Indeed, if this is false, i.e. $\omega_{\mathcal{H}(F)}(\Pi_2\mathfrak{A}) \not\subset \mathcal{A}_1$. Let $\mu \in \omega_{\mathcal{H}(F)}(\Pi_2\mathfrak{A}) \setminus \mathcal{A}_1$, then there exist $\{\nu_n\} \subset \Pi_2\mathfrak{A}$, $\{F_n\} \subset \mathcal{H}(F)$ and $\{t_n\} \subset \mathbb{R}_+$ such that $t_n \to +\infty$ and $P^*(t_n, F_n, \nu_n) \to \mu$ as $n \to +\infty$. Hence we have

$$
\begin{aligned}
0 < d(\mu, \mathcal{A}_1) &\leq \lim_{n\to+\infty} d(P^*(t_n, F_n, \nu_n), \mathcal{A}_1) \\
&\leq \lim_{n\to+\infty} \mathrm{dist}_{\mathrm{Pr}_2(\mathrm{L}^2(\mathbb{T}^\mathrm{d}))}(P^*(t_n, F_n, \Pi_2\mathfrak{A}), \mathcal{A}_1) \\
&\leq \lim_{n\to+\infty} \sup_{\tilde{F}\in\mathcal{H}(F)} \mathrm{dist}_{\mathrm{Pr}_2(\mathrm{L}^2(\mathbb{T}^\mathrm{d}))}(P^*(t_n, \tilde{F}, \Pi_2\mathfrak{A}), \mathcal{A}_1) = 0,
\end{aligned}
$$

a contradiction. On the other hand, for any $(\tilde{F}, \mu) \in \omega(\mathcal{H}(F) \times \Pi_2\mathfrak{A}) = \mathfrak{A}$, there exist $\{\nu_n\} \subset \Pi_2\mathfrak{A}$, $\{F_n\} \subset \mathcal{H}(F)$, $\{t_n\} \subset \mathbb{R}_+$ such that $P^*(t_n, F_n, \nu_n) \to \mu$ and $\sigma_{t_n} F_n \to \tilde{F}$ as $n \to +\infty$. Then $\mu \in \omega_{\mathcal{H}(F)}(\Pi_2\mathfrak{A})$. Therefore, $\Pi_2\mathfrak{A} \subset \mathcal{A}_1$.

The proof is complete. □

Remark 5.5. It is known that $\mathcal{H}(F)$ is compact provided F is Birkhoff recurrent, which includes periodic, quasi-periodic, almost periodic, almost automorphic as special cases.

Theorem 5.6. *Suppose that f, g satisfy the conditions* (G1)–(G2) *and* (H1)–(H4). *Assume further that $\lambda_* - \lambda_f - \frac{L_g^2}{2} > 0$. If $\mathcal{H}(F)$ is compact, then*

(i) *the cocycle P_ε^* associated with stochastic CGL equation* (3.2) *has a uniform attractor $\mathfrak{A}^\varepsilon$ for any $0 < \varepsilon \leq 1$;*

(ii) *the cocycle $\bar{P}^*$ associated with averaged equation* (3.3) *has a uniform attractor $\bar{\mathfrak{A}}$, which is a singleton set;*

(iii) *for arbitrary large R_1, small $\rho > 0$ and $\tilde{F} \in \mathcal{H}(F)$ there exist $\varepsilon_0 = \varepsilon_0(R_1, \rho)$ and $T = T(R_1, \rho)$ such that for all $\varepsilon \leq \varepsilon_0,\ t \geq T$*

$$
P_\varepsilon^*(t, \tilde{F}, B_{R_1}) \subset \mathcal{O}_\rho\left(\bar{\mathfrak{A}}\right). \tag{5.1}
$$

In particular,

$$
\lim_{\varepsilon\to 0} \mathrm{dist}_{Pr_2(L^2(\mathbb{T}^d))}\left(\mathfrak{A}^\varepsilon, \bar{\mathfrak{A}}\right) = 0. \tag{5.2}
$$

Proof. (i)–(ii) According to Proposition 5.4, P_ε^* and $\bar{P}^*$ admit uniform attractors $\mathfrak{A}^\varepsilon$ and $\bar{\mathfrak{A}}$, where $\bar{\mathfrak{A}} = \{\mathcal{L}(\bar{u}(0))\} \in Pr_2(L^2(\mathbb{T}^d))$. Here $\bar{u}(t), t \in \mathbb{R}$ is the unique stationary solution to averaged equation (3.3).

(iii) In view of Proposition 4.5, there exists δ, $0 < \delta < \frac{\rho}{2}$ such that

$$
\bar{P}^*\left(t, \mathcal{O}_\delta(\bar{\mathfrak{A}})\right) \subset \mathcal{O}_{\frac{\rho}{2}}(\bar{\mathfrak{A}})
$$

for all $t \geq 0$. Fix R_1 large enough. Employing (4.1), there exists $T_0 > 0$ satisfying

$$
P_\varepsilon^*(t, \tilde{F}, B_{R_1}) \subset B_{R_1} \tag{5.3}
$$

for all $t \geq T_0$. Since $\bar{\mathfrak{A}}$ is attractor, we can choose $T_1 = T_1(R_1, \rho)$ so large such that

$$
\bar{P}^*(t, B_{R_1}) \subset \mathcal{O}_{\frac{\delta}{2}}(\bar{\mathfrak{A}}) \tag{5.4}
$$

for all $t \geq T_1$. Set $T := \max\{T_0, T_1\}$. In view of (3.24), we have

$$
\sup_{0\leq t\leq T} W_2\left(P_\varepsilon^*(t, \tilde{F}, \mu), \bar{P}^*(t, \mu)\right) < \eta(T, R_1)(\varepsilon) \tag{5.5}
$$

for all $\mu \in B_{R_1}$ and $\tilde{F} \in \mathcal{H}(F)$, where $\eta(T, R_1)(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then, there exists $\varepsilon_0 = \varepsilon_0(T, R_1)$ such that $\eta(T, R_1)(\varepsilon) < \frac{\delta}{2}$ for all $\varepsilon \leq \varepsilon_0$.

For any $\mu \in B_{R_1}$, in view of (5.3)–(5.5), we have

$$
P_\varepsilon^*(T, \tilde{F}, \mu) \in \mathcal{O}_\delta(\bar{\mathfrak{A}}) \cap B_{R_1}
$$

for all $\varepsilon \le \varepsilon_0$. It can be verified that $P_\varepsilon^*(t, \tilde{F}, \mu) \in \mathcal{O}_\rho(\bar{\mathfrak{A}})$ for all $t \ge T$ and $\varepsilon \le \varepsilon_0$. To this end, define $\mu_1^\varepsilon := P_\varepsilon^*(T, \tilde{F}, \mu)$. Then $\bar{P}^*(t, \mu_1^\varepsilon) \in \mathcal{O}_{\frac{\rho}{2}}(\bar{\mathfrak{A}})$ and $P_\varepsilon^*(t+T, \tilde{F}, \mu) = P_\varepsilon^*(t, \sigma_T \tilde{F}, \mu_1^\varepsilon)$ for all $t \ge 0$. Therefore, according to (5.4)–(5.5), we get

$$P_\varepsilon^*(2T, \tilde{F}, \mu) \in \mathcal{O}_\delta\left(\bar{\mathfrak{A}}\right) \cap B_{R_1}$$

and

$$P_\varepsilon^*(t+T, \tilde{F}, \mu) \in \mathcal{O}_{\frac{\rho}{2}+\frac{\delta}{2}}(\bar{\mathfrak{A}}) \subset \mathcal{O}_\rho(\bar{\mathfrak{A}})$$

for all $t \in [0, T]$. Repeating the above procedure, we have

$$P_\varepsilon^*(t, \tilde{F}, \mu) \in \mathcal{O}_\rho(\bar{\mathfrak{A}})$$

for all $t \ge T$ and $\varepsilon \le \varepsilon_0$.

Take R_1 large enough so that $\mathfrak{A}^\varepsilon \subset B_{R_1}$, then (5.2) follows. □

Appendix A

A.1. **Poisson stable functions.** Let $(\mathcal{X}, \rho)$ and $(\mathcal{Y}, \rho_1)$ be two complete metric spaces. Denote by $C(\mathbb{R}, \mathcal{X})$ the space of all continuous functions $\varphi : \mathbb{R} \to \mathcal{X}$ equipped with the distance

$$d(\varphi_1, \varphi_2) := \sum_{k=1}^\infty \frac{1}{2^k} \frac{d_k(\varphi_1, \varphi_2)}{1 + d_k(\varphi_1, \varphi_2)},$$

where

$$d_k(\varphi_1, \varphi_2) := \sup_{|t| \le k} \rho(\varphi_1(t), \varphi_2(t)),$$

which generates the compact-open topology on $C(\mathbb{R}, \mathcal{X})$. The space $(C(\mathbb{R}, \mathcal{X}), d)$ is a complete metric space (see, e.g. [34, 36, 38, 39]).

Remark A.1 ([39])**.** Let $\{\varphi_n\}_{n=1}^\infty, \varphi \in C(\mathbb{R}, \mathcal{X})$. Then the following statements are equivalent.

(i) $\lim_{n\to\infty} d(\varphi_n, \varphi) = 0$.
(ii) $\lim_{n\to\infty} \max_{|t| \le l} \rho(\varphi_n(t), \varphi(t)) = 0$ for any $l > 0$.
(iii) There exists a sequence $l_n \to +\infty$ such that $\lim_{n\to\infty} \max_{|t| \le l_n} \rho(\varphi_n(t), \varphi(t)) = 0$.

Let us now introduce a shift dynamical system. We say that φ^τ is the τ*-translation* of φ if $\varphi^\tau(t) := \varphi(t+\tau)$ for any $t \in \mathbb{R}$ and $\varphi \in C(\mathbb{R}, \mathcal{X})$. For any $(\tau, \varphi) \in \mathbb{R} \times C(\mathbb{R}, \mathcal{X})$, define the mapping $\sigma : \mathbb{R} \times C(\mathbb{R}, \mathcal{X}) \to C(\mathbb{R}, \mathcal{X})$ by $\sigma(\tau, \varphi) := \varphi^\tau$. Then the triplet $(C(\mathbb{R}, \mathcal{X}), \mathbb{R}, \sigma)$ is a dynamical system which is called *shift dynamical system* or *Bebutov's dynamical system*. Indeed, it is easy to check that $\sigma(0, \varphi) = \varphi$ and $\sigma(\tau_1 + \tau_2, \varphi) = \sigma(\tau_2, \sigma(\tau_1, \varphi))$ for any $\varphi \in C(\mathbb{R}, \mathcal{X})$ and $\tau_1, \tau_2 \in \mathbb{R}$. And it can be proved that the mapping $\sigma : \mathbb{R} \times C(\mathbb{R}, \mathcal{X}) \to C(\mathbb{R}, \mathcal{X})$ is continuous, see, e.g. [34, 36, 39].

We write $\mathcal{H}(\varphi)$ to mean the *hull* of φ, which is the set of all the limits of φ^{τ_n} in $C(\mathbb{R}, \mathcal{X})$, i.e.

$$\mathcal{H}(\varphi) := \{\psi \in C(\mathbb{R}, \mathcal{X}) : \psi = \lim_{n\to\infty} \varphi^{\tau_n} \text{ for some sequence } \{\tau_n\} \subset \mathbb{R}\}.$$

Notice that $\mathcal{H}(\varphi) \subset C(\mathbb{R}, \mathcal{X})$ is closed and translation invariant. Consequently, it naturally defines on $\mathcal{H}(\varphi)$ a shift dynamical system $(\mathcal{H}(\varphi), \mathbb{R}, \sigma)$.

Definition A.2. We say that $\varphi \in C(\mathbb{R}, \mathcal{X})$ is *periodic* if there exists a constant $T > 0$ such that $\varphi(t+T) = \varphi(t)$ for all $t \in \mathbb{R}$. In particular, φ is called *stationary* provided $\varphi(t) = \varphi(0)$ for all $t \in \mathbb{R}$.

Definition A.3. We say that $\varphi \in C(\mathbb{R}, \mathcal{X})$ is *quasi-periodic with the spectrum of frequencies* $\nu_1, \nu_2, \ldots, \nu_k$ if it satisfies the following conditions:

(i) the numbers $\nu_1, \nu_2, \dots, \nu_k$ are rationally independent;
(ii) there exists a continuous function $\Phi : \mathbb{R}^k \to \mathcal{X}$ such that $\Phi(t_1 + 2\pi, t_2 + 2\pi, \dots, t_k + 2\pi) = \Phi(t_1, t_2, \dots, t_k)$ for all $(t_1, t_2, \dots, t_k) \in \mathbb{R}^k$;
(iii) $\varphi(t) = \Phi(\nu_1 t, \nu_2 t, \dots, \nu_k t)$ for $t \in \mathbb{R}$.

Definition A.4. We say that $\varphi \in C(\mathbb{R}, \mathcal{X})$ is *Bohr almost periodic* if the set $\mathcal{T}(\varphi, \varepsilon)$ of ε-almost periods of φ is relatively dense for each $\varepsilon > 0$, i.e. for each $\varepsilon > 0$ there exists a constant $l = l(\varepsilon) > 0$ such that $\mathcal{T}(\varphi, \varepsilon) \cap [a, a + l] \neq \emptyset$ for all $a \in \mathbb{R}$, where

$$\mathcal{T}(\varphi, \varepsilon) := \left\{ \tau \in \mathbb{R} : \sup_{t \in \mathbb{R}} \rho(\varphi(t + \tau), \varphi(t)) < \varepsilon \right\},$$

and $\tau \in \mathcal{T}(\varphi, \varepsilon)$ is called *ε-almost period* of φ.

Definition A.5. We say that $\varphi \in C(\mathbb{R}, \mathcal{X})$ is *pseudo-periodic in the positive* (respectively, *negative*) *direction* if for each $\varepsilon > 0$ and $l > 0$ there exists a ε-almost period $\tau > l$ (respectively, $\tau < -l$) of the function φ. The function φ is called *pseudo-periodic* if it is pseudo-periodic in both directions.

Definition A.6. We say that $\varphi \in C(\mathbb{R}, \mathcal{X})$ is *Levitan almost periodic* if there exists a almost periodic function $\psi \in C(\mathbb{R}, \mathcal{Y})$ such that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\mathcal{T}(\psi, \delta) \subseteq \mathfrak{T}(\varphi, \varepsilon)$, where $\mathfrak{T}(\varphi, \varepsilon) := \{\tau \in \mathbb{R} : d(\varphi^\tau, \varphi) < \varepsilon\}$. And $\tau \in \mathfrak{T}(\varphi, \varepsilon)$ is said to be *ε-shift* for φ.

Definition A.7. A function $\varphi \in C(\mathbb{R}, \mathcal{X})$ is called *almost recurrent (in the sense of Bebutov)* if the set $\mathfrak{T}(\varphi, \varepsilon)$ is relatively dense for every $\varepsilon > 0$.

Definition A.8. (i) We say that a function $\varphi \in C(\mathbb{R}, \mathcal{X})$ is *Lagrange stable* provided $\{\varphi^h : \ h \in \mathbb{R}\}$ is a relatively compact subset of $C(\mathbb{R}, \mathcal{X})$.
(ii) We say that a function $\varphi \in C(\mathbb{R}, \mathcal{X})$ is *Birkhoff recurrent* if it is almost recurrent and Lagrange stable.

Definition A.9. We say that a function $\varphi \in C(\mathbb{R}, \mathcal{X})$ is *almost automorphic* if it is Levitan almost periodic and Lagrange stable.

Definition A.10. ([35, 36, 38]) A function $\varphi \in C(\mathbb{R}, \mathcal{X})$ is called *pseudo-recurrent* if for any $\varepsilon > 0$ and $l \in \mathbb{R}$ there exists $L \geq l$ such that for any $\tau_0 \in \mathbb{R}$ we can find a number $\tau \in [l, L]$ satisfying

$$\sup_{|t| \leq 1/\varepsilon} \rho(\varphi(t + \tau_0 + \tau), \varphi(t + \tau_0)) \leq \varepsilon.$$

Definition A.11. We say that $\varphi \in C(\mathbb{R}, \mathcal{X})$ is *Poisson stable in the positive* (respectively, *negative*) *direction* if for every $\varepsilon > 0$ and $l > 0$ there exists $\tau > l$ (respectively, $\tau < -l$) such that $d(\varphi^\tau, \varphi) < \varepsilon$. The function φ is called *Poisson stable* if it is Poisson stable in both directions.

Remark A.12. ([35, 36, 38, 39])
(i) Every Birkhoff recurrent function is pseudo-recurrent, but not vice versa.
(ii) Suppose that $\varphi \in C(\mathbb{R}, \mathcal{X})$ is pseudo-recurrent, then every function $\psi \in \mathcal{H}(\varphi)$ is pseudo-recurrent.
(iii) Suppose that $\varphi \in C(\mathbb{R}, \mathcal{X})$ is Lagrange stable and every function $\psi \in \mathcal{H}(\varphi)$ is Poisson stable, then φ is pseudo-recurrent.

Finally, we remark that a Lagrange stable function is not Poisson stable in general, but all other types of functions introduced above are Poisson stable.

A.2. **Shcherbakov's comparability method by character of recurrence.** Let $\varphi \in C(\mathbb{R}, \mathcal{X})$. Denote by $\mathfrak{N}_\varphi$ (respectively, $\mathfrak{M}_\varphi$) the space of all sequences $\{t_n\}_{n=1}^\infty$ such that $\varphi(\cdot + t_n)$ converges to $\varphi(\cdot)$ (respectively, $\varphi(\cdot + t_n)$ converges) uniformly on any bounded interval.

Definition A.13. A function $\varphi \in C(\mathbb{R}, \mathcal{X})$ is called *comparable* (respectively, *strongly comparable*) *by character of recurrence* with $\psi \in C(\mathbb{R}, \mathcal{Y})$ provided $\mathfrak{N}_\psi \subseteq \mathfrak{N}_\varphi$ (respectively, $\mathfrak{M}_\psi \subseteq \mathfrak{M}_\varphi$).

Theorem A.14. ([36, ChII], [37])

(i) $\mathfrak{M}_\psi \subseteq \mathfrak{M}_\varphi$ *implies* $\mathfrak{N}_\psi \subseteq \mathfrak{N}_\varphi$, *and hence strong comparability implies comparability.*

(ii) *Assume that* $\varphi \in C(\mathbb{R}, \mathcal{X})$ *is comparable by character of recurrence with* $\psi \in C(\mathbb{R}, \mathcal{Y})$. *If* ψ *is stationary (respectively,* T*-periodic, Levitan almost periodic, almost recurrent, Poisson stable), then so is* φ.

(iii) *Assume that* $\varphi \in C(\mathbb{R}, \mathcal{X})$ *is strongly comparable by character of recurrence with* $\psi \in C(\mathbb{R}, \mathcal{Y})$. *If* ψ *is quasi-periodic with the spectrum of frequencies* $\nu_1, \nu_2, \ldots, \nu_k$ *(respectively, almost periodic, almost automorphic, Birkhoff recurrent, Lagrange stable), then so is* φ.

(iv) *Assume that* $\varphi \in C(\mathbb{R}, \mathcal{X})$ *is strongly comparable by character of recurrence with* $\psi \in C(\mathbb{R}, \mathcal{Y})$. *And suppose further that* ψ *is Lagrange stable. If* ψ *is pseudo-periodic (respectively, pseudo-recurrent), then so is* φ.

A.3. BUC **space.** ([7]) Denote by $BUC(\mathbb{R} \times \mathcal{X}, \mathcal{Y})$ the space of all continuous functions $f : \mathbb{R} \times \mathcal{X} \to \mathcal{Y}$ that satisfy the following conditions:

(i) f is bounded on every bounded subset from $\mathbb{R} \times \mathcal{X}$;

(ii) f is continuous in $t \in \mathbb{R}$ uniformly w.r.t. x on each bounded subset $Q \subset \mathcal{X}$.

We endow $BUC(\mathbb{R} \times \mathcal{X}, \mathcal{Y})$ with the following d distance

$$d(f, g) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(f, g)}{1 + d_k(f, g)}, \tag{A.1}$$

where $d_k(f, g) := \sup_{|t| \le k, x \in Q_k} \rho_1(f(t, x), g(t, x))$. Here $Q_k \subset \mathcal{X}$ is bounded, $Q_k \subset Q_{k+1}$ and $\cup_{k \in \mathbb{N}} Q_k = \mathcal{X}$. Note that d generates the topology of uniform convergence on bounded subsets on $BUC(\mathbb{R} \times \mathcal{X}, \mathcal{Y})$ and $(BUC(\mathbb{R} \times \mathcal{X}, \mathcal{Y}), d)$ is a complete metric space.

Let $f \in BUC(\mathbb{R} \times \mathcal{X}, \mathcal{Y})$ and $\tau \in \mathbb{R}$. Recall that f^τ means the τ-translation of f i.e. $f^\tau(t, x) := f(t + \tau, x)$ for all $(t, x) \in \mathbb{R} \times \mathcal{X}$. Note that $BUC(\mathbb{R} \times \mathcal{X}, \mathcal{Y})$ is invariant w.r.t. translations. Define a mapping $\sigma : \mathbb{R} \times BUC(\mathbb{R} \times \mathcal{X}, \mathcal{Y}) \to BUC(\mathbb{R} \times \mathcal{X}, \mathcal{Y})$, $(\tau, f) \mapsto f^\tau$. Then it can be proved that the triplet $(BUC(\mathbb{R} \times \mathcal{X}, \mathcal{Y}), \mathbb{R}, \sigma)$ is a dynamical system. Given $f \in BUC(\mathbb{R} \times \mathcal{X}, \mathcal{Y})$, $\mathcal{H}(f) \subset BUC(\mathbb{R} \times \mathcal{X}, \mathcal{Y})$ is closed and translation invariant. Consequently, it naturally defines on $\mathcal{H}(f)$ a shift dynamical system $(\mathcal{H}(f), \mathbb{R}, \sigma)$.

We write $BC(\mathcal{X}, \mathcal{Y})$ to mean the space of all continuous functions $f : \mathcal{X} \to \mathcal{Y}$ which are bounded on every bounded subset of $\mathcal{X}$ and equipped with the following metric

$$d(f, g) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(f, g)}{1 + d_k(f, g)},$$

where $d_k(f, g) := \sup_{x \in Q_k} \rho_1(f(x), g(x))$. Note that $(BC(\mathcal{X}, \mathcal{Y}), d)$ is a complete metric space. For any $f \in BUC(\mathbb{R} \times \mathcal{X}, \mathcal{Y})$, define the mapping $\mathcal{F} : \mathbb{R} \to BC(\mathcal{X}, \mathcal{Y})$ by $\mathcal{F}(t) := f(t, \cdot) : \mathcal{X} \to \mathcal{Y}$. Clearly, $\mathcal{F} \in C(\mathbb{R}, BC(\mathcal{X}, \mathcal{Y}))$.

Definition A.15. (i) We say that a function $\varphi \in C(\mathbb{R}, \mathcal{X})$ *possesses the property A* if the motion $\sigma(\cdot, \varphi)$ through φ with respect to the Bebutov dynamical system $(C(\mathbb{R} \times \mathcal{X}), \mathbb{R}, \sigma)$ possesses the property A.

(ii) Similarly, we say that $f \in BUC(\mathbb{R} \times \mathcal{X}, \mathcal{Y})$ *possesses the property A in* $t \in \mathbb{R}$ *uniformly with respect to* x *on each bounded subset* $Q \subset \mathcal{X}$, if the motion $\sigma(\cdot, f) : \mathbb{R} \to BUC(\mathbb{R} \times \mathcal{X}, \mathcal{Y})$ through f with respect to the Bebutov dynamical system $(BUC(\mathbb{R} \times \mathcal{X}, \mathcal{Y}), \mathbb{R}, \sigma)$ possesses the property A.

Here the property A may be stationary, periodic, Bohr/Levitan almost periodic, etc.

A.4. **Proof of Theorem 3.3.**

Proof. Let $\{e_i, i \in \mathbb{N}\} \subset H^1$ be the eigenfunctions of $-\Delta$ forming an orthonormal basis of $L^2(\mathbb{T}^d)$ and $H_n := \operatorname{span}\{e_1, ..., e_n\}$. Let $P_n : L^2(\mathbb{T}^d) \to H_n$ be defined by

$$P_n u := \sum_{i=1}^{n} \langle u, e_i \rangle e_i, \quad u \in L^2(\mathbb{T}^d).$$

Let $\{w_k, k \in \mathbb{N}\}$ be an orthonormal basis of U and set $W^n(t) := \sum_{k=1}^{n} \langle W(t), w_i \rangle_U w_i$. We set $f^n(t, u) := P_n f(t, u)$ and $g^n(t, u) := P_n g(t, u)$. Consider the following equations

(A.2)
$$\begin{cases} \mathrm{d}u^n(t) = \left[(1 + i\alpha)\Delta u^n(t) - (1 + i\beta)P_n|u^n(t)|^2 u^n(t) + f^n(t, u^n(t))\right] \mathrm{d}t + g^n(t, u^n(t))\mathrm{d}W^n(t) \\ u^n(s) = P_n \zeta_s. \end{cases}$$

It follows from Theorem 3.1.1 in [28] that there exists a unique solution $u^n(t), t \geq s$ to (A.2) for any $n \in \mathbb{N}$. Similar to the proof of (3.4), we have

$$\mathbb{E} \sup_{s \leq t \leq s+T} \|u^n(t)\|^2 + \mathbb{E} \int_s^{s+T} \|u^n(t)\|_{H^1}^2 \mathrm{d}t + \mathbb{E} \int_s^{s+T} \|u^n(t)\|_{L^4(\mathbb{T}^d)}^4 \mathrm{d}t \leq C_T(1 + \mathbb{E}\|\zeta_s\|^2) \tag{A.3}$$

for all $T > 0$, where C_T is independent of n.

Then there exists a subsequence of $\{u^n\}$, which we still denote by $\{u^n\}$, such that

(i) $u^n \to u$ weakly in $L^2([s, s+T] \times \Omega, \mathrm{d}t \otimes \mathbb{P}; L^2(\mathbb{T}^d))$, $L^2([s, s+T] \times \Omega, \mathrm{d}t \otimes \mathbb{P}; H^1)$ and $L^4([s, s+T] \times \Omega, \mathrm{d}t \otimes \mathbb{P}; L^4(\mathbb{T}^d))$.

(ii) $|u^n(\cdot)|^2 u^n(\cdot) \to Y^1(\cdot)$ weakly in $L^{\frac{4}{3}}\left([s, s+T] \times \Omega, \mathrm{d}t \otimes \mathbb{P}; L^{\frac{4}{3}}(\mathbb{T}^d)\right)$.

(iii) $f(\cdot, u^n(\cdot)) \to Y^2(\cdot)$ weakly in $L^2\left([s, s+T] \times \Omega, \mathrm{d}t \otimes \mathbb{P}; L^2(\mathbb{T}^d)\right)$.

(iv) $g^n(\cdot, u^n(\cdot)) \to Z(\cdot)$ weakly in $L^2\left([s, s+T] \times \Omega, \mathrm{d}t \otimes \mathbb{P}; L_2(U, L^2(\mathbb{T}^d))\right)$ and hence

$$\int_s^t g^n(\sigma, u^n(\sigma))\mathrm{d}W^n(\sigma) \to \int_s^t Z(\sigma)\mathrm{d}W(\sigma)$$

weakly* in $L^\infty\left([s, s+T], \mathrm{d}t; L^2(\Omega, \mathbb{P}; L^2(\mathbb{T}^d))\right)$.

For all $v \in \cup_{n \geq 1} H_n$, $\phi \in L^\infty([s, s+T] \times \Omega, \mathrm{d}t \otimes \mathbb{P}; \mathbb{R})$, it follows from Fubini's theorem that

$$\begin{aligned}
&\mathbb{E}\left(\int_s^{s+T} \langle u(t), \phi(t)v \rangle \mathrm{d}t\right) \\
&= \lim_{n\to\infty} \mathbb{E}\left(\int_s^{s+T} \langle u^n(t), \phi(t)v \rangle \mathrm{d}t\right) \\
&= \lim_{n\to\infty} \mathbb{E}\Bigg(\int_s^{s+T} \left\langle P_n\zeta_s + \int_s^t \left[(1 + i\alpha)\Delta u^n(\sigma) - (1 + i\beta)P_n|u^n(\sigma)|^2 u^n(\sigma)\right] \mathrm{d}\sigma, \phi(t)v \right\rangle \mathrm{d}t \\
&\qquad + \left\langle \int_s^t f^n(\sigma, u^n(\sigma))\mathrm{d}\sigma + \int_s^t g^n(\sigma, u^n(\sigma))\mathrm{d}W^n(\sigma), \phi(t)v \right\rangle \mathrm{d}t\Bigg) \\
&= \mathbb{E}\left\langle \zeta_s, \int_s^{s+T} \phi(t)v\mathrm{d}t \right\rangle + \mathbb{E}\left(\int_s^{s+T} \left\langle \int_s^t (1 + i\alpha)u(\sigma)\mathrm{d}\sigma, \phi(t)\Delta v \right\rangle \mathrm{d}t\right)
\end{aligned}$$

$$+\mathbb{E}\left(\int_s^{s+T}\left\langle\int_s^t\left(-(1+i\beta)Y^1(\sigma)+Y^2(\sigma)\right)\mathrm{d}\sigma,\phi(t)v\right\rangle\mathrm{d}t\right)$$
$$+\mathbb{E}\left(\int_s^{s+T}\left\langle\int_s^t Z(\sigma)\mathrm{d}W(\sigma),\phi(t)v\right\rangle\mathrm{d}t\right).$$

Therefore, we have

$$u(t)=\zeta_s+\int_s^t\left((1+i\alpha)\Delta u(\sigma)-(1+i\beta)Y^1(\sigma)+Y^2(\sigma)\right)\mathrm{d}t+\int_s^t Z(\sigma)\mathrm{d}W(\sigma)\quad \mathrm{d}t\otimes\mathbb{P}-\text{a.s.}$$

Now we prove that $-(1+i\beta)Y^1+Y^2=-(1+i\beta)|u|^2u+f(\cdot,u)$ and $Z=g(\cdot,u)$ $\mathrm{d}t\otimes\mathbb{P}$-a.s. Note that for any nonnegative $\psi\in L^\infty([s,s+T],\mathrm{d}t;\mathbb{R})$ we have

$$\begin{aligned}&\mathbb{E}\left(\int_s^{s+T}\psi(t)\|u(t)\|^2\mathrm{d}t\right)\\&=\lim_{n\to\infty}\mathbb{E}\left(\int_s^{s+T}\langle\psi(t)u(t),u^n(t)\rangle\,\mathrm{d}t\right)\\&\le\left(\mathbb{E}\int_s^{s+T}\psi(t)\|u(t)\|^2\mathrm{d}t\right)^{\frac12}\liminf_{n\to\infty}\left(\mathbb{E}\int_s^{s+T}\psi(t)\|u^n(t)\|^2\mathrm{d}t\right)^{\frac12}<\infty.\end{aligned}$$

Then

$$\mathbb{E}\left(\int_s^{s+T}\psi(t)\|u(t)\|^2\mathrm{d}t\right)\le\liminf_{n\to\infty}\mathbb{E}\left(\int_s^{s+T}\psi(t)\|u^n(t)\|^2\mathrm{d}t\right).\tag{A.4}$$

According to the product rule and Itô's formula we get

$$\begin{aligned}&\mathbb{E}\left(e^{-c(t-s)}\|u(t)\|^2\right)-\mathbb{E}\|u(s)\|^2\\&=\mathbb{E}\Bigg(\int_s^t e^{-c(\sigma-s)}\Big(2\left\langle(1+i\alpha)\Delta u(\sigma)-(1+i\beta)Y^1(\sigma)+Y^2(\sigma),u(\sigma)\right\rangle\\&\qquad+\|Z(\sigma)\|^2_{L_2(U,L^2(\mathbb{T}^d))}-c\|u(\sigma)\|^2\Big)\mathrm{d}\sigma\Bigg)\end{aligned}\tag{A.5}$$

for any constant c. Let $c=2\lambda_f+L_g^2$ and

$$K_i:=L^2\left([s,s+T]\times\Omega,\mathrm{d}t\otimes\mathbb{P};V_i\right),\quad K_3:=L^4\left([s,s+T]\times\Omega,\mathrm{d}t\otimes\mathbb{P};L^4(\mathbb{T}^d)\right)$$

where $i=1,2$, $V_1:=L^2(\mathbb{T}^d)$ and $V_2:=H^1$. For any $\phi\in K_1\cap K_2\cap K_3$, we obtain

$$\begin{aligned}&\mathbb{E}\left(e^{-c(t-s)}\|u^n(t)\|^2\right)-\mathbb{E}\|u^n(s)\|^2\\&\le\mathbb{E}\Bigg(\int_s^t e^{-c(\sigma-s)}\Big(2\left\langle(1+i\alpha)\Delta(u^n(\sigma)-\phi(\sigma)),u^n(\sigma)-\phi(\sigma)\right\rangle\\&\quad-2\left\langle(1+i\beta)\left(|u^n(\sigma)|^2u^n(\sigma)-|\phi(\sigma)|^2\phi(\sigma)\right),u^n(\sigma)-\phi(\sigma)\right\rangle\\&\quad+2\left\langle f(\sigma,u^n(\sigma))-f(\sigma,\phi(\sigma)),u^n(\sigma)-\phi(\sigma)\right\rangle\\&\quad+\|g(\sigma,u^n(\sigma))-g(\sigma,\phi(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}-c\|u^n(\sigma)-\phi(\sigma)\|^2\Big)\mathrm{d}\sigma\Bigg)\\&\quad+\mathbb{E}\Bigg(\int_s^t e^{-c(\sigma-s)}\Big(2\left\langle(1+i\alpha)\Delta\phi(\sigma),u^n(\sigma)\right\rangle+2\left\langle(1+i\alpha)\Delta(u^n(\sigma)-\phi(\sigma)),\phi(\sigma)\right\rangle\\&\quad-2\left\langle(1+i\beta)|\phi(\sigma)|^2\phi(\sigma),u^n(\sigma)\right\rangle-2\left\langle(1+i\beta)\left(|u^n(\sigma)|^2u^n(\sigma)-|\phi(\sigma)|^2\phi(\sigma)\right),\phi(\sigma)\right\rangle\end{aligned}$$

$$
\begin{aligned}
&+ 2\left\langle f(\sigma,\phi(\sigma)), u^n(\sigma)\right\rangle + 2\left\langle f(\sigma,u^n(\sigma)) - f(\sigma,\phi(\sigma)), \phi(\sigma)\right\rangle - \|g(\sigma,\phi(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\\
&+ 2\left\langle g(\sigma,u^n(\sigma)), g(\sigma,\phi(\sigma))\right\rangle_{L_2(U,L^2(\mathbb{T}^d))} - 2c\langle u^n(\sigma),\phi(\sigma)\rangle + c\|\phi(\sigma)\|^2\Big)\mathrm{d}\sigma\Bigg)\\
\leq{}& \mathbb{E}\Bigg(\int_s^t e^{-c(\sigma-s)}\Big(2\left\langle (1+i\alpha)\Delta\phi(\sigma), u^n(\sigma)\right\rangle + 2\left\langle (1+i\alpha)\Delta(u^n(\sigma)-\phi(\sigma)), \phi(\sigma)\right\rangle\\
&- 2\left\langle (1+i\beta)|\phi(\sigma)|^2\phi(\sigma), u^n(\sigma)\right\rangle - 2\left\langle (1+i\beta)\left(|u^n(\sigma)|^2u^n(\sigma) - |\phi(\sigma)|^2\phi(\sigma)\right), \phi(\sigma)\right\rangle\\
&+ 2\left\langle f(\sigma,\phi(\sigma)), u^n(\sigma)\right\rangle + 2\left\langle f(\sigma,u^n(\sigma)) - f(\sigma,\phi(\sigma)), \phi(\sigma)\right\rangle - \|g(\sigma,\phi(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\\
&+ 2\left\langle g(\sigma,u^n(\sigma)), g(\sigma,\phi(\sigma))\right\rangle_{L_2(U,L^2(\mathbb{T}^d))} - 2c\langle u^n(\sigma),\phi(\sigma)\rangle + c\|\phi(\sigma)\|^2\Big)\mathrm{d}\sigma\Bigg).
\end{aligned}
$$

Letting $n\to\infty$, in view of (A.4) we have

$$
\begin{aligned}
&\mathbb{E}\left(\int_s^{s+T}\psi(t)\left(e^{-c(t-s)}\|u(t)\|^2 - \|u(s)\|^2\right)\mathrm{d}t\right)\\
&\leq \mathbb{E}\Bigg(\int_s^{s+T}\psi(t)\Bigg(\int_s^t e^{-c(\sigma-s)}\Big[2\left\langle (1+i\alpha)\Delta\phi(\sigma), u(\sigma)\right\rangle + 2\left\langle (1+i\alpha)\Delta(u(\sigma)-\phi(\sigma)), \phi(\sigma)\right\rangle\\
&\quad - 2\left\langle (1+i\beta)|\phi(\sigma)|^2\phi(\sigma), u(\sigma)\right\rangle - 2\left\langle (1+i\beta)\left(Y^1(\sigma) - |\phi(\sigma)|^2\phi(\sigma)\right), \phi(\sigma)\right\rangle\\
&\quad + 2\left\langle f(\sigma,\phi(\sigma)), u(\sigma)\right\rangle + 2\left\langle Y^2(\sigma) - f(\sigma,\phi(\sigma)), \phi(\sigma)\right\rangle - \|g(\sigma,\phi(\sigma))\|^2_{L_2(U,L^2(\mathbb{T}^d))}\\
&\quad + 2\left\langle Z(\sigma), g(\sigma,\phi(\sigma))\right\rangle_{L_2(U,L^2(\mathbb{T}^d))} - 2c\langle u(\sigma),\phi(\sigma)\rangle + c\|\phi(\sigma)\|^2\Big]\mathrm{d}\sigma\Bigg)\mathrm{d}t\Bigg).
\end{aligned}
$$

Combining (A.5), we get

$$
\begin{aligned}
\mathbb{E}\Bigg(\int_s^{s+T}\psi(t)\Bigg(\int_s^t e^{-c(\sigma-s)}\Big[&-2\left\langle (1+i\beta)\left(Y^1(\sigma) - |\phi(\sigma)|^2\phi(\sigma)\right), u(\sigma)-\phi(\sigma)\right\rangle\\
&+ 2\left\langle Y^2(\sigma) - f(\sigma,\phi(\sigma)), u(\sigma)-\phi(\sigma)\right\rangle\\
&+ \|g(\sigma,\phi(\sigma)) - Z(\sigma)\|^2_{L_2(U,L^2(\mathbb{T}^d))} - c\|u(\sigma)-\phi(\sigma)\|^2\Big]\mathrm{d}\sigma\Bigg)\mathrm{d}t\Bigg) \leq 0.
\end{aligned}
\tag{A.6}
$$

Letting $\phi = u$ in (A.6), we have $Z = g(\cdot,u)$ $\mathrm{d}t\otimes\mathbb{P}$-a.s. Then letting $\phi = u - \varepsilon\tilde{\phi}v$ for $\varepsilon>0$ and $\tilde{\phi}\in L^\infty([s,s+T]\times\Omega, \mathrm{d}t\times\mathbb{P};\mathbb{R})$ we have

$$
\begin{aligned}
&\mathbb{E}\Bigg(\int_s^{s+T}\psi(t)\Bigg(\int_s^t e^{-c(\sigma-s)}\Big[-2\Big\langle (1+i\beta)\Big(Y^1(\sigma) - |u(\sigma)-\varepsilon\tilde{\phi}(\sigma)v|^2(u(\sigma)-\varepsilon\tilde{\phi}(\sigma)v)\Big), \varepsilon\tilde{\phi}(\sigma)v\Big\rangle\\
&\quad + 2\Big\langle Y^2(\sigma) - f(\sigma, u(\sigma)-\varepsilon\tilde{\phi}(\sigma)v), \varepsilon\tilde{\phi}(\sigma)v\Big\rangle - c\varepsilon^2\|\tilde{\phi}v\|^2\Big]\mathrm{d}\sigma\Bigg)\mathrm{d}t\Bigg) \leq 0.
\end{aligned}
$$

Dividing both side by ε and letting $\varepsilon\to 0$, in view of Lebesgue's dominated convergence theorem, we obtain

$$
\mathbb{E}\Bigg(\int_s^{s+T}\psi(t)\Bigg(\int_s^t e^{-c(\sigma-s)}\Big[-\Big\langle (1+i\beta)\left(Y^1(\sigma) - |u(\sigma)|^2u(\sigma)\right), \tilde{\phi}(\sigma)v\Big\rangle
$$

$$+ \left\langle Y^2(\sigma) - f(\sigma, u(\sigma)), \tilde{\phi}(\sigma) v \right\rangle \Big] \mathrm{d}\sigma \Big) \mathrm{d}t \Bigg) \leq 0.$$

Therefore, we have $-(1+i\beta)Y^1 + Y^2 = -(1+i\beta)|u|^2 u + f(\cdot, u)$ $\mathrm{d}t \otimes \mathbb{P}$-a.s.

Now we show the uniqueness of solutions. Suppose that there exist two solutions u_1 and u_2, then

$$\begin{aligned}
&\mathbb{E}\left(\mathrm{e}^{(2\lambda_* - 2\lambda_f - L_g^2)(t-s)} \|u_1(t,s,\zeta_s) - u_2(t,s,\zeta_s)\|^2\right) \\
&= \int_s^t (2\lambda_* - 2\lambda_f - L_g^2) \mathrm{e}^{(2\lambda_* - 2\lambda_f - L_g^2)(\sigma - s)} \mathbb{E}\|u_1(\sigma,s,\zeta_s) - u_2(\sigma,s,\zeta_s)\|^2 \mathrm{d}\sigma \\
&\quad + \mathbb{E}\int_s^t \mathrm{e}^{(2\lambda_* - 2\lambda_f - L_g^2)(\sigma - s)} \Big(2\langle (1+i\alpha)\Delta\left(u_1(\sigma,s,\zeta_s) - u_2(\sigma,s,\zeta_s)\right), u_1(\sigma,s,\zeta_s) - u_2(\sigma,s,\zeta_s)\rangle \\
&\qquad + 2\langle -(1+i\beta)\left(|u_1(\sigma,s,\zeta_s)|^2 u_1(\sigma,s,\zeta_s) - |u_2(\sigma,s,\zeta_s)|^2 u_2(\sigma,s,\zeta_s)\right), u_1(\sigma,s,\zeta_s) - u_2(\sigma,s,\zeta_s)\rangle \\
&\qquad + 2\langle f(\sigma, u_1(\sigma,s,\zeta_s)) - f(\sigma, u_2(\sigma,s,\zeta_s)), u_1(\sigma,s,\zeta_s) - u_2(\sigma,s,\zeta_s)\rangle \\
&\qquad + \|g(\sigma, u_1(\sigma,s,\zeta_s)) - g(\sigma, u_2(\sigma,s,\zeta_s))\|^2_{L_2(U, L^2(\mathbb{T}^d))} \Big) \mathrm{d}\sigma \leq 0.
\end{aligned}$$

Similar to the proof of (3.4), we have

$$\mathbb{E} \sup_{s \leq t \leq s+T} \|u^n(t)\|^2_{H^1} \leq C_T(1 + \mathbb{E}\|\zeta_s\|^2_{H^1}). \tag{A.7}$$

Here C_T is independent of n. Therefore, $u^n(t) \to u(t)$ weakly in $L^2\left(\Omega, \mathbb{P}; H^1\right)$ and

$$\sup_{s \leq t \leq s+T} \mathbb{E}\|u(t)\|^2_{H^1} \leq C_T(1 + \mathbb{E}\|\zeta_s\|^2_{H^1}).$$

The proof is complete. □

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M. Cheng: School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, P. R. China; Fakultat für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany

E-mail address: mengyucheng@mail.dlut.edu.cn; mengyu.cheng@hotmail.com

Z. Liu: School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, P. R. China

E-mail address: zxliu@dlut.edu.cn

M. Röckner: Fakultat für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany

E-mail address: roeckner@math.uni-bielefeld.de