

Markov uniqueness and Fokker-Planck-Kolmogorov equations

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Dedicated to Masatoshi Fukushima for his 88th birthday.

Abstract. In this paper we show that Markov uniqueness for symmetric pre-Dirichlet operators L follows from the uniqueness of the corresponding Fokker-Planck-Kolmogorov equation (FPKE). Since in recent years a considerable number of uniqueness results for FPKE's have been achieved, we obtain new Markov uniqueness results in concrete cases. A selection of such will be presented in this paper. They include cases with killing and with degenerate diffusion coefficients.

1 Introduction and framework

In this paper we fix a σ -finite measure space (E, \mathcal{B}, m) . Let $L^p := L^p(m) = L^p(E, m)$, $p \in [1, \infty]$ be the corresponding (real) L^p -spaces with their usual norms $\|\cdot\|_p$ and inner product $(\cdot, \cdot)_2$ if $p = 2$. On $L^p(m)$ we shall consider linear operators

$$L: D(L) \subset L^p(m) \rightarrow L^p(m)$$

with their usual partial order defined by

$$L_1 \subset L_2 \stackrel{\text{Def.}}{\Leftrightarrow} \Gamma(L_1) \subset \Gamma(L_2),$$

where $D(L)$ is a linear subspace of L^p , called domain of L , and

$$\Gamma(L) := \{(u, Lu) \in L^p \times L^p : u \in D(L)\}$$

is the graph of L . In particular, we shall consider those L which generate a (unique) strongly continuous semigroup of (everywhere defined) continuous linear operators on L^p , denoted by e^{tL} , $t \geq 0$. Henceforth such L will be shortly called generator (on L^p). We refer to [Pa1985] for the notions and the well-known characterization of such generators.

We recall that a generator L is always closed, i.e. $\Gamma(L)$ is a closed subset of $L^p \times L^p$, with domain $D(L)$ dense in L^p and that for $p \in (1, \infty)$, its adjoint operator $(L^*, D(L^*))$ on $L^{p'}$, with $p' := \frac{p}{p-1}$, generates a strongly continuous semigroup of linear operators, e^{tL^*} , $t \geq 0$, on $L^{p'}$. This satisfies

$$e^{tL^*} = (e^{tL})^*, \quad t \geq 0, \quad (1.1)$$

(see [Pa1985, Chapter 1, Corollary 10.6]). We consider three cases of sets of generators on L^p for $p \in (1, \infty)$:

- (1) Let D_0^* be a dense linear subspace of $L^{p'}$ and $L_0^* : D_0^* \subset L^{p'} \rightarrow L^{p'}$ a linear operator. Define $\mathcal{M} := \mathcal{M}(L_0^*, D_0^*)$ to be the set of all linear operators $L : D(L) \subset L^p \rightarrow L^p$ such that $L_0^* \subset L^*$ and L is a generator on L^p .
- (2) Let D_0 be a dense linear subspace of L^2 and $L_0 : D_0 \subset L^2 \rightarrow L^2$ a symmetric linear operator, i.e., $L_0 \subset L_0^*$, which is upper bounded, i.e.

$$\sup_{u \in D_0 \setminus \{0\}} (L_0 u, u)_2 \|u\|_2^{-2} < \infty.$$

Define $\mathcal{M}_{sym} := \mathcal{M}_{sym}(L_0, D_0)$ to be the set of all linear operators $L : D(L) \subset L^2 \rightarrow L^2$ such that $L_0 \subset L$, L is a generator on L^2 and L is symmetric, i.e., $L \subset L^*$.

- (3) Let (L_0, D_0) be as in (2) and define $\mathcal{M}_{sym,M} := \mathcal{M}_{sym,M}(L_0, D_0)$ to be the subset of all $(L, D(L))$ in $\mathcal{M}_{sym}(L_0, D_0)$ such that each e^{tL} , $t \geq 0$, is sub-Markovian, i.e., if $u \in L^2$ such that $0 \leq u \leq 1$, then $0 \leq e^{tL}u \leq 1$.

Concerning (2) we note that by [Pa1985, Theorems 4.2 and 5.3] it obviously follows that $\mathcal{M}_{sym}(L_0, D_0)$ coincides with the set of all linear operators $L : D(L) \subset L^2 \rightarrow L^2$ such that $L_0 \subset L$ and L is upper bounded and self-adjoint, i.e., $L = L^*$. Furthermore, $\mathcal{M}_{sym}(L_0, D_0)$ is not empty, because the Friedrichs extension of (L_0, D_0) is self-adjoint and upper bounded (see e.g. [FOT2011, p. 131]).

Concerning (3) we refer to [FOT2011] and [MR1992] for more details on such sub-Markovian operator semigroups.

The first aim of this paper is to derive a "parabolic" condition in each of the cases (1),(2),(3) which implies that the respective sets \mathcal{M} , \mathcal{M}_{sym} , $\mathcal{M}_{sym,M}$ contain at most one element. Here, "parabolic" means in terms of the corresponding Fokker-Planck-Kolmogorov equation (FPKE). The second aim of this paper is (by refining this "parabolic condition") to use uniqueness results from [BKRS2015] to obtain new results on "Markov uniqueness" in the sense of the following definition:

Definition 1.1. *Let (L_0, D_0) be as in (2) above. (L_0, D_0) is called Markov unique if $\mathcal{M}_{sym,M}$ contains exactly one element.*

Let us note that our notion of "Markov uniqueness" is in fact stronger than the one extensively studied in the literature, since there, uniqueness is studied in the subset of all linear operators $(L, D(L))$ in $\mathcal{M}_{sym,M}(L_0, D_0)$, which are nonpositive definite, i.e., $\sup_{u \in D(L)} (Lu, u)_2 \leq 0$, while also assuming that (L_0, D_0) is nonpositive definite.

The literature on Markov uniqueness is quite extensive and a number of types of state spaces E , as e.g. \mathbb{R}^d or infinite dimensional vector spaces or manifolds have been considered. To the best of our knowledge the first paper on this subject is [Ta1985] by Masayoshi Takeda. To give an overview of the entire literature is beyond the scope of this paper. Instead, we refer to the references in [FOT2011], [Eb1999], [EL2006], [RS2011], [AR1995], [ARZ1993] and the more recent papers [AMR2014], [RZZ2017].

It seems, however, that the method to prove Markov uniqueness proposed in this paper, i.e., by using the corresponding FPKE, is new, though it is very natural. Furthermore our applications and examples in Section 3, even though they are all in the classical case $E := \mathbb{R}^d$, appear to be not covered by the existing literature, in particular, since they include cases with degenerate diffusion coefficients and we can allow "killing", more precisely in our applications, where L is a partial differential operator on \mathbb{R}^d , this operator is allowed to have a (negative) zero order coefficient.

Finally, we would like to recall the notion of "strong uniqueness" which is different from Markov uniqueness. In our context here it means that the larger set $\mathcal{M}_{sym}(L_0, D_0)$ contains exactly one element which is equivalent to the fact that the closure of (L_0, D_0) is self adjoint on L^2 . For more details we refer to [AKR1995], [Eb1999] and as a very recent paper to [AKMR2020], in particular to the lists of references in them.

2 The main idea and a parabolic condition for uniqueness

For a set \mathcal{F} of real-valued functions on E and $T \in (0, \infty)$ we define \mathcal{F}_T to be the set of all functions of the form

$$[0, T] \times E \ni (t, x) \mapsto f(t)\varphi(x) =: (f \otimes \varphi)(t, x),$$

where $\varphi \in \mathcal{F}$ and $f \in C^1([0, T]; \mathbb{R})$ with $f(T) = 0$.

Let us start with case (1) from the introduction and consider $(L, D(L)) \in \mathcal{M}(L_0^*, D_0^*)$.

For $t \geq 0$ we set

$$T_t^L := e^{tL}, \quad T_t^{L^*} = e^{tL^*}$$

(cf. (1.1)). Then for all $\varphi \in D_0^*$, $u \in L^p$ and $t \geq 0$ we have

$$\begin{aligned} \int \varphi T_t^L u \, dm &= \int T_t^{L^*} \varphi u \, dm \\ &= \int \varphi u \, dm + \int_0^t \int T_s^{L^*} L^* \varphi u \, dm \, ds \\ &= \int \varphi u \, dm + \int_0^t \int L_0^* \varphi T_s^L u \, dm \, ds. \end{aligned} \tag{2.1}$$

Hence defining the (signed) measure $\mu_t(dx) := T_t^L u(x) m(dx)$, $t \geq 0$, by the (integral) product rule for all $f \otimes \varphi \in D_{0,T}^*$ (defined as above with $\mathcal{F} := D_0^*$) we have

$$\begin{aligned}
& \int (f \otimes \varphi)(t, x) \mu_t(dx) = f(t) \int \varphi(x) \mu_t(dx) \\
& = f(0) \int \varphi d\mu_0 + \int_0^t f(s) \int L_0^* \varphi d\mu_s ds + \int_0^t f'(s) \int \varphi d\mu_s ds \\
& = \int (f \otimes \varphi)(0, x) \mu_0(dx) + \int_0^t \int \left(\frac{\partial}{\partial s} + L_0^* \right) (f \otimes \varphi) d\mu_s ds.
\end{aligned}$$

In particular, for $t = T$

$$\int_0^T \int \left(\frac{\partial}{\partial s} + L_0^* \right) (f \otimes \varphi) d\mu_s ds = - \int (f \otimes \varphi)(0, x) \mu_0(dx). \quad (2.2)$$

(2.1) (equivalently (2.2)) means that $\mu_t = T_t^L u \cdot m, u \in L^p, t \geq 0$, solves the FPKE (up to time T for every $T \in (0, \infty)$) corresponding to $(L_0^*, D(L_0^*))$ (see [BKRS2015]).

Now it is very easy to prove the following "parabolic condition" that ensures that

$$\#\mathcal{M}(L_0^*, D_0^*) \leq 1$$

(where as usual $\#$ is an abbreviation for cardinality).

Proposition 2.1. *Assume that for every $T \in (0, \infty)$*

$$\left(\frac{\partial}{\partial s} + L_0^* \right) D_{0,T}^* \text{ is dense in } L^{p'}([0, T] \times E, dt \otimes m). \quad (2.3)$$

Then $\mathcal{M}(L_0^, D_0^*)$ contains at most one element.*

Proof. Let $(\tilde{L}, D(\tilde{L})) \in \mathcal{M}_{sym}(L_0^*, D_0^*)$. Then, as seen above, $\tilde{\mu}_t := T_t^{\tilde{L}} u m, u \in L^p, t \geq 0$, also satisfies (2.1), hence (2.2). So, (by subtracting) for $g(t, \cdot) := T_t^L u - T_t^{\tilde{L}} u, t \geq 0$, we obtain for all $T \in (0, \infty)$

$$\int_0^T \int \left(\frac{\partial}{\partial s} + L_0^* \right) (f \otimes \varphi) g(s, \cdot) dm ds = 0$$

for all $f \otimes \varphi \in D_{0,T}^*$. Since $g \in L^p([0, T] \times E, dt \otimes m)$, by (2.3) it follows that $g = 0$, and the assertion follows, since $u \in L^p$ was arbitrary. \square

Now let us consider case (2) from the introduction. So, let $(L, D(L)) \in \mathcal{M}_{sym}(L_0, D_0)$. Then using the same notation as in case (1) we analogously obtain for all $\varphi \in D_0, u \in L^2$ and $t \geq 0$

$$\int \varphi T_t^L u dm = \int \varphi u dm + \int_0^t \int L_0 \varphi T_s^L u dm ds, \quad (2.4)$$

hence for $\mu_t := T_t^L u m$ and for all $f \otimes \varphi \in D_{0,T}, T \in (0, \infty)$ we have

$$\int_0^T \left(\frac{\partial}{\partial s} + L_0^* \right) (f \otimes \varphi) d\mu_s ds = - \int (f \otimes \varphi)(0, x) \mu_0(dx), \quad (2.5)$$

i.e., $\mu_t, t \geq 0$, solves the FPKE corresponding to (L_0, D_0) .

Analogously to Proposition 2.1 we then prove the following result.

Proposition 2.2. *Assume that for every $T \in (0, \infty)$*

$$\left(\frac{\partial}{\partial s} + L_0\right)D_{0,T} \text{ is dense in } L^2([0, T] \times E, dt \otimes m). \quad (2.6)$$

Then $\mathcal{M}_{sym}(L_0, D_0)$ consists of exactly one element.

In case (3) if $(L, D(L)) \in \mathcal{M}_{sym, M}(L_0, D_0)$, then obviously $T_t^L(L^2 \cap L^\infty) \subset L^2 \cap L^\infty$, and since $L^2 \cap L^\infty$ is dense in L^2 , T_t^L is uniquely determined on $L^2 \cap L^\infty$.

Proposition 2.3. *Suppose that $L_0(D_0) \subset L^1$ (which automatically holds if $m(E) < \infty$) and that for every $T \in (0, \infty)$*

$$\left(\frac{\partial}{\partial s} + L_0\right)D_{0,T} \text{ is dense in } L^1([0, T] \times E, dt \otimes m). \quad (2.7)$$

Then $\mathcal{M}_{sym, M}(L_0, D_0)$ consists of at most one element. If the semigroup generated by the Friedrichs extension of (L_0, D_0) is sub-Markovian, then this extension is the unique element in $\mathcal{M}_{sym, M}(L_0, D_0)$.

Proof. We repeat the proof of Proposition 2.2 (respectively, 2.1) with $u \in L^2 \cap L^\infty$. Then for all $T \in (0, \infty)$ we have for g as in the proof of Proposition 2.3 that $g \in L^\infty([0, T] \times E, dt \otimes m)$. Then by (2.7) we conclude again that $g = 0$. Since T_t^L is uniquely determined on $L^2 \cap L^\infty$, the assertion follows. \square

Clearly, conditions (2.3), (2.6) and (2.7) are not easy to check in applications and certainly too strong, at least in case (3). So, let us discuss a weaker condition in this case.

Let $(L, D(L)) \in \mathcal{M}_{sym, M}(L_0, D_0)$ and fix $u \in L^\infty$ such that $u \geq 0$ and $\int u \, dm = 1$. Then $\mu_t^L := T_t^L u \, m$, $t \geq 0$, are subprobability measures on (E, \mathcal{B}) , i.e., $\mu_t^L(E) \leq 1$ for all $t \geq 0$. We note that obviously each T_t^L is uniquely determined by its values on such u . We have seen that $\mu_t := \mu_t^L$, $t \geq 0$, solves the corresponding FPKE

$$\int \varphi \, d\mu_t = \int \varphi \, d\mu_0 + \int_0^t \int L_0 \varphi \, d\mu_s \, ds, \quad t \geq 0, \forall \varphi \in D_0, \quad (2.8)$$

hence for all $T \in (0, \infty)$, $f \otimes \varphi \in D_{0,T}$

$$\int_0^T \int \left(\frac{\partial}{\partial s} + L_0\right)(f \otimes \varphi) \, d\mu_s \, ds = - \int (f \otimes \varphi)(0, x) \mu_0(dx). \quad (2.9)$$

(2.8) and (2.9) are equations for paths $(\mu_t)_{t \geq 0}$ of subprobability measures on (E, \mathcal{B}) such that $[0, \infty) \ni t \rightarrow \mu_t(A)$ is Lebesgue measurable for all $A \in \mathcal{B}$. Define \mathcal{SP} to be the set of all such paths, and $\mathcal{SP}(T)$ the set of their restrictions to $[0, T]$

Now the following result is obvious.

Theorem 2.4. *If, for every probability density $u \in L^1 \cap L^\infty$, (2.8) (or (2.9)) has a unique solution $(\mu_t)_{t \geq 0} \in \mathcal{SP}$ such that each μ_t , $t \geq 0$, is absolutely continuous with respect to m and such that $\mu_0 = u \cdot m$, then $\mathcal{M}_{sym, M}(L_0, D_0)$ consists of at most one element. If, in addition, the semigroup generated by the Friedrichs extension of $(L_0, D(L_0))$ is sub-Markovian, then this extension is the unique element in $\mathcal{M}_{sym, M}(L_0, D_0)$.*

In Theorem 2.4, it is enough to prove uniqueness for (2.9) (or (2.10)) in the subclass of all $(\mu_t)_{t \geq 0} \in \mathcal{SP}$ for which each $\mu_t, t \geq 0$, is absolutely continuous with respect to m with bounded density, i.e., one only needs uniqueness in a convex subset of \mathcal{SP} . Therefore, the following result, which was first observed in [BDPRSt2007], is useful and goes beyond absolutely continuous solutions.

Proposition 2.5. *Let $T \in (0, \infty)$ and ζ be a subprobability measure on (E, \mathcal{B}) and let $\mathcal{K}_\zeta \subset \mathcal{SP}(T)$ be a non-empty convex set such that each $(\mu)_{t \in [0, T]} \in \mathcal{K}_\zeta$ is a solution to (2.8) (hence to (2.9)) with $\mu_0 = \zeta$. Suppose that for every $(\mu_t)_{t \in [0, T]} \in \mathcal{K}_\zeta$*

$$\left(\frac{\partial}{\partial s} + L_0\right)(D_{0, T}) \text{ is dense in } L^1([0, T] \times E, \mu_t dt). \quad (2.10)$$

Let $(\mu_t)_{t \in [0, T]}, (\tilde{\mu}_t)_{t \in [0, T]} \in \mathcal{K}_\zeta$. Then $\mu_t = \tilde{\mu}_t$ for dt -a.e. $t \in [0, T]$.

Remark 2.6. Clearly for $(L, D(L)) \in \mathcal{M}_{sym, M}(L_0, D_0)$ and the corresponding solutions $(\mu_t^L)_{t \geq 0}$ defined above, condition (2.10) is weaker than condition (2.7) in Proposition 2.3, since $\sup_{t \in [0, T]} \|T_t^L u\|_\infty < \infty$.

Proof of Proposition 2.5. Since \mathcal{K}_ζ is convex, we have that $\nu_t := \frac{1}{2}\mu_t + \frac{1}{2}\tilde{\mu}_t, t \geq 0$, is again in \mathcal{K}_ζ and

$$\mu_t dt = g \nu_t dt, \quad \tilde{\mu}_t dt = \tilde{g} \nu_t dt \quad (2.11)$$

for some $g, \tilde{g} \in L^\infty([0, T] \times E, \nu_t dt)$.

Furthermore, by (2.9) it follows that for all $f \otimes \varphi \in D_{0, T}$

$$\int_0^T \int \left(\frac{\partial}{\partial s} + L_0\right)(f \otimes \varphi)(g - \tilde{g}) d\nu_s ds = 0. \quad (2.12)$$

Hence by (2.10) this implies that $g = \tilde{g}$ and the assertion follows. \square

Proposition 2.5 and the observation that (at least in many cases) it suffices to check (2.10) for just one solution in \mathcal{K}_ζ , are the core of the proof of many results on uniqueness of solutions in \mathcal{SP} to concrete FPKEs in Chapter 9 of [BKRS2015], which thus can be applied to prove Markov uniqueness for many examples of given operators (L_0, D_0) on $L^2(m)$ as above. We shall present a selection of such in the next section. We shall restrict ourselves to the symmetric case, i.e., $p = 2$ and $L_0 \subset L_0^*$, though also nonsymmetric cases (as in case (1) from Section 1) can be treated if one has enough knowledge about the dual operator (L_0^*, D_0^*) on $(L^p)'$ for $p \in (1, \infty)$ (see Remark 4.2 below).

3 Some uniqueness results for FPKEs

In the rest of the paper we shall concentrate on the case where the state space E is equal to \mathbb{R}^d . By the same ideas it is, however, possible to obtain Markov uniqueness from uniqueness results of FPKEs on more general state spaces, including infinite dimensional vector spaces or manifolds. This will be done in future work.

3.1 Fokker-Planck-Kolmogorov equations

As already mentioned we shall use the uniqueness results on FPKEs from [BKRS2015, Chapter 9]. So, let us briefly recall the framework there, but for simplicity restricting to solutions in \mathcal{SP} , since we shall only use these in our applications below.

Below (E, \mathcal{B}) from the previous sections will always be $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $d \in \mathbb{N}$, where $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ -algebra of \mathbb{R}^d . Consider a partial differential operator of the form

$$L_0\varphi = a^{ij}\partial_{x_i}\partial_{x_j}\varphi + b^i\partial_{x_i}\varphi + c\varphi, \quad \varphi \in D_0 := C_0^\infty(\mathbb{R}^d), \quad (3.1)$$

where we use Einstein's summation convention, $\partial_{x_i} := \frac{\partial}{\partial x_i}$, $1 \leq i \leq d$, a^{ij} , b^i , $c : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$, with $c \leq 0$, are $\mathcal{B}(\mathbb{R}^d)$ -measurable functions, $A(t, x) := (a^{ij}(t, x))_{1 \leq i, j \leq d}$ is a nonnegative definite matrix for all $(t, x) \in [0, T] \times \mathbb{R}^d$ and $T \in (0, \infty)$ is fixed. For some of the results below we need to assume local boundedness and local strict ellipticity of A , i.e.:

(H1) For each ball $U \subset \mathbb{R}^d$ there exist $\gamma(U)$, $M(U) \in (0, \infty)$ such that

$$\gamma(U) \cdot I \leq A(t, x) \leq M(U) \cdot I \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d,$$

where I denotes the $d \times d$ identity matrix.

Let \mathcal{SP} be defined as in Section 2. We say that $(\mu_t)_{t \geq 0} \in \mathcal{SP}$ satisfies the FPKE (up to time T for L_0) if a^{ij} , b^i , $c \in L_{loc}^1([0, T] \times \mathbb{R}^d, \mu_t dt)$ and for every $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\int \varphi d\mu_t = \int \varphi d\mu_0 + \int_0^t \int L_0\varphi d\mu_s ds \quad \text{for } dt\text{-a.e. } t \in [0, T]. \quad (3.2)$$

In Subsection 3.2 - 3.4 below we shall only be interested in the so-called subprobability solutions to (3.2), i.e., we a priori restrict to a class $\mathcal{SP}_\nu \subset \mathcal{SP}$ in which we search for a (hopefully unique) solution to (3.2). So, given a subprobability measure ν on $\mathcal{B}(\mathbb{R}^d)$ (i.e., $\nu \geq 0$ and $\nu(\mathbb{R}^d) \leq 1$), \mathcal{SP}_ν is defined to be the set of all $(\mu_t)_{t \in [0, T]} \in \mathcal{SP}(T)$ with the following properties:

$$(\mu_t)_{t \in [0, T]} \text{ solves (3.2),} \quad (3.3)$$

$$c \in L^1([0, T] \times \mathbb{R}^d, \mu_t dt), \quad (3.4)$$

$$b \in L^2([0, T] \times U, \mu_t dt; \mathbb{R}^d) \text{ for all balls } U \subset \mathbb{R}^d, \quad (3.5)$$

$$\mu_0 = \nu \text{ and } \mu_t(\mathbb{R}^d) \leq \nu(\mathbb{R}^d) + \int_0^t \int c(x, s) \mu_s(dx) ds \text{ for } dt\text{-a.e. } t \in [0, T]. \quad (3.6)$$

Clearly, if $\nu \neq 0$, by dividing by $\nu(\mathbb{R}^d)$, we may assume, without loss of generality concerning the uniqueness of solutions in \mathcal{SP}_ν for (3.2), that $\nu(\mathbb{R}^d) = 1$. Below we fix a probability measure ν on $\mathcal{B}(\mathbb{R}^d)$.

Now let us recall several uniqueness results for (3.2) from [BKRS2015, Chapter 9,]. Below let dx denote Lebesgue measure on \mathbb{R}^d .

3.2 Nondegenerate VMO diffusion coefficients

Let us recall the definition of the VMO(=vanishing mean oscillation)-property of a function (see [K2007] and the references therein), which is a vast generalization of local Lipschitzianity.

Let g be a bounded Borel-measurable function on \mathbb{R}^{d+1} . Set

$$O(g, R) := \sup_{(x,t) \in \mathbb{R}^{d+1}} \sup_{r \leq R} r^{-2} |U(x, r)|^{-2} \times \\ \times \int_t^{t+r^2} \int \int_{y,z \in U(x,r)} |g(s, y) - g(s, z)| dy dz ds.$$

If $\lim_{R \rightarrow 0} O(g, R) = 0$, then we say that the function g belongs to the class $VMO_x(\mathbb{R}^{d+1})$.

Suppose that a Borel-measurable function g is defined on $[0, T] \times \mathbb{R}^d$ and bounded on $[0, T] \times U$ for every ball U . We extend g by zero to the whole space \mathbb{R}^{d+1} . If for every function $\zeta \in C_0^\infty(\mathbb{R}^d)$ the function $g\zeta$ belongs to the class $VMO_x(\mathbb{R}^{d+1})$, then we say that g belongs to the class $VMO_{x,loc}([0, T] \times \mathbb{R}^d)$.

Theorem 3.1. *Let (H1) hold and assume that*

$$a^{ij} \in VMO_{x,loc}([0, T] \times \mathbb{R}^d), \quad 1 \leq i, j \leq d.$$

Then the set

$$\{(\mu_t)_{t \in [0, T]} \in \mathcal{SP}_\nu : a^{ij}, b^i \in L^1([0, T] \times \mathbb{R}^d, \mu_t dt)\} \quad (3.7)$$

contains at most one element.

Proof. See [BKRS2015, Theorem 9.3.6]. □

3.3 Nondegenerate locally Lipschitz diffusion coefficients

In this subsection and the next one we use the following condition:

(H2) For every ball $U \subset \mathbb{R}^d$ there exists $\Lambda(U) \in (0, \infty)$ such that for all $1 \leq i, j \leq d$

$$|a^{ij}(t, x) - a^{ij}(t, y)| \leq \Lambda(U)|x - y| \quad \forall t \in [0, T], x, y \in U.$$

Theorem 3.2. *Suppose that conditions (H1) and (H2) hold, that $c \leq 0$ and that $b \in L_{loc}^p([0, T] \times \mathbb{R}^d, dt dx; \mathbb{R}^d)$, $c \in L_{loc}^{\frac{p}{2}}([0, T] \times \mathbb{R}^d, dt dx)$ for some $p > d + 2$. Assume also that there exists $(\mu_t)_{t \geq 0} \in \mathcal{SP}_\nu$ satisfying the condition*

$$|a^{ij}|/(1 + |x|^2) + |b^i|/(1 + |x|) \in L^1([0, T] \times \mathbb{R}^d, \mu_t dt), \quad 1 \leq i, j \leq d.$$

Then the set \mathcal{SP}_ν consists of exactly one element.

Proof. See [BKRS2015, Theorem 9.4.3]. □

3.4 Nondegenerate diffusion coefficients and the Lyapunov function condition

The function V in the following theorem is called a Lyapunov function.

Theorem 3.3. *Suppose that conditions (H1) and (H2) hold, $c \leq 0$ and that $b \in L^p_{loc}([0, T] \times \mathbb{R}^d, dt dx; \mathbb{R}^d)$, $c \in L^{\frac{p}{2}}_{loc}([0, T] \times \mathbb{R}^d, dt dx)$ for some $p > d + 2$. Suppose also that there exists a positive function $V \in C^2(\mathbb{R}^d)$ such that $V(x) \rightarrow +\infty$ as $|x| \rightarrow +\infty$ and for some $C \in (0, \infty)$ and all $(t, x) \in [0, T] \times \mathbb{R}^d$ we have*

$$L_0V(t, x) \leq C + CV(x).$$

Then the set \mathcal{SP}_ν contains at most one element.

Proof. See [BKRS2015, Theorem 9.4.6]. \square

Example 3.4. *Let $V(x) = \ln(|x|^2 + 1)$ if $|x| > 1$. Then the condition $L_0V \leq C + CV$ is equivalent to the inequality*

$$\begin{aligned} 2 \operatorname{tr} A(t, x) - 4 \frac{\langle A(t, x)x, x \rangle}{|x|^2 + 1} + c(t, x)(|x|^2 + 1) \ln(|x|^2 + 1) + 2 \langle b(t, x), x \rangle \\ \leq C(|x|^2 + 1) + C(|x|^2 + 1) \ln(|x|^2 + 1). \end{aligned} \quad (3.8)$$

Proof. See [BKRS2015, Theorem 9.4.7]. \square

3.5 Degenerate diffusion coefficients

3.5.1 A uniqueness result of LeBris/Lions

Here we assume that $c = 0$ in (3.1), i.e., we consider a partial differential operator of the form

$$L_0\varphi = a^{ij} \partial_{x_i} \partial_{x_j} \varphi + b^i \partial_{x_i} \varphi, \quad \varphi \in D_0 := C_0^\infty(\mathbb{R}^d), \quad (3.9)$$

where $a^{ij}, b^i, 1 \leq i, j \leq d$, are as in (3.1), and its corresponding FPKE (3.2).

Let $\sigma^{ij} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ be $\mathcal{B}(\mathbb{R}^d)$ -measurable functions such that $A = \sigma \sigma^*$, where $\sigma := (\sigma^{ij})_{1 \leq i, j \leq d}$. Set

$$\beta^i := b^i - \partial_{x_j} a^{ij}, \quad 1 \leq i, j \leq d.$$

The following result is due to C. LeBris and P.L. Lions (see [LBL2008, Proposition 5], and also [BKRS2015, Theorem 9.8.1]).

Theorem 3.5. *Suppose that in the natural notation*

$$\sigma^{ij} \in L^2([0, T]; W_{loc}^{1,2}(\mathbb{R}^d, dx)), \quad \beta^i \in L^1([0, T]; W_{loc}^{1,1}(\mathbb{R}^d, dx)),$$

$$\operatorname{div} \beta \in L^1([0, T]; L^\infty(\mathbb{R}^d, dx)), \quad \frac{|\beta|}{1 + |x|} \in L^1([0, T]; L^1(\mathbb{R}^d, dx)) + L^1([0, T]; L^\infty(\mathbb{R}^d, dx)),$$

$$\frac{\sigma^{ij}}{1+|x|} \in L^2([0, T]; L^2(\mathbb{R}^d, dx)) + L^2([0, T]; L^\infty(\mathbb{R}^d, dx)).$$

Then, for every initial condition given by density ρ_0 from $L^1(\mathbb{R}^d, dx) \cap L^\infty(\mathbb{R}^d, dx)$ there exists a unique solution to (3.2) with $\mu_0 := \rho_0 dx$ in the class

$$\{\rho: \rho \in L^\infty([0, T]; L^1(\mathbb{R}^d, dx) \cap L^\infty(\mathbb{R}^d, dx)), \sigma^* \nabla \rho \in L^2([0, T]; L^2(\mathbb{R}^d, dx)).\}$$

3.5.2 Uniqueness in the class of absolutely continuous paths of probability measures

Here we assume

(H3) (H1) is satisfied with $\gamma = \gamma(U)$, $M = M(U)$, independent of the ball U and $(t, x) \mapsto A(t, x)$ is Lipschitz in t and x on $[0, T] \times \mathbb{R}^d$, $T > 0$.

(H4) $b \in L^\infty([0, \infty) \times \mathbb{R}^d, dt dx; \mathbb{R}^d)$.

Furthermore, we fix a $\mathcal{B}([0, T] \times \mathbb{R}^d)$ -measurable non-negative function $\tilde{\rho}: [0, \infty) \times \mathbb{R}^d \rightarrow [0, \infty)$.

Consider the operator

$$L_0 \varphi = \tilde{\rho} \operatorname{div}(A \nabla \varphi) + \sqrt{\tilde{\rho}} \langle b, \nabla \varphi \rangle, \quad \varphi \in D_0 := C_0^\infty(\mathbb{R}^d). \quad (3.10)$$

and its corresponding FPKE (3.2).

Define \mathcal{Z}_ν to be the set of all $(\mu_t)_{t \in [0, T]} \in \mathcal{SP}(T)$ such that $\mu_0 = \nu$ and $\mu_t dt$ is absolutely continuous w.r.t. $dx dt$ with density $z := \frac{d(\mu_t dt)}{dx dt}$ satisfying the following properties:

$$(\mu_t)_{t \in [0, T]} \text{ solves the FPKE corresponding to (3.10).} \quad (3.11)$$

$$\mu_t(\mathbb{R}^d) = 1 \text{ for } dt\text{-a.e. } t \in [0, T]. \quad (3.12)$$

$$\tilde{\rho} z \in L^2([0, T] \times U, dt dx) \text{ for all balls } U \subset \mathbb{R}^d. \quad (3.13)$$

$$\lim_{N \rightarrow \infty} \int_0^T \int_{N \leq |x| \leq 2N} \left[\frac{\sqrt{\tilde{\rho}(t, x)} + \tilde{\rho}(t, x)}{1 + |x|} z(t, x) + \frac{\tilde{\rho}^2(t, x)}{1 + |x|^2} z^2(t, x) \right] dx dt = 0. \quad (3.14)$$

Theorem 3.6. *Suppose that (H3) and (H4) hold. Then \mathcal{Z}_ν contains at most one element.*

Proof. This follows from [BKRS2015, Theorem 9.8.2]. \square

4 Applications to the Markov uniqueness problem

4.1 The Framework

Also in this section we take $(E, \mathcal{B}) := (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and $m := \rho dx$, where

$$\rho \in L_{loc}^1(\mathbb{R}^d, dx), \quad \rho > 0 \text{ } dx\text{-a.e.}$$

We consider the following partial differential operator:

$$L_0\varphi = \frac{1}{\rho}\partial_{x_i}(\rho a^{ij}\partial_{x_j}\varphi) + c\varphi, \quad \varphi \in D_0 := C_0^\infty(\mathbb{R}^d), \quad (4.1)$$

where a^{ij} , $1 \leq i, j \leq d$, and c satisfy assumption (A) below, which we assume to hold throughout this section:

(A) $a^{ij}, c : \mathbb{R}^d \rightarrow \mathbb{R}$ are $\mathcal{B}(\mathbb{R}^d)$ measurable, $c \leq 0$, and $A(x) := (a^{ij}(x))_{1 \leq i, j \leq d}$ is a nonnegative definite matrix for all $x \in \mathbb{R}^d$. Furthermore,

$$a^{ij} \in W_{loc}^{1,1}(\mathbb{R}^d, dx) \cap L_{loc}^2(\mathbb{R}^d, \rho dx); c, \partial_{x_i}a^{ij} \in L_{loc}^2(\mathbb{R}^d, \rho dx), \rho^{\frac{1}{2}} \in W_{loc}^{1,1}(\mathbb{R}^d, dx)$$

such that

$$a^{ij}\rho^{-\frac{1}{2}}\partial_{x_i}\rho^{\frac{1}{2}} \in L_{loc}^2(\mathbb{R}^d, \rho dx)$$

for all $1 \leq i, j \leq d$.

Remark 4.1. We note that (A) is a standard a priori assumption on L_0 in (4.1), because it implies the following:

(i) for every $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$L_0\varphi = a^{ij}\partial_{x_i}\partial_{x_j}\varphi + (\partial_{x_i}a^{ij})\partial_{x_j}\varphi + 2\rho^{-\frac{1}{2}}\partial_{x_i}\rho^{\frac{1}{2}}a^{ij}\partial_{x_j}\varphi + c\varphi, \quad (4.2)$$

and $(L_0, C_0^\infty(\mathbb{R}^d))$ is symmetric on $L^2(\mathbb{R}^d, \rho dx)$, i.e., $L_0 \subset L_0^*$, where the adjoint is taken in $L^2(\mathbb{R}^d, \rho dx)$.

(ii) The nonnegative definite symmetric bilinear form

$$\begin{aligned} \mathcal{E}_0(\psi, \varphi) &:= - \int \psi L_0 \varphi \rho dx \\ &= \int \langle A \nabla \psi, \nabla \varphi \rangle_{\mathbb{R}^d} \rho dx - \int c \psi \varphi \rho dx; \quad \psi, \varphi \in C_0^\infty(\mathbb{R}^d), \end{aligned}$$

is a symmetric pre-Dirichlet form, hence its closure $(\mathcal{E}_F, D(\mathcal{E}_F))$ is a symmetric Dirichlet form, whose corresponding generator $(-L_F, D(L_F))$ is just the Friedrichs extension of $(L_0, C_0^\infty(\mathbb{R}^d))$. In particular, $T_t^{L_F} := e^{tL_F}$, $t \geq 0$, is sub-Markovian. We refer to [FOT2011, Section 3.3] and [MR1992, Chapter II, Section 1a) and 1c)] for details on the standard proofs for the above claims. In particular, for (L_0, D_0) as above

$$\mathcal{M}_{sym, M}(L_0, D_0) \neq \emptyset.$$

Below we shall present various sets of additional assumptions on a^{ij} , $1 \leq i, j \leq d$, and c so that a respective theorem from the previous section will apply to imply

$$\# \mathcal{M}_{sym, M}(L_0, D_0) = 1,$$

i.e., to imply Markov uniqueness for $(L_0, C_0^\infty(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d, \rho dx)$. We briefly repeat the set-up in each subsection to ease selective reading.

Remark 4.2. As mentioned above, we only consider time-independent coefficients for the operator in (4.1) and assume symmetry of L_0 on some weighted L^2 -space over \mathbb{R}^d . As shown in Section 2, however, our approach is much more general and could be applied also to non-symmetric cases and for more general state spaces than merely $E = \mathbb{R}^d$. By time-space homogenization one can also find applications of the theorems in Section 3 to the cases of time-dependent coefficients (and the associated generalized Dirichlet forms; see [St1999] and [T2000]). A starting point for the nonsymmetric case could be the case of an operator L_0 as in (3.1) with time-independent coefficients and with $c \equiv 0$, which has an infinitesimally invariant measure μ , or equivalently has a stationary solution μ to its corresponding FPKE (3.2). This case has been studied intensively in [BKRS2015] in Chapters 1-5. In particular, it has been shown there that under broad conditions μ has a reasonably regular density with respect to Lebesgue measure and L_0 can be written as the sum of a symmetric operator L_{sym} on $L^2(\mathbb{R}^d, \mu)$ and a vector field b which has divergence zero with respect to μ . In this case L_0^* on $L^2(\mathbb{R}^d, \mu)$, calculated on $D_0(= C_0^\infty(\mathbb{R}^d))$, is just given by $L_{sym} - \langle b, \nabla \rangle_{\mathbb{R}^d}$ and then one can proceed analogously as in the symmetric case to obtain Markov uniqueness results in this nonsymmetric case, which falls into the class (1) introduced in the Introduction.

4.2 Nondegenerate VMO diffusion coefficients

Let (L_0, D_0) be as in (4.1) (respectively, (4.2)) and assume that assumption (A) holds. Let

$$\mathcal{M}_{sym, M} := \mathcal{M}_{sym, M}(L_0, D_0)$$

be as defined in Section 1.

Theorem 4.3. *Suppose (A) and (H1) hold and that $a^{ij} \in VMO_{x, loc}([0, T] \times \mathbb{R}^d)$, $1 \leq i, j \leq d$. Additionally, assume that for $1 \leq i, j \leq d$*

$$a^{ij}, \partial_{x_i} a^{ij} + a^{ij} \rho^{-\frac{1}{2}} \partial_{x_i} \rho^{\frac{1}{2}}, c \in L^1(\mathbb{R}^d, \rho dx) + L^\infty(\mathbb{R}^d, \rho dx). \quad (4.3)$$

Then

$$\mathcal{M}_{sym, M} = \{L_F\},$$

i.e. Markov uniqueness holds for $(L_0, C_0^\infty(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d, \rho dx)$.

Proof. Let $L \in \mathcal{M}_{sym, M}$ and $T_t^L := e^{tL}$, $t \geq 0$. Let $u \in L^\infty(\mathbb{R}^d, \rho dx)$, $u \geq 0$, $\int u \rho dx = 1$ and $\mu_t^L := T_t^L u \rho dx$, $t \geq 0$. Then $(\mu_t^L)_{t \geq 0} \in \mathcal{SP}$ for all $t \geq 0$ and $\mu_0 = u \rho dx =: \nu$. Now let us check that $(\mu_t^L)_{t \geq 0} \in \mathcal{SP}_\nu$, i.e., satisfies (3.3) - (3.6). We have seen in (2.8) that $(\mu_t^L)_{t \geq 0}$ solves the FPKE (3.2), hence (3.3) holds.

From (4.2) it follows that L_0 in this section is of type (3.1) with

$$b^j := \partial_{x_i} a^{ij} + 2a^{ij} \rho^{-\frac{1}{2}} \partial_{x_i} \rho^{\frac{1}{2}}, 1 \leq j \leq d. \quad (4.4)$$

Since $T_t^L u \in (L^1 \cap L^\infty)(\rho dx)$, it follows from (A) and condition (4.3) that also (3.4), (3.5) holds, and additionally we have that

$$a^{ij}, b^j \in L^1([0, T] \times \mathbb{R}^d; \mu_t^L dt) 1 \leq i, j \leq d. \quad (4.5)$$

So, it remains to check the second half of (3.6). To this end let $\chi_n \in C_0^\infty(\mathbb{R}^d)$, $n \in \mathbb{N}$, such that $\mathbb{1}_{B_n} \leq \chi_n \leq \mathbb{1}_{B_{n+1}}$ for all $n \in \mathbb{N}$, $\sup_{n \in \mathbb{N}} \|\chi_n'\|_\infty < \infty$, $\sup_{n \in \mathbb{N}} \|\chi_n''\|_\infty < \infty$, and $\chi_n \nearrow \mathbb{1}$ in n , where B_n denotes the ball in \mathbb{R}^d with center 0 and radius n .

Then by (4.3) for all $t \geq 0$

$$\begin{aligned}
\nu(\mathbb{R}^d) - \mu_t^L(\mathbb{R}^d) &= \int u \rho \, dx - \lim_{n \rightarrow \infty} \int \chi_n T_t^L u \rho \, dx \\
&= \lim_{n \rightarrow \infty} \int (1 - T_t^L \chi_n) u \rho \, dx \\
&= \lim_{n \rightarrow \infty} \int (1 - \chi_n - \int_0^t T_s^L L_0 \chi_n \, ds) u \rho \, dx \\
&= - \lim_{n \rightarrow \infty} \int_0^t \int L_0 \chi_n T_s^L u \rho \, dx \, ds \\
&= - \int_0^t \int c \, d\mu_s^L \, ds
\end{aligned} \tag{4.6}$$

and the second part of (3.6) follows even with equality sign. Hence $(\mu_t^L)_{t \geq 0} \in \mathcal{SP}_\nu$. By (4.5) it thus follows that $(\mu_t^L)_{t \geq 0}$ also lies in the set defined in (3.7). Since T_t^L , $t \geq 0$, is uniquely determined by its values on all functions u as above and $L \in \mathcal{M}_{sym, M}$ was arbitrary, Theorem 3.1 implies that

$$\#\mathcal{M}_{sym, M} \leq 1.$$

Now the assertion follows by Remark 4.1(ii). \square

4.3 Nondegenerate locally Lipschitz diffusion coefficients

Let (L_0, D_0) be as in (4.1) (respectively, (4.2)) such that assumption (A) holds and let $\mathcal{M}_{sym, M} := \mathcal{M}_{sym, M}(L_0, D_0)$ be defined as in Section 1. In the following result we shall assume (H2) for our a^{ij} , $1 \leq i, j \leq d$, which is stronger than the local VMO-condition in Theorem 4.3. As a reward we can relax the global conditions in (4.3). We need, however, to restrict to the case $c \equiv 0$.

Theorem 4.4. *Suppose that $c \equiv 0$ and that conditions (A), (H1) and (H2) hold. Additionally, assume that for $1 \leq i, j \leq d$, and some $p > d + 2$*

$$\rho^{-\frac{1}{2}} \partial_{x_i} \rho^{\frac{1}{2}} \in L_{loc}^p(\mathbb{R}^d, dx), \tag{4.7}$$

and that

$$\frac{|a^{ij}|}{(1 + |x|^2)} + \frac{|\partial_{x_i} a^{ij} + a^{ij} \rho^{-\frac{1}{2}} \partial_{x_i} \rho^{\frac{1}{2}}|}{(1 + |x|)} \in L^1(\mathbb{R}^d, \rho \, dx) + L^\infty(\mathbb{R}^d, \rho \, dx) \tag{4.8}$$

Then

$$\mathcal{M}_{sym, M} = \{L_F\},$$

i.e., Markov uniqueness holds for $(L_0, C_0^\infty(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d, \rho \, dx)$.

Proof. Define $b = (b^j)_{1 \leq j \leq d}$ as in (4.4). We note that by (H2) we have $\partial_{x_i} a^{ij} \in L_{loc}^\infty(\mathbb{R}^d, dx)$ for $1 \leq i, j \leq d$. Let $L \in \mathcal{M}_{sym, M}$ and let $(\mu_t^L)_{t \geq 0}$, ν , χ_n , $n \in \mathbb{N}$, be as defined in the proof of Theorem 4.3. Then for every $t \geq 0$, since T_t^L is sub-Markovian, we have

$$\begin{aligned} \mu_t^L(\mathbb{R}^d) &= \int T_t^L u \rho \, dx \\ &= \lim_{n \rightarrow \infty} \int \chi_n T_t^L u \rho \, dx \\ &= \lim_{n \rightarrow \infty} \int T_t^L \chi_n u \rho \, dx \\ &\leq \int u \rho \, dx = \nu(\mathbb{R}^d). \end{aligned}$$

Hence (3.6) holds and then exactly as in the proof of Theorem 4.3 one checks (without using (4.8)) that by assumption (A) also (3.3)–(3.5) hold to conclude that $(\mu_t^L)_{t \geq 0} \in \mathcal{SP}_\nu$. Furthermore, since $T_t^L u \in (L^1 \cap L^\infty)(\mathbb{R}^d, \rho \, dx)$, the left-hand side of (4.8) is also an element of $L^1([0, T] \times \mathbb{R}^d, \mu_t^L dt)$, hence by (4.7) all assumptions of Theorem 3.2 are fulfilled. So, $\#\mathcal{M}_{sym, M} \leq 1$, and Remark 4.1(ii) implies the assertion. \square

Remark 4.5. Let us mention the uniqueness problem studied [H1954] for the one-dimensional Fokker-Planck-Kolmogorov equation. For simplicity we consider the case of the unit diffusion coefficient (note that in [H1954] the opposite notation is used, the drift is denoted by a , but we follow our notation). The problem posed in [H1954, §8, p. 116] (in the case of the equation on the whole real line) is this: to find necessary and sufficient conditions in order that for every function $h \in L^1(\mathbb{R})$ with $Lh = h'' - (bh)' \in L^1(\mathbb{R})$ there is a unique solution $T(x, t, h)$ of the equation $\partial_t u = \partial_x^2 u - \partial_x(ub)$ with initial condition h in the sense of the relation $\|T(\cdot, t, h) - h\|_{L^1} \rightarrow 0$ as $t \rightarrow 0$. This setting is called Problem L_0 , and in Problem L it is required in addition that the solutions with probability initial densities from the domain of definition of the operator L must be probabilistic. According to [H1954, Theorems 8.5 and 8.7], where the drift coefficient is assumed to be continuous, a necessary and sufficient condition for the solvability of Problem L_0 is the divergence of the integral

$$\int_0^x \exp B(y) \int_0^y \exp(-B(u)) \, du \, dy, \quad \text{where} \quad B(y) = \int_0^y b(s) \, ds$$

at $-\infty$ and $+\infty$, and for the solvability of Problem L the divergence of the integral

$$\int_0^x \exp(-B(y)) \int_0^y \exp(B(u)) \, du \, dy$$

at $-\infty$ and $+\infty$ is additionally required. This is the previous condition for the drift $-b$, which makes the conditions for b and $-b$ the same. In both cited theorems of Hille the closure of the operator L generates a semigroup on $L^1(\mathbb{R})$. It is proved in [BKS21] that a probability solution is always unique in the one-dimensional case (under the stated assumptions about a and b). However, an example constructed in [BKS21] shows that the situation is possible where for an initial condition that is a probability measure there exists

a unique probability solution of the Cauchy problem, but there are also other solutions. It is worth noting that it is asserted in Remark 4.6 in [BKS21] that if Hille's condition is violated, then for some initial condition there is no solution, but this does follow from the results in [BKS21], because they ensure uniqueness only for probability solutions, so that one cannot rule out the possibility that existence holds for all initial solutions, but uniqueness fails in the class of signed solutions.

4.4 Nondegenerate diffusion coefficients and Lyapunov function conditions

Let (L_0, D_0) be as in (4.1) (respectively, (4.2)) and assume that assumption (A) holds. Let $\mathcal{M}_{sym,M} := \mathcal{M}_{sym,M}(L_0, D_0)$ be as defined in Section 1.

Theorem 4.6. *Suppose that $c \equiv 0$ and that conditions (A), (H1) and (H2) hold. Additionally, assume that (4.7) holds and that (3.8) holds with $b = (b^j)_{1 \leq j \leq d}$ defined as in (4.4). Then*

$$\mathcal{M}_{sym,M} = \{L_F\},$$

i.e., Markov uniqueness holds for $(L_0, C_0^\infty(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d, \rho dx)$.

Proof. The proof is completely analogous to the proof of Theorem 4.4 except for applying Theorem 3.3 and Example 3.4 instead of Theorem 3.2 and replacing condition (4.8) by (3.8). \square

Remark 4.7. We would like to point out that Theorem 4.6 is close to Corollary 2.3 in [St1999a] and to Proposition 2.9.4 in [BRSt2000]. However, it is not covered by them, since ρ is not a probability density. The function ρ is not even assumed to be in $L^1(\mathbb{R}^d, dx)$ here.

4.5 Degenerate diffusion coefficients

4.5.1 Markov uniqueness as a consequence of the results of Le Bris and Lions

Theorem 4.8. *Let $\sigma := (\sigma^{ij})_{1 \leq i, j \leq d}$, $A := \sigma \sigma^*$ and $a^{ij} = (\sigma \sigma^*)^{ij}$, $1 \leq i, j \leq d$, where*

$$\sigma^{ij} \in W_{loc}^{1,2}(\mathbb{R}^d, dx), \partial_{x_i} a^{ij} \in W_{loc}^{1,2}(\mathbb{R}^d, dx) \quad (4.9)$$

and

$$\sigma^{ij}, \partial_{x_j} \partial_{x_i} a^{ij} \in L^\infty(\mathbb{R}^d, dx), \frac{\partial_{x_i} a^{ij}}{1 + |x|} \in L^1(\mathbb{R}^d, dx) + L^\infty(\mathbb{R}^d, dx). \quad (4.10)$$

Then condition (A) holds for $\rho \equiv 1, c \equiv 0$, and the corresponding operator (L_0, D_0) from (4.2) is symmetric on $L^2(\mathbb{R}^d, dx)$. Let $\mathcal{M}_{sym,M} := \mathcal{M}_{sym,M}(L_0, D_0)$ be as defined in Section 1. Then

$$\mathcal{M}_{sym,M} = \{L_F\},$$

i.e., Markov uniqueness holds for $(L_0, C_0^\infty(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d, dx)$.

Proof. Let $L \in \mathcal{M}_{sym,M}$ and $\mu_t^L := T_t^L u \, dx, t \geq 0$, with u as in the proof of Theorem 4.3. Then by assumptions (4.9), (4.10), we can apply Theorem 3.5 with $\rho_0 := u$, since $T_t^L u \in (L^1 \cap L^\infty)(\mathbb{R}^d, dx)$ and $\sigma^* \nabla T_t^L u \in L^2(\mathbb{R}^d, dx; \mathbb{R}^d)$, because $\nabla T_t^L u \in L^2(\mathbb{R}^d, dx; \mathbb{R}^d)$ and $\sigma^{ij} \in L^\infty(\mathbb{R}^d, dx), 1 \leq i, j \leq d$. Hence $\#\mathcal{M}_{sym,M} \leq 1$ and by Remark 4.1(ii) the assertion follows. \square

4.5.2 Markov uniqueness in another degenerate case

Let $\rho \in (L^1 \cap L^3)(\mathbb{R}^d, dx)$ such that

$$\rho > 0, \int \rho dx = 1, \rho^{\frac{1}{2}} \in W_{loc}^{1,1}(\mathbb{R}^d, dx)$$

and $\nabla \rho^{\frac{1}{2}} \in L^\infty(\mathbb{R}^d, dx; \mathbb{R}^d)$, and assume that (H3) holds. Consider the operator

$$L_0 \varphi := \rho \operatorname{div}(A \nabla \varphi) + \sqrt{\rho} \langle A \nabla \sqrt{\rho}, \nabla \varphi \rangle_{\mathbb{R}^d}, \varphi \in D_0 := C_0^\infty(\mathbb{R}^d), \quad (4.11)$$

and its corresponding FPKE (3.2). Note that by our assumptions on A and ρ we have that $L_0: D_0 \subset L^2(\mathbb{R}^d, \rho \, dx) \rightarrow L^2(\mathbb{R}^d, \rho \, dx)$ and $L_0 \varphi = \frac{1}{\rho} \operatorname{div}(\rho^2 A \nabla \varphi)$ for all $\varphi \in D_0 = C_0^\infty(\mathbb{R}^d)$, hence (L_0, D_0) is symmetric on $L^2(\mathbb{R}^d, \rho \, dx)$. Let $\mathcal{M}_{sym,M} := \mathcal{M}_{sym,M}(L_0, D_0)$ be as defined in Section 1.

Theorem 4.9. *Assume that (H3) holds and let ρ satisfy the assumptions specified above. Then*

$$\mathcal{M}_{sym,M} = \{L_F\},$$

i.e., Markov uniqueness holds for $(L_0, C_0^\infty(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d, \rho \, dx)$.

Proof. Let $L \in \mathcal{M}_{sym,M}$ and $\mu_t^L := T_t^L u \, \rho \, dx, t \geq 0$, with u as in the proof of Theorem 4.3. We have seen in (2.8) that $(\mu_t^L)_{t \geq 0}$ solves the FPKE associated with (L_0, D_0) in (4.11). To show that it is the only such solution we are going to apply Theorem 3.6. So, let us check its assumptions for $z(t, \cdot) := T_t^L u \, \rho$ and $\tilde{\rho} := \rho$. First of all, (3.11) holds as just seen. So, let us show (3.12). As in (4.6) we have for every $t \geq 0$

$$\mu_t^L(\mathbb{R}^d) = \int u \rho \, dx + \lim_{n \rightarrow \infty} \int_0^t \int L_0 \chi_n T_s^L u \, \rho \, dx ds.$$

By our assumptions about A and since $\nabla \sqrt{\rho} \in L^\infty(\mathbb{R}^d, dx; \mathbb{R}^d)$, we have that for some $C \in (0, \infty)$ and all $s \geq 0$

$$\sup_n |L_0 \chi_n T_s^L u| \leq C \|u\|_\infty (\rho + 1), \quad dx - \text{a.e.}$$

Since $\rho \in (L^1 \cap L^2)(\mathbb{R}^d, dx)$ and $L_0 \chi_n \rightarrow 0 \, dx$ -a.e. as $n \rightarrow \infty$, we conclude that

$$\mu_t^L(\mathbb{R}^d) = \int u \rho \, dx = 1 \quad \text{for all } t \geq 0.$$

Next, (3.13) is clear, since $T_t^L u \in L^\infty(\mathbb{R}^d, dx)$ and $\rho \in L_{loc}^\infty(\mathbb{R}^d, dx)$, because $\nabla \rho^{\frac{1}{2}} \in L^\infty(\mathbb{R}^d, dx)$.

Finally, let us show (3.14). It suffices to show that all functions under the integral in (3.14) are in $L^1(\mathbb{R}^d, dx)$ in our case, due to our assumptions. For the first summand this is immediate, since

$$\begin{aligned} (\rho^{\frac{1}{2}} + \rho) z(t, \cdot) &= (\rho^{\frac{1}{2}} + \rho) \rho T_t^L u \\ &\leq (1 + 2\rho) \rho \|u\|_\infty \in L^1(\mathbb{R}^d, dx), \end{aligned}$$

since $\rho \in (L^1 \cap L^2)(\mathbb{R}^d, dx)$ by assumption. For the second summand we note that $\rho^{\frac{1}{2}}$ has a Lipschitz dx -version on \mathbb{R}^d , since $\nabla \sqrt{\rho} \in L^\infty(\mathbb{R}^d, dx; \mathbb{R}^d)$ by assumption. Hence $\rho^{\frac{1}{2}}$ is of at most linear growth and thus ρ of at most quadratic growth. Hence, since $\rho \in L^3(\mathbb{R}^d, dx)$, for some $C \in (0, \infty)$ and all $t \geq 0$ we have

$$\frac{\rho^2(x)}{1 + |x|^2} z^2(t, \cdot) \leq C \rho^3 \|u\|_\infty \in L^1(\mathbb{R}^d, dx),$$

and altogether (3.14) follows. Since (H4) also holds by our assumptions about A and $\nabla \sqrt{\rho} \in L^\infty(\mathbb{R}^d, dx; \mathbb{R}^d)$, we can apply Theorem 3.6 and conclude that $\#\mathcal{M}_{sym, M} \leq 1$ and again by Remark 4.1(ii) the assertion follows. \square

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References

- [AKMR2020] S. Albeverio, H.Kawabi, S.-R. Mihalache, M. Röckner, *Strong uniqueness for Dirichlet operators related to stochastic quantization under exponential/trigonometric interactions on the two-dimensional torus* arxiv:2004.12383, to appear in Ann. Scuola Norm. Sup. Pisa Cl. Sci., 28pp. , 2021+.
- [AKR1995] S. Albeverio, Yu.G. Kondratiev, M. Röckner, *Dirichlet operators via stochastic analysis*. J. Funct. Anal. 128 (1995), no. 1, pp. 102–138.
- [AMR2014] S. Albeverio, Z.-M. Ma, M. Röckner, *Quasi regular Dirichlet forms and the stochastic quantization problem*. Festschrift Masatoshi Fukushima, pp. 27–58, Interdiscip. Math Sci., 17, World Sci. Publ. Hackensack, NJ, 2015.
- [AR1995] S. Albeverio, M. Röckner, *Dirichlet form methods for uniqueness of martingale problems and applications*. Stochastic Analysis. Proceedings of Symposia in Pure Mathematics (Providence, Rhode Island) (M. C. Cranston and M. A. Pinsky, eds.), vol. 57, Amer. Math. Soc., 1995, pp. 513–528.

- [ARZ1993] S. Albeverio, M. Röckner, T. S. Zhang, *Markov uniqueness for a class of infinite dimensional Dirichlet operators*. Stochastic Processes and Optimal Control (H. J. Engelbert et al., eds.), Stochastic Monographs, vol. 7, Gordon & Breach, 1993, pp. 1–26.
- [BDPRSt2007] V. I. Bogachev, G. Da Prato, W. Stannat, *Uniqueness of solutions to weak parabolic equations for measures*, Bull. London Math. Soc. 39 (2007), no. 4, 631–640.
- [BKS21] V. I. Bogachev, T. I. Krasovitskii, S. V. Shaposhnikov, *On uniqueness of probability solutions of the Fokker-Planck-Kolmogorov equation*. Sbornik Math. 212 (2021), no. 6, 745–781.
- [BKRS2015] V. I. Bogachev, N. V. Krylov, M. Röckner, S. V. Shaposhnikov, *Fokker–Planck–Kolmogorov equations*. Amer. Math. Soc., Providence, RI, 2015, pp. xii+479.
- [BRSt2000] V. I. Bogachev, M. Röckner, W. Stannat, *Uniqueness of invariant measures and essential m -dissipativity of diffusion operators on L^1* . Infinite dimensional Stochastic Analysis (P. Clément et al., eds.), Royal Netherlands Academy of Arts and Sciences, Amsterdam, 2000, pp. 39–54.
- [Eb1999] A. Eberle, *Uniqueness and non-uniqueness of singular diffusion operators*. Lecture Notes in Math. V. 1718, Springer, Berlin, 1999; 262 pp.
- [EL2006] K.D. Elworthy, X.-M. Li, *Interwining and the Markov uniqueness problem on path spaces*. Stochastic partial differential equations and applications–VII, pp. 89–95, Lect. Notes Pure Appl. Math., 245, Chapman & Hall/CRC, Boca Raton, FL 2006.
- [FOT2011] M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet forms and symmetric Markov processes*. 2nd ed. Walter de Gruyter, Berlin – New York, 2011; x+489 pp.
- [H1954] E. Hille, *The abstract Cauchy problem and Cauchy’s problem for parabolic differential equations*. J. Anal. Math. 3 (1954), 81–196.
- [K2007] N. V. Krylov, *Parabolic and elliptic equations with VMO coefficients*. Comm. Part. Differ. Equ. 32 (2007), 453–475.
- [LBL2008] C. Le Bris, P. L. Lions, *Existence and uniqueness of solutions to Fokker-Planck type equations with irregular coefficients*. Comm. Partial Differ. Equ. 33 (2008), 1272–1317.

- [MR1992] Z. M. Ma, M. Röckner, *Introduction to the theory of (non-symmetric) Dirichlet forms*. Universitext, Springer, Berlin, 1992, viii+209 pp.
- [Pa1985] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*. Springer-Verlag, New York, 1983; viii+279 pp.
- [RS2011] D.W. Robinson, A. Sikora, *Markov uniqueness of degenerate elliptic operators*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 10(2011), no.3, pp. 683–710.
- [RZZ2017] M. Röckner, R. Zhu, X. Zhu, *Restricted Markov uniqueness for the stochastic quantization of $P(\varphi)_2$ and its applications*. J. Funct. Anal. 272 (2017), no. 10, 4263–4303.
- [St1999] W. Stannat, *The theory of generalized Dirichlet forms and its applications in analysis and stochastics*. Mem. Amer. Math. Soc. V. 142 (1999), No. 678; viii+101 pp.
- [St1999a] W. Stannat, *(Nonsymmetric) Dirichlet operators on L^1 : existence, uniqueness and associated Markov processes*. Ann. Sc. Norm. Sup. Pisa Cl. Sci. (4) 28 (1999), no. 1, 99–140.
- [Ta1985] M. Takeda, *On the uniqueness of Markovian-self-adjoint extension of diffusion operators on infinite dimensional spaces*. Osaka J. Math. 22 (1985), 733–742.
- [T2000] G. Trutnau, *Stochastic calculus of generalized Dirichlet forms and applications to stochastic differential equations in infinite dimensions*. Osaka J. Math. 37 (2000), no. 2, 315–343.